A study of the lex plus powers conjecture

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Abstract

An \{a_1, \ldots, a_n\}-lex plus powers ideal is a monomial ideal in \( I \subset k[x_1, \ldots, x_n] \) which minimally contains the regular sequence \( x_1^{a_1} \ldots, x_n^{a_n} \) and such that whenever \( m \in R_t \) is a minimal generator of \( I \) and \( m' \in R_t \) is greater than \( m \) in lex order, then \( m' \in I \). Conjectures of Eisenbud et al. and Charalambous and Evans predict that after restricting to ideals containing a regular sequence in degrees \( \{a_1, \ldots, a_n\} \), then \{a_1, \ldots, a_n\}-lex plus powers ideals have extremal properties similar to those of the lex ideal. That is, it is proposed that a lex plus powers ideal should give maximum possible Hilbert function growth (Eisenbud et al.), and, after fixing a Hilbert function, that the Betti numbers of a lex plus powers ideal should be uniquely largest (Charalambous, Evans).

The first of these assertions would extend Macaulay’s theorem on Hilbert function growth, while the second improves the Bigatti, Hulett, Pardue theorem that lex ideals have largest graded Betti numbers. In this paper we explore these two conjectures. First we give several equivalent forms of each statement. For example, we demonstrate that the conjecture for Hilbert functions is equivalent to the statement that for a given Hilbert function, lex plus powers ideals have the most minimal generators in each degree. We use this result to prove that it is enough to show that lex plus powers ideals have the most minimal generators in the highest possible degree. A similar result holds for the stronger conjecture. In this paper we also prove that if the weaker conjecture holds, then lex plus powers ideals are guaranteed to have largest socles. This suffices to show that the two conjectures are equivalent in dimension \( \leq 3 \), which proves the monomial case of the conjecture for Betti numbers in those degrees. In dimension 2, we prove both conjectures outright.

\( \text{MSC: 13D02} \)

1. Introduction

It is well known that lexicographic ideals in \( k[x_1, \ldots, x_n] \) exhibit interesting extremal properties. Macaulay \[11\] showed a long time ago that quotients of lex ideals attain
the largest possible Hilbert function growth. That is, given an ideal \( I \) and a degree \( d \), there exists a lex ideal \( L \) such that the Hilbert functions of \( L \) and \( I \) agree in degree \( d \), and the Hilbert function of \( L \) bounds that of \( I \) in degree \( d + 1 \). This is equivalent to the statement that lex ideals attain all possible Hilbert functions. More precisely, the set of valid Hilbert functions of cyclic modules is exactly the set of Hilbert functions of lex ideals.

Macaulay’s theorem can also be formulated in terms of generators. The theorem is equivalent to the statement that \( L \) has more generators than \( I \) in each degree for any \( I \subset R \) and lex ideal \( L \) such that \( I \) and \( L \) share the same Hilbert function. This version of Macaulay’s theorem bounds the graded Betti numbers of the \( L \)rst syzygy of \( I \) with those of \( L \), after fixing a Hilbert function.

An even stronger theorem, due to Bigatti [1] and Hulett [10], fixes a Hilbert function and compares all the graded Betti numbers of \( I \) and \( L \). Given a Hilbert function \( \mathcal{H} \), they show that the lex ideal attaining \( \mathcal{H} \) has the unique largest graded Betti numbers among all ideals attaining \( \mathcal{H} \). Pardue [12] later extended this result to positive characteristic. That the Bigatti–Hulett–Pardue result implies Macaulay’s theorem is clear from the third formulation given above.

It is suspected that the extremal behavior of lex ideals is actually an example (albeit, a special one) of the extremal behavior of the larger set of lex plus powers ideals. A lex plus powers ideal \( L \) is an ideal with \( \{ x_1^{a_1}, \ldots, x_n^{a_n} \} \) among its minimal generators for some \( 1 \leq a_1 \leq \cdots \leq a_n \), such that for each pair of monomials \( m_1 >_{\text{lex}} m_2 \) in \( R_d \), if \( m_2 \) is a non-pure-power minimal generator of \( L \), then \( m_1 \in L \). Clearly any lex ideal is lex plus powers. Conjectures of Eisenbud et al. [6], and Charalambous and Evans [7] predict that lex plus powers ideals should exhibit extremal properties. Given \( a_1 \leq \cdots \leq a_n \), Eisenbud, Green, and Harris claim that after restricting to ideals containing a regular sequence of forms of degrees \( \{a_1, \ldots, a_n\} \), the \( \{a_1, \ldots, a_n\} \)-lex plus powers ideal (when it exists) should attain the largest possible Hilbert function growth. More specifically, given a degree \( d \) and an ideal \( I \) containing a \( \{a_1, \ldots, a_n\} \)-regular sequence, then if there exists an \( \{a_1, \ldots, a_n\} \)-lex plus powers ideal \( L \) such that the Hilbert functions of \( L \) and \( I \) agree in degree \( d \), then there is a \( \{a_1, \ldots, a_n\} \)-lex plus powers ideal \( L' \) such that the Hilbert functions of \( L' \) and \( I \) agree in degree \( d \) and the Hilbert function of \( L' \) bounds that of \( I \) in degree \( d + 1 \). We will refer to this conjecture as the \textit{lex plus powers conjecture for Hilbert functions} (LPPH). Eisenbud et al. first considered this problem in relation to certain questions of importance to algebraic geometers. It turns out that in the special case that \( a_1 = \cdots = a_n = 2 \), the conjecture about lex plus powers ideals implies the generalized Cayley–Bacharach conjecture [6].

Lex plus powers ideals are also thought to generalize the extremal behavior of the graded Betti numbers of lex ideals. Charalambous and Evans [7] have conjectured that after fixing a Hilbert function \( \mathcal{H} \) and a list of degrees \( \{a_1, \ldots, a_n\} \), then the graded Betti numbers of the \( \{a_1, \ldots, a_n\} \)-lex plus powers ideal attaining \( \mathcal{H} \) should be everywhere larger then the graded Betti numbers of any \( I \subset R \) such that \( I \) contains an \( \{a_1, \ldots, a_n\} \)-regular sequence, and \( H(R/I) = \mathcal{H} \). We will refer to this statement as the \textit{lex plus power conjecture} (LPP). It is not difficult to see that as well as generalizing the Bigatti–Hulett–Pardue result for lex ideals, this implies the lex plus powers conjecture for Hilbert functions.
The lex plus powers conjecture seems to be known in only one case. Francisco [8] showed that the bound holds when the lex plus powers ideal in question is an almost complete intersection. Note that the monomial case and low-dimensional cases remain unsolved. The weaker conjecture for Hilbert functions has been slightly more yielding. In addition to almost complete intersections, it is known for monomial ideals, or even if one restricts to ideals containing $x_1^{a_1}, \ldots, x_n^{a_n}$ [6,9]. In the special case that $a_1 = \cdots = a_n = 2$, I was able to prove the conjecture for $n \leq 5$ [13]. Bruns and Popescu [2] showed that if all the $a_i = 2$, then the bound holds for ideals for which all generators other than the regular sequence are in general position.

Our first task in this paper is to explore several equivalent statements of the lex plus powers conjecture for Hilbert functions. In fact, in Section 2 we give three different formulations, mimicking the case for lex ideals described above. We also demonstrate that if the conjecture is true, then the extremal behavior of lex ideals is just a special case of the extremal behavior for lex plus powers ideals. We conclude Section 2 with a discussion of the stronger lex plus powers conjecture, and note that it would similarly generalize the Bigatti–Hulett–Pardue result.

In Section 3 we give another equivalent formulation for each of the LPP and LPPH conjectures. Fix a Hilbert function $H$, a list of degrees $A$, and let $\rho$ be the Castelnuovo–Mumford regularity of the $A$-lex plus powers ideal $L$ which attains $H$. For the lex plus powers conjecture, we show in Theorem 3.5 that to prove that $L$ has uniquely largest graded Betti numbers it is enough to show that $L$ has largest graded Betti numbers in degree $\rho$. This simply means, in the notation of the computer algebra system Macaulay II [15], that it is enough to show that the last row of the Betti diagram of $L$ is largest. The same proof suffices to show that the lex plus powers conjecture for Hilbert functions is true if $L$ can be shown to have the most generators in degree $\rho + 1$.

In Section 4, we turn our attention to the relationship between the LPP and LPPH conjectures. We want to know under what circumstances the lex plus powers conjecture for Hilbert functions implies that the other graded Betti numbers of lex plus powers ideals are largest. The hope is to eventually understand exactly when LPPH implies LPP. This is interesting because the former conjecture is known in more cases than the latter. In Theorem 4.4 we show that if LPPH holds, then lex plus powers ideals have largest socles. That is, given a Hilbert function $H$, a list of degrees $\{a_1, \ldots, a_n\} = A$, and assuming that the conjecture for Hilbert functions is true, then we prove that in each degree the socle of the $A$-lex plus powers ideal attaining $H$ is larger than the socle of any $I \subset R$ attaining $H$ and containing an $A$-regular sequence.

In Section 5 we explore the implications of Theorem 4.4 in low-dimensional cases. First we demonstrate (in Theorem 5.1) that the lex plus powers conjecture for Hilbert functions holds for $n = 2$. Next we show in Theorem 5.2 that if $n \leq 3$ then the lex plus powers conjecture and the lex plus powers conjecture for Hilbert functions are equivalent. There are two immediate corollaries. First, we can conclude that the lex plus powers conjecture holds for $n = 2$. For $n = 3$, the equivalence implies that the lex plus powers conjecture holds for monomial ideals.
2. The lex plus powers conjectures

We begin with several definitions. Let \( R = k[x_1, \ldots, x_n] \) be the polynomial ring in \( n \) variables over a field \( k \), let \( m = (x_1, \ldots, x_n) \), and fix an order on the monomials, \( x_1 > \cdots > x_n \).

**Definition 2.1.** Let \( \{a_1, \ldots, a_n\} \) be a set of integers such that \( 1 \leq a_1 \leq \cdots \leq a_n \). Then we call \( \{f_1, \ldots, f_n\} \) an \( \{a_1, \ldots, a_n\} \)-regular sequence if \( \{f_1, \ldots, f_n\} \) is a regular sequence such that \( \deg(f_i) = a_i \) for \( i = 1, \ldots, n \).

We are especially interested in certain monomial ideals containing an \( \{a_1, \ldots, a_n\} \)-regular sequence, the lex plus powers ideals.

**Definition 2.2** (Charalambous and Evans [3]). Suppose that \( \mathbb{A} = \{a_1, \ldots, a_n\} \) is a non-decreasing list of integers, \( a_1 \geq 1 \). Then a monomial ideal \( L \) is a lex plus powers ideal with respect to \( \mathbb{A} \), also called an \( \mathbb{A} \)-lex plus powers ideal, if \( L \) is minimally generated by monomials \( x_1^{m_1} \cdots x_n^{m_l} \), \( m_1, \ldots, m_l \) such that for each \( j = 1, \ldots, l \), all monomials of degree \( \deg(m_j) \) which are larger than \( m_j \) in lex order are contained in \( L \). We will abbreviate the terminology “lex plus powers with respect to \( \mathbb{A} \)” by saying that \( L \) is \( \text{LPP}(\mathbb{A}) \).

To compare the properties of \( \mathbb{A} \)-lex plus powers ideals and ideals containing an \( \mathbb{A} \)-regular sequence will require some subtlety. Thus we make the following definition.

**Definition 2.3.** We say that \( I \) minimally contains an \( \mathbb{A} \)-regular sequence when \( I \) contains an \( \mathbb{A} \)-regular sequence, but fails to contain an \( \mathbb{A}' \)-regular sequence for each \( \mathbb{A}' \prec \mathbb{A} \) (we say that \( \{a'_1, \ldots, a'_n\} \prec \{a_1, \ldots, a_n\} \) if \( a'_i \leq a_i \) for all \( i = 1, \ldots, n \) and \( a'_j < a_j \) for some \( j \)).

Recall that the Hilbert function \( H(R/I, d) \) of an ideal \( I \) is \( \dim_k(R/I)_d \). A central concern will be comparing the Hilbert functions of \( \{a_1, \ldots, a_n\} \)-lex plus powers ideals with the Hilbert functions of ideals containing an \( \{a_1, \ldots, a_n\} \)-regular sequence, or, having fixed a Hilbert function, comparing the graded Betti numbers of these ideals. In light of this, we make the following definition.

**Definition 2.4.** Suppose that \( \mathcal{H} \) is a Hilbert function and \( \mathbb{A} = \{a_1, \ldots, a_n\} \) is a non-decreasing list of integers, \( a_1 \geq 1 \). We call \( \mathcal{H} \) an \( \mathbb{A} \)-valid Hilbert function if there exists an ideal \( I \) such that \( I \) minimally contains an \( \{a_1, \ldots, a_n\} \)-regular sequence and \( H(R/I) = \mathcal{H} \). We call \( \mathcal{H} \) an \( \mathbb{A} \)-lpp valid Hilbert function if there exists an \( \text{LPP}(\mathbb{A}) \) ideal \( L \) such that \( H(R/L) = \mathcal{H} \). Note that if an \( \text{LPP}(\mathbb{A}) \) ideal \( L \) attaining a given Hilbert function \( \mathcal{H} \) exists, then it is clearly unique. We will often refer to this ideal as \( L_{\mathcal{H}, \mathbb{A}} \).

We will also need a degree-wise notation which distinguishes between the “powers” of the lex plus powers ideal, and the rest of the minimal generators. Thus we make the following definition.
Definition 2.5. Given an \( \text{LPP}(\mathbb{A}) \) ideal \( L \), we write \( L_{(d)} \) to be the ideal generated by \( L_{d} + (x^{a_1}, \ldots, x^{a_n}) \).

With these preliminaries disposed of, we now state the conjecture due to Eisenbud et al. [6].

Conjecture 2.6 (The lex plus powers conjecture for Hilbert functions). Let \( I \subset R \) contain an \( \mathbb{A} \)-regular sequence and suppose there exists an \( \text{LPP}(\mathbb{A}) \) ideal \( L \) such that \( H(R/I, d) = H(R/L, d) \). Then

\[
H(R/L_{(d)}, d + 1) \geq H(R/I, d + 1).
\]

Remark 2.7. Note that given \( I \) as above, such an \( \text{LPP}(\mathbb{A}) \) ideal may not exist. Consider the case \( I = m \), for example. \( I \) does contain an \( \mathbb{A} = \{2, \ldots, 2\} \)-regular sequence, but there is clearly no \( \text{LPP}(\mathbb{A}) \) ideal \( L \) with \( H(R/L, 1) = H(R/I, 1) = 0 \).

Remark 2.8. We will refer to the lex plus powers conjecture for Hilbert functions as LPPH.

We now give two equivalent formulations of LPPH. The first is rather simple. In fact, the proof (that it is equivalent) is the same as for lex ideals, and is thus omitted.

Conjecture 2.9. Given an \( \mathbb{A} \)-lpp valid Hilbert function \( \mathcal{H} \), then \( \beta_{L, \mathcal{H}}^{L_{(d)}, \mathbb{A}} \geq \beta_{L, \mathcal{H}}^{L, \mathbb{A}} \) for all \( i \) whenever \( I \subset R \) attains \( \mathcal{H} \) and contains an \( \mathbb{A} \)-regular sequence.

In Section 3, we will show that it is enough to check that \( \beta_{L, \mathcal{H}}^{L_{(d)}, \mathbb{A}} \geq \beta_{L, \mathcal{H}}^{L, \mathbb{A}} \) for all \( i \) whenever \( I \subset R \) attains \( \mathcal{H} \) and contains an \( \mathbb{A} \)-regular sequence.

The next version of LPPH is slightly more subtle.

Conjecture 2.10. Given any non-decreasing list of positive integers \( \mathbb{A} = \{a_1, \ldots, a_n\} \), then the set of \( \mathbb{A} \)-valid Hilbert functions is equal to the set of \( \mathbb{A} \)-lpp valid Hilbert functions.

You will note that Conjecture 2.6 says nothing about minimal containment, while the preceding conjecture appears to deal exclusively with this concept (because it shows up in the definition of an \( \mathbb{A} \)-valid Hilbert function). Nevertheless, the two statements can be shown to be equivalent. We isolate the central argument needed to prove this fact in the following lemma.

Lemma 2.11. Let \( L \) be an \( \text{LPP}(\mathbb{A}) \) ideal and suppose that for some \( d \), \( L_d \) is a lex segment. Let \( i \) and \( j \) be such that \( a_i < d \leq a_{i+1} \) and \( a_j \leq d < a_{j+1} \). For any \( I \subset R \), if \( I \) contains a regular sequence \( f_1, \ldots, f_i \) in degrees \( a_1, \ldots, a_i \) and \( H(R/L, d) = H(R/I, d) \), then \( I \) contains a regular sequence in degrees \( a_1, \ldots, a_j \).
Proof. Note that \(a_i = d\) for \(i < t \leq j\). So it is enough to show that depth\((I_{\leq d}) \geq j\), because then we may extend the regular sequence \(f_1, \ldots, f_j\) to a longer regular sequence \(f_1, \ldots, f_j\) using \(j - i\) elements from \(I_d\) (note that following Eisenbud [5] we use depth\((I)\) to indicate the length of a maximal regular sequence in \(I\)). Let lt\((f)\) denote the leading term of a form \(f \in R\). It is not difficult to show that if \(\{\text{lt}(h_1), \ldots, \text{lt}(h_p)\}\) is a regular sequence, then so is \(h_1, \ldots, h_p\). Thus, writing \(\text{Gin}(I_{\leq d})\) to be the generic initial ideal of \(I_{\leq d}\), we have that depth\((\text{Gin}(I_{\leq d})) \leq \text{depth}(I_{\leq d})\).

If \((\text{Gin}(I_{\leq d}))_d = L_d\), then \(\text{depth}(I_{\leq d}) \geq \text{depth}(\text{Gin}(I_{\leq d})) \geq \text{depth}(\text{Gin}(I_{\leq d}))_d = \text{depth}(L_{\leq d})\). If \((\text{Gin}(I_{\leq d}))_d \neq L_d\) then because \(H(R/\text{Gin}(I_{\leq d}), d) = H(R/L_{\leq d}, d)\), \((\text{Gin}(I_{\leq d}))_d\) must contain an element outside of the lex segment of \(L_d\). The smallest element in \(L_d\) is of the form \(x_{d_1}^j \cdots x_{d_n}^j\), and thus \(x_{d_1}^j \cdots x_{d_n}^j > x_{d_1}^i \cdots x_{d_n}^i\) for some \(x_{d_1}^j \cdots x_{d_n}^j \in (\text{Gin}(I_{\leq d}))_d\).

But \(\text{Gin}(I_{\leq d})\) is Borel fixed, so \(x_{d_p}^j \in (\text{Gin}(I_{\leq d}))_d\) for all \(p \leq t\), giving a lower bound on depth. Note that \(t \geq j\). Thus \(\text{depth}(I_{\leq d}) \geq \text{depth}(\text{Gin}(I_{\leq d})) \geq t \geq j\), which gives the result. □

There are two immediate corollaries.

**Corollary 2.12.** Suppose that \(L\) is LPP\((\mathbb{A}_d)\), \(i\) and \(j\) are such that \(a_i < d \leq a_{i+1}\) and \(a_j \leq d < a_{j+1}\) for some \(d\), and \(L_d\) is a lex segment. Let \(I \subset R\) be such that \(H(R/L, d) = H(R/I, d)\). If \(I\) contains an \(\mathbb{A}' = \{a'_1, \ldots, a'_n\}\)-regular sequence where \(a'_i = a_i\) for \(1 \leq t \leq i\) and \(j + 1 \leq t \leq n\), then \(I\) contains an \(\mathbb{A}\)-regular sequence.

**Proof.** By the lemma, \(I\) contains a regular sequence in degrees \(a_1, \ldots, a_j\). To extend this sequence to an \(\mathbb{A}\)-regular sequence it is enough to note that depth\((I_{\leq a_i}) \geq i\) for \(i \geq j + 1\). □

**Corollary 2.13.** Given a Hilbert function \(\mathcal{H}\), let \(\mathbb{A}\) be the list of degrees such that the \(\mathbb{A}\)-lex plus powers ideal attaining \(\mathcal{H}\) is lex. Then every ideal attaining \(\mathcal{H}\) contains an \(\mathbb{A}\)-regular sequence.

**Proof.** Let \(I\) be an ideal attaining \(\mathcal{H}\). Then it is clear that \(I\) contains a regular element in degree \(a_1\), and this starts the induction. Suppose that \(I\) can be shown to contain an \(\{a_1, \ldots, a_{p-1}\}\)-regular sequence \(f_1, \ldots, f_{p-1}\) for \(p - 1 \geq 1\). It is enough to find an element in \(I_{d_p}\) which is regular mod \(f_1, \ldots, f_{p-1}\). That we can do this is exactly the conclusion of Lemma 2.11.

**Remark 2.14.** This second result shows that if LPPH is true, then Macaulay’s theorem is simply a special case of the extremal growth of lex plus powers ideals.

Note that given a degree \(d\), and an LPP\((\mathbb{A}_d)\) ideal \(L\) with \(\mathbb{A}_d = \{a_1, \ldots, a_n\}\), then it is easy to see that \(L_d\) consists of a lex segment, along with other \(d\) forms which are
divisible by $x_i^{d_j}$ for some $i$. An implication of this observation is that any minimal generator of $L$ in degree $d$ is either part of the lex segment in $L_d$, or a pure power. Equally important, if the largest element in $R_d - L_d$ is the pure power $x_i^{d_j}$, then $(L + (x_i^{d_j}))_d$ is a lex segment.

We can now prove the proposition.

**Proposition 2.15.** Conjectures 2.6 and 2.10 are equivalent.

**Proof.** First suppose that Conjecture 2.6 is true and let $\mathcal{H}$ be an $\mathbb{A}$-valid Hilbert function. Let $I$ be an ideal with Hilbert function $\mathcal{H}$ and minimally containing an $\mathbb{A}$-regular sequence. We construct $L_{\mathcal{H},\mathbb{A}}$ inductively. Let $L(d-1)$ be an LPP($\mathbb{A}$) ideal such that $L(d-1)(d-1) = L(d-1)$ and $H(R/L(d-1), i) = \mathcal{H}(i)$ for $i \leq d - 1$ (note that $L(0) = (x_1^{d_1}, \ldots, x_n^{d_n})$ clearly exists). By hypothesis, $q = H(R/L(d-1), d) - \mathcal{H}(d) \geq 0$, so we add $q$ generators we added to $R_d - L(d-1)_d$ to $L(d-1)$ in lex order to form a new ideal $L'(d-1)$. We want to show that $L'(d-1) = L(d)$, which will complete the proof.

To do this, we must demonstrate that $L'(d-1) = \text{LPP}(\mathbb{A})$.

The trouble is that one of the $q$ generators we added to $L(d-1)$ may have been a pure power. Suppose that $L'(d-1)$ is LPP($\mathbb{B}$) for some $\mathbb{B} < \mathbb{A}$. The fact that we added $x_i^{d_j}$ to $L(d-1)$ implies that $L'(d-1)$ is a lex segment in degree $d$. Then $\mathbb{A}$ differs from $\mathbb{B}$ as required to satisfy the hypothesis of Corollary 2.12. This is a contradiction because $I$ does not contain any $\mathbb{B}$-regular sequences for $\mathbb{B} < \mathbb{A}$.

Suppose now that Conjecture 2.10 holds and let $I$ be an ideal containing an $\mathbb{A}$-regular sequence. If $I$ minimally contains an $\mathbb{A}$-regular sequence, or if $H(R/I)$ is not $\mathbb{A}$-lpp valid, then the result is obvious. So write $L = L_{H(R/I),\mathbb{A}}$ and $\mathbb{A}' = \{a'_1, \ldots, a'_n\}$ such that $I$ minimally contains an $\mathbb{A}'$-regular sequence and $\mathbb{A}' < \mathbb{A}$. By hypothesis there is an $\mathbb{A}'$-lex plus powers ideal $L'$ such that $H(R/I) = H(R/L')$. It is enough to show that $H(R/L(d), d+1) \geq H(R/L', d+1)$. This follows because $L'$ contains an $\mathbb{A}$-regular sequence and Conjecture 2.6 is known for monomial ideals [6,9].

We now turn our attention to the lex plus powers conjecture for graded Betti numbers (due to Charalambous and Evans [7]).

Recall that the $i,j$th graded Betti number of $I$ is defined to be

$$\beta_{i,j}^I := \dim_k(\text{Tor}_i(R/I, k))_j.$$

We will often refer to the set of graded Betti numbers of an ideal by considering its Betti diagram. Using the notation of the computer algebra system Macaulay II, the Betti diagram of $I$ is a table listing the graded Betti numbers of $I$. Counting from zero, the entry in the $i,j$th position in this table is $\beta_{i,j}^I$.

**Definition 2.16.** We write $\mathcal{LPH}_\mathbb{A}$ to be the set of all sets of graded Betti numbers of ideals $I \subset R$ containing an $\mathbb{A}$-regular sequence and attaining $\mathcal{H}$. Equivalently, this is the set of all Betti diagrams of such ideals.
There is an obvious partial order on $L^A_{\mathcal{H}}$; for $\beta^I, \beta^J \in L^A_{\mathcal{H}}$, we say that $\beta^I \succeq \beta^J$ if $\beta^I_{i,j} \geq \beta^J_{i,j}$ for all $i,j \in \mathbb{N}$.

**Conjecture 2.17** (The lex plus powers conjecture). If $\mathcal{H}$ is $\mathfrak{A}$-lpp valid, then $\beta^L_{\mathfrak{A},\mathcal{H}}$ is the unique largest element in $L^A_{\mathcal{H}}$.

**Remark 2.18.** We will refer to the lex plus powers conjecture as LPP.

It is obvious that LPP implies the second formulation of LPPH (Conjecture 2.9) given above. Thus an equivalent statement of the lex plus powers conjecture is:

**Conjecture 2.19.** If $\mathcal{H}$ is $\mathfrak{A}$-valid, then $\beta^L_{\mathfrak{A},\mathcal{H}}$ is the unique largest element in $L^A_{\mathcal{H}}$.

**Remark 2.20.** If the lex plus powers conjecture is true then Corollary 2.13 is sufficient to show that LPP implies the Bigatti–Hulett–Pardue theorem (that lex ideals have extremal resolutions).

### 3. An equivalent formulation

In this section we present another equivalence for each of LPP and LPPH. Given a Hilbert function $\mathcal{H}$, and an ideal $I$ attaining it, we let $\rho(\mathcal{H})$ be the Castelnuovo–Mumford regularity of $I$. This is, of course, the largest degree for which $R/I$ has a socle element, and it tells us the row number of the last non-zero entry in the Betti diagram of $I$. In particular, we can write the Betti diagram of $I$ as

<table>
<thead>
<tr>
<th>$\beta^I$</th>
<th>0</th>
<th>1</th>
<th>$\cdots$</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$\beta^I_{0,0}$</td>
<td>$\beta^I_{1,1}$</td>
<td>$\cdots$</td>
<td>$\beta^I_{n,n}$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$\rho(\mathcal{H}) - 2$</td>
<td>$\beta^I_{0,\rho(\mathcal{H})-2}$</td>
<td>$\beta^I_{1,\rho(\mathcal{H})-1}$</td>
<td>$\cdots$</td>
<td>$\beta^I_{n,\rho(\mathcal{H})+n-2}$</td>
</tr>
<tr>
<td>$\rho(\mathcal{H}) - 1$</td>
<td>$\beta^I_{0,\rho(\mathcal{H})-1}$</td>
<td>$\beta^I_{1,\rho(\mathcal{H})}$</td>
<td>$\cdots$</td>
<td>$\beta^I_{n,\rho(\mathcal{H})+n-1}$</td>
</tr>
<tr>
<td>$\rho(\mathcal{H})$</td>
<td>$\beta^I_{0,\rho(\mathcal{H})}$</td>
<td>$\beta^I_{1,\rho(\mathcal{H})+1}$</td>
<td>$\cdots$</td>
<td>$\beta^I_{n,\rho(\mathcal{H})+n}$</td>
</tr>
</tbody>
</table>

**Definition 3.1.** Given an $\mathfrak{A}$-valid Hilbert function $\mathcal{H}$, let $\mathcal{L}^\mathfrak{A}(\rho(\mathcal{H}))$ be the set of all $\rho(\mathcal{H})$ rows of Betti diagrams in $L^\mathfrak{A}_{\mathcal{H}}$. If $\beta^I \in L^\mathfrak{A}_{\mathcal{H}}$, we refer to the $\rho(\mathcal{H})$th row of $\beta^I$ as $\beta^I_{\rho(\mathcal{H})}$.

It is clear that the partial order on $L^\mathfrak{A}_{\mathcal{H}}$ extends naturally to $\mathcal{L}^\mathfrak{A}(\rho(\mathcal{H}))$, which leads us to the following.
Conjecture 3.2. If \( \mathcal{H} \) is \( A \)-lpp valid, then \( \beta_{p_\mathcal{H}}^{L_{x,\mathcal{H}}} \) is the unique largest element in \( \mathcal{L}_{\mathcal{P}_{\mathcal{H}}^{\mathcal{H}}} (p_{\mathcal{H}}) \).

Remark 3.3. We refer to this conjecture as the lex plus powers conjecture for last rows (LPPL).

The surprising thing is that the LPPL conjecture is equivalent to LPP. The main tool used in the proof of this fact is a theorem due to Stanley [14].

Theorem 3.4 (Stanley). For every graded \( R \)-module \( M \),

\[
\sum_{d=0}^{\infty} H(M,d) t^d = \frac{\sum_{d=0}^{\infty} \sum_{i=0}^{\infty} (-1)^i \beta_{i,d}^M t^d}{(1-t)^n}.
\]

Stanley’s theorem implies that fixing a Hilbert function \( \mathcal{H} \) has the effect of fixing the alternating sums along the diagonals from southwest to northeast in the Betti diagrams of ideals attaining \( \mathcal{H} \).

Theorem 3.5. The lex plus powers conjecture for last rows is equivalent to the lex plus powers conjecture.

Proof. It is obvious that LPP implies Conjecture 3.2. Assume then that Conjecture 3.2 is true. We proceed by induction on \( p_{\mathcal{H}} \). If \( p_{\mathcal{H}} = 0 \), then \( \mathcal{H} = \{1\} \), and the equivalence becomes trivial.

So suppose that \( \mathcal{H} \) is \( A \)-lpp valid, \( p_{\mathcal{H}} > 0 \), and \( \mathcal{H} \) is the Hilbert function \( \mathcal{H} (i) = \mathcal{H} (i) \) for \( 0 \leq i < p_{\mathcal{H}} \) and \( \mathcal{H} (p_{\mathcal{H}}) = 0 \). Now if \( A = \{a_1,\ldots,a_n\} \) and \( a_n \leq p_{\mathcal{H}} \), then \( \mathcal{H} \) is also \( A \)-lpp valid. If for some \( i \) we have that \( a_{i-1} < p_{\mathcal{H}} \) and \( a_i,\ldots,a_n = p_{\mathcal{H}} + 1 \), then \( \mathcal{H} \) is \( \{a_1,\ldots,a_{i-1},a_i-1,\ldots,a_n-1\} = \{a_1,\ldots,a_{i-1},p_{\mathcal{H}},\ldots,p_{\mathcal{H}}\} \)-lpp valid. In either case simply add the \( p_{\mathcal{H}} \)th power of the maximal ideal to \( L_{\mathcal{H},A} \). If we write \( \tilde{A} \) to be either \( \{a_1,\ldots,a_n\} \) or \( \{a_1,\ldots,a_{i-1},p_{\mathcal{H}},\ldots,p_{\mathcal{H}}\} \), depending on whether \( a_n \leq p_{\mathcal{H}} \) or not, then \( L_{\mathcal{H},A} + (x_1,\ldots,x_n)^{p_{\mathcal{H}}} = L_{\tilde{A},\tilde{A}} \). Note that \( p_{\tilde{A}} = p_{\mathcal{H}} - 1 \).

Write \( \tilde{I} \) to be \( I + (x_1,\ldots,x_n)^{p_{\mathcal{H}}} \) and note that if \( \beta^l \in \mathcal{L}_{\mathcal{P}_{\mathcal{H}}^{\mathcal{H}}} \), then \( \beta_{\mathcal{H}}^{L_{x,\mathcal{H}}} \in \mathcal{L}_{\mathcal{P}_{\mathcal{H}}^{\mathcal{H}}} \). By induction, \( \beta_{\mathcal{H}}^{L_{x,\mathcal{H}}} \) is the unique largest element of \( \mathcal{L}_{\mathcal{P}_{\mathcal{H}}^{\mathcal{H}}} \). This means that the Betti diagram of \( L_{\mathcal{H},A} \) is larger then the Betti diagram of \( \tilde{I} \). We know, however, that the only effect of adding the \( p_{\mathcal{H}} \)th power of the maximal ideal to an ideal whose largest socle degree is \( p_{\mathcal{H}} \) is to disturb the final two rows of its Betti diagram. Thus we conclude that for each \( \beta^l \in \mathcal{L}_{\mathcal{P}_{\mathcal{H}}^{\mathcal{H}}} \), and each row except the final two, \( \beta_{\mathcal{H}}^{L_{x,\mathcal{H}}} \) is larger then \( \beta^l \). We are assuming that Conjecture 3.2 holds, so by hypothesis \( \beta_{\mathcal{H}}^{L_{x,\mathcal{H}}} \geq \beta_{\mathcal{H}}^l \).

Thus, it remains to check that the \((p_{\mathcal{H}} - 1)\)st row of \( \beta_{\mathcal{H}}^{L_{x,\mathcal{H}}} \) is larger than the \((p_{\mathcal{H}} - 1)\)st row of \( \beta^l \).
In order to make the argument clearer, we write down copies of $\beta_{j,\rho_{\varphi,\varpi}}^L$ and $\hat{\beta}_{j,\rho_{\varphi,\varpi}}^L$.

<table>
<thead>
<tr>
<th>$\beta_{j,\rho_{\varphi,\varpi}}^L$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>...</th>
<th>$\rho_{\varphi,\varpi}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_{0,0}^L$</td>
<td>$\beta_{1,1}^L$</td>
<td>$\beta_{2,2}^L$</td>
<td>...</td>
<td>$\beta_{n,n}^L$</td>
<td></td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td></td>
</tr>
<tr>
<td>$\rho_{\varphi,\varpi} - 3$</td>
<td>$\beta_{0,\rho_{\varphi,\varpi} - 3}^L$</td>
<td>$\beta_{1,\rho_{\varphi,\varpi} - 2}^L$</td>
<td>...</td>
<td>$\beta_{n,n}^L$</td>
<td></td>
</tr>
<tr>
<td>$\rho_{\varphi,\varpi} - 2$</td>
<td>$\beta_{0,\rho_{\varphi,\varpi} - 2}^L$</td>
<td>$\beta_{1,\rho_{\varphi,\varpi} - 1}^L$</td>
<td>...</td>
<td>$\beta_{n,n}^L$</td>
<td></td>
</tr>
<tr>
<td>$\rho_{\varphi,\varpi} - 1$</td>
<td>$\beta_{0,\rho_{\varphi,\varpi} - 1}^L$</td>
<td>$\beta_{1,\rho_{\varphi,\varpi} + 1}^L$</td>
<td>...</td>
<td>$\beta_{n,n}^L$</td>
<td></td>
</tr>
<tr>
<td>$\rho_{\varphi,\varpi}$</td>
<td>$\beta_{0,\rho_{\varphi,\varpi} + 1}^L$</td>
<td>$\beta_{1,\rho_{\varphi,\varpi} + 2}^L$</td>
<td>...</td>
<td>$\beta_{n,n}^L$</td>
<td></td>
</tr>
</tbody>
</table>

Write $B_{L_{\varphi,\varpi}}(i) = \sum_{j=i+2}^{\rho_{\varphi,\varpi}}(-1)^j \beta_{j,\rho_{\varphi,\varpi} + i}^L$ and $B_I(i) = \sum_{j=i+2}^{\rho_{\varphi,\varpi}}(-1)^j \hat{\beta}_{j,\rho_{\varphi,\varpi} + i}^L$. Then by Stanley’s theorem, we know that

\[
(-1)^i \beta_{i,\rho_{\varphi,\varpi} + i}^L + (-1)^{i+1} \beta_{i+1,\rho_{\varphi,\varpi} + i}^L + B_{L_{\varphi,\varpi}}(i) = (-1)^i \hat{\beta}_{i,\rho_{\varphi,\varpi} + i}^L + (-1)^{i+1} \hat{\beta}_{i+1,\rho_{\varphi,\varpi} + i}^L + B_I(i).
\]

Rearranging this expression, we find that

\[
\beta_{i+1,\rho_{\varphi,\varpi} + i}^L - \beta_{i,\rho_{\varphi,\varpi} + i}^L = \beta_{i,\rho_{\varphi,\varpi} + i}^L - \beta_{i,\rho_{\varphi,\varpi} + i}^L + (-1)^{i+1} B_I(i) - (-1)^i B_{L_{\varphi,\varpi}}(i).
\]

By hypothesis we have that $\beta_{i,\rho_{\varphi,\varpi} + i}^L - \beta_{i,\rho_{\varphi,\varpi} + i}^L \geq 0$. Then because the $\beta_{i+1,\rho_{\varphi,\varpi} + i}^L$ and $\beta_{i,\rho_{\varphi,\varpi} + i}^L$ constitute the entries in the $(\rho_{\varphi,\varpi} - 1)$st rows of the Betti diagrams of $L_{\varphi,\varpi}$ and $I$, it is enough to show that

\[
(-1)^{i+1} B_I(i) - (-1)^i B_{L_{\varphi,\varpi}}(i) \geq 0
\]

for all $i$.

Now write $B_{L_{\varphi,\varpi}}(i) = \sum_{j=i+1}^n(-1)^j \beta_{j,\rho_{\varphi,\varpi} + i}^L$ and $B_I(i) = \sum_{j=i+1}^n(-1)^j \hat{\beta}_{j,\rho_{\varphi,\varpi} + i}^L$. Again by Stanley’s theorem

\[
(-1)^{i+1} \beta_{i+1,\rho_{\varphi,\varpi} + i + 1}^L + B_{L_{\varphi,\varpi}}(i + 1) = (-1)^i \beta_{i+1,\rho_{\varphi,\varpi} + i + 1}^L + B_I(i + 1)
\]
and by hypothesis this yields the inequality

$$0 \leq \beta_{t+1, \rho_{\mathcal{H}}} \leq \beta_{t+1, \rho_{\mathcal{H}}} = (-1)^{i+1} B_{t} - (-1)^{i+1} B_{L_{\mathcal{H}}} (i + 1).$$

Of course, $B_{L_{\mathcal{H}}} (i + 1) = \sum_{j=i+1+1}^{n} (-1)^{j} B_{j, \rho_{\mathcal{H}}} = \sum_{j=i+2}^{n} (-1)^{j} B_{j, \rho_{\mathcal{H}}} + 1 = B_{L_{\mathcal{H}}} (i)$ and $B_{j} (i + 1) = \sum_{j=i+1+1}^{n} (-1)^{j} B_{j, \rho_{\mathcal{H}}} + 1 = \sum_{j=i+2}^{n} (-1)^{j} B_{j, \rho_{\mathcal{H}}} + 1 = B_{j} (i)$, because only the last two rows of the Betti diagram are perturbed when passing to $L_{\mathcal{P}}$. Thus, we have that

$$0 \leq (-1)^{i+1} B_{t} - (-1)^{i+1} B_{L_{\mathcal{H}}} (i + 1) = (-1)^{i+1} B_{t} - (-1)^{i+1} B_{L_{\mathcal{H}}} (i)$$

for all $i$ as required. This completes the proof. □

In the proof given above, we actually prove something stronger.

**Theorem 3.6.** Suppose that there exists $i$ such that $L_{\mathcal{H}}$ has $\beta_{t, \rho_{\mathcal{H}}} \geq \beta_{t, \rho_{\mathcal{H}}} + 1$ for all $0 \leq t \leq i$ and all ideals $I$ containing an $\mathcal{H}$-regular sequence attaining $\mathcal{H}$. Then $\beta_{t, \rho_{\mathcal{H}}}$ for all $0 \leq t \leq i$ and $j = 0, \ldots, \rho_{\mathcal{H}}$.

This theorem says that if the Betti numbers in the last row of the first $i$ columns of the Betti diagram of $L_{\mathcal{H}}$ are uniquely largest, then all the Betti numbers in the first $i$ columns of the Betti diagram of $L_{\mathcal{H}}$ are uniquely largest. An obvious corollary has important implications for the lex plus powers conjecture for Hilbert functions.

**Corollary 3.7.** To prove LPPH, it is enough to show that whenever $L$ is LPP($\mathcal{A}$) for some $\mathcal{A}$, $I$ contains an $\mathcal{A}$-regular sequence, and $H(R/I) = H(R/I)$, then $\beta_{t, \rho_{\mathcal{H}}} \geq \beta_{t, \rho_{\mathcal{H}}} + 1$ where as usual $\rho_{\mathcal{H}}$ is the Castelnuovo–Mumford regularity of $L$. That is, it is enough to show that lex plus powers ideals have the most generators in degree $\rho_{\mathcal{H}} + 1$.

**4. Socle dimensions**

As pointed out in Section 2, it is clear that LPP implies LPPH. It would be very helpful to know when these two conjectures are equivalent (one obvious reason for this is that LPPH is known in more cases than is LPP). We show in this section that if LPPH holds, then lex plus powers ideals have largest socles. In the language of Betti diagrams: if the first column of the Betti diagram of $L_{\mathcal{H}}$ is known to be largest, then the final column of its Betti diagram is also largest. This has immediate consequences in low dimension, which we discuss in Section 5.

**Definition 4.1.** Let $I \subset R$. Then we denote the socle of $R/I$ in degree $d$ by the notation $S_d(R/I)$. We write $\hat{S}_d(R/I)$ to be the preimage of $S_d(R/I)$ in $R$. There are a few facts we need before we are ready to prove the theorem. First, we need to show that if $L$ is LPP($\mathcal{A}$), then $L + \hat{S}_d(R/L)$ is a lex plus powers ideal.
This allows us to take a statement about socles, and translate it to a statement about generators, thus providing the vehicle for the proof of the main theorem.

Lemma 4.2. Suppose that $L \subset R$ is $\text{LPP}(\mathbb{A})$ for some $(\mathbb{A})$. Then $L + \hat{S}_d(R/L)$ is a lex plus powers ideal.

Proof. It is enough to show that if $x_1^{a_1} \cdots x_n^{a_n} \in \hat{S}_d(R/L)$, and $x_1^{b_1} \cdots x_n^{b_n} \in R_d - L_d$ such that $x_1^{b_1} \cdots x_n^{b_n} > x_1^{a_1} \cdots x_n^{a_n}$, then $x_1^{b_1} \cdots x_n^{b_n} \in \hat{S}_d(R/L)$. Let $j$ be such that $\gamma_i = \alpha_i$ for all $1 \leq t < j$, and $\gamma_i > \alpha_j$. If $x_j x_1^{b_1} \cdots x_n^{b_n}$ lands outside of the lex segment of $L_{d+1}$, then $x_j^{b_j}$ divides $x_1 x_1^{b_1} \cdots x_n^{b_n}$ whence $x_j^{b_j}$ divides $x_1^{b_1} \cdots x_n^{b_n}$ and $x_1^{b_1} \cdots x_n^{b_n} \in L_d$, a contradiction. If $x_j x_1^{b_1} \cdots x_n^{b_n}$ lands in the lex segment of $L_d$, then we are finished because $x_j x_1^{b_1} \cdots x_n^{b_n} > x_1 x_1^{b_1} \cdots x_n^{b_n} \in L_{d+1}$ for all $1 \leq i \leq n$. So suppose that $x_j x_1^{b_1} \cdots x_n^{b_n}$ lands in the lex segment of $L_d$, while $x_{j+1} x_1^{b_1} \cdots x_n^{b_n}$ lands outside. If $\gamma_j - 1 > \alpha_j$, then we are done because $x_j x_1^{b_1} \cdots x_n^{b_n} > x_{j+1} x_1^{b_1} \cdots x_n^{b_n} \in L_{d+1}$ for all $1 \leq i \leq n$, so assume that $\gamma_j = \alpha_j + 1$. If $\gamma_j > \alpha_j$ for some $t > j$, then because $x_j^{b_j}$ divides $x_1 x_1^{b_1} \cdots x_n^{b_n}$, and since $\alpha_j + 1 \leq \gamma_j$, we know that $x_1^{b_1} \cdots x_n^{b_n} \in L_d$, a contradiction. So assume that $\gamma_j \leq \alpha_j$ for all $t > j$. Because $x_1^{b_1} \cdots x_n^{b_n}$ have the same degree there is only one integer $p > j$ such that $\gamma_i < \alpha_i$ and $\gamma_p + 1 = \alpha_p$. Now we have that $x_1 x_1^{b_1} \cdots x_n^{b_n} > x_j x_1^{b_1} \cdots x_n^{b_n}$ is in the lex segment of $L_{d+1}$, for all $1 \leq i \leq j$, $x_p x_1^{b_1} \cdots x_n^{b_n} = x_1 x_1^{b_1} \cdots x_n^{b_n}$ is in the lex segment of $L_{d+1}$, and $x_p^{b_p}$ divides $x_1 x_1^{b_1} \cdots x_n^{b_n}$ for all $i > j$, $i \neq p$. We conclude that $x_1 x_1^{b_1} \cdots x_n^{b_n}$ is in $\hat{S}_d(R/L)$, as required. 

The next lemma will help us identify the socle elements of lex plus powers ideals. It shows that after adding all socle elements of degree $d$ to a lex plus powers ideal, the result cannot have any non-pure-power minimal generators in degree $d+1$. We will use this in the proof of Theorem 4.4 to force a needed contradiction.

Lemma 4.3. Suppose that $L$ is $\text{LPP}(\mathbb{A})$ for some $\mathbb{A} = \{a_1, \ldots, a_n\}$ and let $x_1^{a_1} \cdots x_n^{a_n}$ be the largest element in $R_d - L_d$. If $R/(L + x_1^{a_1} \cdots x_n^{a_n})$ has minimal generators in degree $d+1$ which are not pure powers, then $x_1^{a_1} \cdots x_n^{a_n}$ is in the socle of $R/L$.

Proof. Note that if $x_1^{b_1} \cdots x_n^{b_n}$ is a degree $d+1$ non-pure-power minimal generator of $L' = L + (x_1^{a_1} \cdots x_n^{a_n})$, then it is also a minimal generator of $L$, and is thus part of the lex segment of $L_{d+1}$. So it is enough to show that $x_1 x_1^{b_1} \cdots x_n^{b_n} > x_1^{a_1} \cdots x_n^{a_n}$ for all $i = 1, \ldots, n$. This must hold, because $x_1 x_1^{b_1} \cdots x_n^{b_n}$ is part of the lex segment of $L'$, and we know that a lex segment in degree $d$ generates a lex segment in degree $d+1$. That is, if $x_1 x_1^{b_1} \cdots x_n^{b_n} < x_1^{a_1} \cdots x_n^{a_n}$ then $x_1^{a_1} \cdots x_n^{a_n}$ could not have been a minimal generator of $L'_{d+1}$. 

We are now ready to prove the theorem.

Theorem 4.4. Let $L$ be $\text{LPP}(\mathbb{A})$ for some $\mathbb{A} = \{a_1, \ldots, a_n\}$ and $I$ be an ideal containing an $\mathbb{A}$-regular sequence such that $H(R/L) = H(R/I)$. If $\text{LPPH}$ holds, then $\dim_k(S_0(R/L)) \geq \dim_k(S_d(R/I))$ for all $d$. 

5. Applications to low-dimensional cases

In this section we explore the implications of Theorem 4.4 in low dimension. In particular, we show that for \( n \leq 3 \), LPPH and LPP are equivalent. First we demonstrate the LPPH is true for \( n = 2 \).

**Theorem 5.1.** If \( n = 2 \), then the lex plus powers conjecture for Hilbert functions (Conjecture 2.6) is true.

**Proof.** Suppose that \( I \) contains an \( \{a_1, a_2\} \)-regular sequence and write \( H(R/I, d) = h_d \).

Now it is enough to show that if there is an LPP(\( \mathbb{A} \)) ideal \( L \) such that \( H(R/L, d) = h_d \), then \( H(R/L(d), d+1) \geq H(R/I, d+1) \).

If \( d < a_2 \), then because \( n = 2 \), \( L \) is lex. Because \( H(R/L(d), d+1) \geq H(R/I, d+1) \), and hence \( H(R/L(d), d+1) \geq H(R/I, d+1) \).

Thus, we can assume that \( d \geq a_2 \). It is a result of Davis [4] that in this case \( H(R/I, d+1) < H(R/I, d) \). So it is enough to find an ideal \( L(d) \) such that \( L(d) \) and that \( H(R/L(d), d+1) = H(R/L(d), d+1) = H(R/L(d), d+1) \).

In two variables this is easy enough to do. Let \( L' = (x_1^{a_1}, x_2^{a_2}) \). Because \( I \) contains an \( \{a_1, a_2\} \)-regular sequence, we know that \( H(R/I', d) \geq H(R/I, d) \). Thus, we can add generators to \( L' \) to lower it’s Hilbert function. Doing so we get the LPPH ideal

\[
L(d) = \langle x_1^{d}, x_1^{d-1}x_2, \ldots, x_1^{d-a_2+h_2}x_2^{a_2-1}, x_1^{d-a_2}x_2^{a_2-1}x_2, \ldots, x_2^{d} \rangle,
\]

that is,

\[
\langle x_1^{d-a_2+h_2}, x_2^{a_2-1} \rangle,
\]

gives a basis of \( (R/L)_d \). So it is clear that

\[
L(d)_{d+1} = \langle x_1^{d+1}, x_1^{d}x_2, \ldots, x_1^{d-a_2+h_2+1}x_2^{a_2-1}, x_1^{d-a_2+h_2}x_2, \ldots, x_2^{d+1} \rangle.
\]
and
\[ \langle x_1^{d-a_1+h_d}x_2^{a_2-h_d+1}, \ldots, x_1^{d-a_2+2:a_2-1} \rangle \]
gives a basis of \( R/L(d)_{d+1} \). We conclude that

\[ H(R/L(d), d + 1) = H(R/L(d), d) - 1 \]
as required.

Note that LPPH and LPP are obviously equivalent for \( n = 1 \). With Theorem 4.4 we now show that they are equivalent for \( n \geq 3 \).

**Theorem 5.2.** If \( n = 2 \) or \( 3 \) then the lex plus powers Conjecture (2.17) and the lex plus powers conjecture for Hilbert functions Conjectures (2.6) are equivalent.

**Proof.** As pointed out before, LPP implies LPPH for any \( n \).

So suppose that LPPH holds (it will be helpful to use the version given in Conjecture 2.9), let \( \mathcal{H} \) and \( \mathcal{A} \) be given such that \( \mathcal{H} \) is \( \mathcal{A} \)-valid, and write \( L \) to be \( L_{\mathcal{H}, \mathcal{A}} \). Suppose first that \( n = 2 \). By hypothesis, \( L \) has \( \beta_{1,i}^L \geq \beta_{1,i}^I \) for all ideals \( I \) such that \( I \) contains an \( \mathcal{A} \)-regular sequence and \( H(R/I) = \mathcal{H} \). Theorem 4.4 tells us that \( \beta_{2,i}^L \geq \beta_{2,i}^I \) as well. So \( L \) has largest graded Betti numbers. If \( n = 3 \), then by hypothesis and Theorem 4.4, \( L \) has \( \beta_{1,i}^L \geq \beta_{1,i}^I \) and \( \beta_{3,i}^L \geq \beta_{3,i}^I \) for all ideals \( I \) such that \( I \) contains an \( \mathcal{A} \)-regular sequence and \( H(R/I) = \mathcal{H} \). Stanley’s theorem now implies that \( -\beta_{1,i}^L + \beta_{2,i}^L - \beta_{3,i}^L = -\beta_{1,i}^I + \beta_{2,i}^I - \beta_{3,i}^I \) for all \( i \), or rearranging things, that \( \beta_{2,i}^L - \beta_{2,i}^I = \beta_{1,i}^L - \beta_{1,i}^I + \beta_{3,i}^L - \beta_{3,i}^I \geq 0 \). This completes the proof.

The following corollaries are immediate (the latter because LPPH is known for ideals containing \( (x_1^{a_1}, \ldots, x_n^{a_n}) \)).

**Corollary 5.3.** LPP holds for the polynomial ring \( k[x_1, x_2] \).

**Corollary 5.4.** LPP holds for \( k[x_1, x_2, x_3] \) if we restrict to ideals containing \( x_1^{a_1}, x_2^{a_2}, x_3^{a_3} \). In particular, LPP hold for monomial ideals in dimension \( n = 3 \).

**References**