Homotopy representations of $SO(7)$ and Spin(7) at the prime 2

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Abstract

A homotopy (complex) representation of a compact Lie group $L$ at the prime $p$ is a map from $BL$ into the $p$-completion (in the sense of Bousfield and Kan) of the classifying space of the unitary group $BU(n)_p^\wedge$. This paper contains the classification of homotopy representations of $SO(7)$ and Spin(7) at the prime 2. The motivation for considering this problem is twofold: first, one may hope that it would help to understand maps between classifying spaces. Secondly, the construction of the suitable homotopy representation of Spin(7) is a crucial step in the construction of a faithful representation of the 2-compact group $DI(4)$ [K. Ziemiański, A faithful unitary representation of the 2-compact group $DI (4)$, Ph.D. Thesis, Uniwersytet Warszawski, 2005].

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1. Introduction

Fix a compact Lie group $L$ and a prime integer $p$. Let $T$ be a maximal torus of $L$ and let $N_p(T)$ be a $p$-normalizer of $T$. A natural question arises: given a map $f : B N_p(T) \to BU(n)_p^\wedge$, does it extend to a homotopy representation $BL \to BU(n)_p^\wedge$? An obvious necessary condition is that $f$ is $R_2(L)$-invariant, i.e. for any subgroups $P, Q \subseteq N_p(T)$ which are $p$-stubborn in $L$ in the sense of [6], and any homomorphism $\alpha : P \to Q$ which is the restriction of a conjugation in $L$ the maps $f|_{BP}$ and $f|_{BO} \circ B\alpha$ are homotopic. The main result of this paper is the following

**Main Theorem.** If $L = SO(7)$ or $L = Spin(7)$, then every $R_2(L)$-invariant map $BN_2(T) \to BU(n)_2^\wedge$ extends to a homotopy representation $BL \to BU(n)_2^\wedge$.

**Outline of the proof.** Fix an $R_2(L)$-invariant map $f : BN_2(T_L) \to BU(n)_2^\wedge$. By Dwyer–Zabrodsky (2.1) $f$ is homotopic to the completion of a unitary representation $\varphi$ of the discrete approximation of $N_2(T_L)$. The proof is based on the subgroup decomposition of Jackowski, McClure and Oliver [6]. By obstruction theory for extending maps from homotopy colimits [14] the problem reduces to prove that the obstructions lying in groups $H^{i+1}(R_2(L), \Pi_1^\varphi)$ vanish, where $\Pi_1^\varphi$ are contravariant functors on $R_2(L)$ whose values are homotopy groups of suitable mapping spaces (cf. (2.5)). For $i > 2$ these groups vanish for dimensional reasons 3.10, and the functor $\Pi_1^\varphi$ is trivial. Thus we need to prove that $H^3(R_2(L); \Pi_2^\varphi) = 0$.

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The main tool is that this calculation is the spectral sequence (2.10). The only one non-trivial entry in the first term of the total degree 3 is \( E_{1}^{3,0} \), thus we have to prove that the differential \( d_{1}^{2,0} \) hitting it is an epimorphism. The isomorphism (2.7) allows for expressing values \( H_{-}^{2} \) in terms of representation rings what enables explicit calculations.

In the case when \( L = G \) we split \( E_{1}^{3,0} \) into a sum of two summands (4.1) and prove that they lie in the image of \( d_{1}^{2,0} \) separately. The proof in the first case (Lemma A) is more difficult and requires an identification of some full subcategory of \( R_{2}(O(7)) \) with the category \( R_{2}(\Sigma_{7}) \). The latter case (Lemma B) is easier. Both these results are obtained by the explicit calculations which use the classification of (quasi)representations of some stubborn subgroups of \( SO(7) \). To prove Main Theorem for \( L = \text{Spin}(7) \) we split \( \varphi \) into an even part (which is induced from \( G \)) and an odd part. For the even part the proof reduces to the case \( L = SO(7) \) and to the even part is relatively easy to handle (Lemma C).

Organization of the paper. In Section 2 we gather well-known facts concerning maps between classifying spaces and present a classical method of constructing homotopy representations. Section 3 contains the explicit description of the category \( R_{2}(\text{Spin}(7)) \cong R_{2}(SO(7)) \). In Section 4 we prove the Main Theorem. The proofs of the crucial technical Lemmas A, B & C are postponed. Section 5 contains the classification of (quasi)representations of some stubborn subgroups of \( G \) which is permanently used in the proof of Lemmas A, B & C contained in Section 6. The paper is concluded with two appendices: in the first we construct the full inclusion \( R_{2}(\Sigma_{n}) \rightarrow R_{2}(O(n)) \) which is used in the proof of Lemma A, and in the latter we prove some elementary facts concerning representations of countable locally finite groups.

Notation. Throughout the whole paper \( G = SO(7) \), \( \tilde{G} = \text{Spin}(7) \). If \( X \) is a space, then \( X_{\varphi}^{p} \) is a \( p \)-completion in the sense of Bousfield and Kan [3]. If \( R \) is a commutative ring and \( C \) is a small category, then \( R[C] \)-module is a contravariant functor \( M \) from \( C \) into the category of \( R \)-modules, and let \( H^{n}(C; M) \) denote the \( n \)th higher inverse limit of \( M \). For any groups \( H_{1}, H_{2} \) we let \( \text{Rep}(H_{1}, H_{2}) := \text{Hom}(H_{1}, H_{2})/\text{Inn}(H_{2}) \).

2. Homotopy representations

In this section we recall the method of constructing homotopy representations of compact Lie groups, based on the subgroup homotopy decomposition due to Jackowski, McClure and Oliver [6] (see also [9] for a detailed discussion). Fix a prime integer \( p \) and a compact Lie group \( L \) with a maximal torus \( T \subseteq L \).

Dwyer–Zabrodsky theorem. If \( L \) is \( p \)-toral (i.e. if it is an extension of a finite \( p \)-group by a torus), then each homotopy representation of \( L \) is the completion of a group homomorphism in the following sense: there is a dense subgroup \( L^{\infty} \subseteq L \) (called a \( p \)-discrete approximation of \( L \)) such that \( T \cap L^{\infty} \) is a subgroup of all elements of rank \( p^{\infty} \) in \( T \) (the subgroup \( L^{\infty} \) is determined uniquely up to conjugacy). By [7, Thm. 1.1] the map

\[
\text{Rep}(L^{\infty}, U(n)) \ni \varphi \mapsto (B\varphi)^{p}_{\infty} \in [BL, BU(n)^{p}_{\infty}]
\]

(2.1)
is a bijection. The group \( L^{\infty} \) is not finite, however it is countable locally finite and a representation theory of such groups is quite similar to the theory if finite groups (see Appendix B). Representations of \( L^{\infty} \) will be called quasirepresentations of \( L \) (following [7]). We let \( \text{QRep}(L, U(n)) \) denote the set of all quasirepresentations \( L^{\infty} \rightarrow U(n) \).

Homotopy decomposition. Now let \( L \) be an arbitrary compact Lie group. A \( p \)-toral subgroup \( P \subseteq L \) is \( p \)-stubborn, if \( N_{P}(P)/P \) is finite and has no non-trivial normal \( p \)-subgroups (if \( L \) is finite then it is \( p \)-stubborn iff it is \( p \)-radical). Let \( R_{P}(L) \) be a category, whose objects are orbits \( L/P \), where \( P \) is \( p \)-stubborn, and morphisms are \( L \)-maps. By [6] the map

\[
\text{hocolim}_{R_{P}(L)} EL \times_{L} L/P \rightarrow BL
\]

(2.2)
induced by projections onto \( BL \) induces isomorphism on mod \( p \)-homology. In particular, there is a bijection

\[
[BL, BU(n)^{p}_{\infty}] \cong [\text{hocolim}_{R_{P}(L)} EL \times_{L} L/P, BU(n)^{p}_{\infty}].
\]

(2.3)

Constructing homotopy representations. Let \( N := N_{p}(T) \) be a maximal \( p \)-toral subgroup of \( L \) (it is an extension of \( T \) by a \( p \)-Sylow subgroup of its Weyl group). Consider the sequence of maps
Let $BL, BU(n)_p \to [\text{hocolim}_{\mathcal{R}_p(L)} EL \times_L L/P, BU(n)_p]$.

$$
\lim_{\mathcal{R}_p(L)} [BP, BU(n)_p] \to \lim_{\mathcal{R}_p(L)} \text{QRep}(P, U(n)) \subseteq \text{QRep}(N, U(n)).$

(2.4)

A quasi-representation of $N$ is $\mathcal{R}_p(L)$-invariant iff it extends to an element of $\lim_{\mathcal{R}_p(L)} \text{QRep}(P, U(n))$; $\mathcal{R}_p(L)$-invariant quasi-representations of $N$ will be called also homotopy characters of $L$. The map $\Theta$ is, in general, neither injective nor surjective; the main goal of this paper is to prove that it is surjective if $L = G$ or $L = \hat{G}$ (and $p = 2$).

Obstruction theory. The problem if a given collection of maps $\{BP \to BU(n)_p\}$ can be extended to a map from the homotopy colimit can be solved by means of obstruction theory. Fix a homotopy character $\varphi$ of $L$. For any $i \geq 1$ define contravariant functors $\Pi_i^\varphi : \mathcal{R}_p(G) \to \text{Groups}$ by

$$
\Pi_i^\varphi(L/P) := \pi_i(\text{map}(EL \times_L L/P, BU(n)_p))_{B\varphi}.
$$

(2.5)

Following [14], if the obstructions lying in groups $H^{i+1}(\mathcal{R}_p(L); \Pi_i^\varphi)$ vanish, then the collection $\{B(\varphi |_P)_p\}$ has an extension to the homotopy colimit. In fact, since the spaces map$(BP, BU(n)_p)_{B\varphi}$ are $p$-complete and $\Pi_i^\varphi$ turns out to be trivial, then the values of $\Pi_i^\varphi$ are $\mathbb{Z}_p$-modules (thus $\Pi_i^\varphi$’s are $\mathbb{Z}_p[\mathcal{R}_p(L)]$-modules).

Description of $\Pi_i$ for $i < 4$. By [7, Thm. 1.1] there is a homotopy equivalence

$$
\text{BC}_U(n)(\varphi(P^\infty))_p \to \text{map}(BP^\infty, BU(n)_p)_{B\varphi}
$$

(2.6)

which is induced by the pairing $\text{BC}_H(\varphi(P^\infty)) \times BP^\infty \to BH$ (hence it is $\mathcal{R}_p(L)$-functorial). By Schur’s lemma $C_U(n)(\varphi(P^\infty))$ is a product of unitary groups with factors corresponding to isomorphism classes of irreducible subrepresentations of $\text{res}_P^{N^\infty} \varphi$. Thus $\Pi_1^\varphi = \Pi_3^\varphi = 0$, and

$$
\Pi_2^\varphi(L/P) = \pi_1(C_U(n)(\varphi(P^\infty))) \cong R(P^\infty, \varphi),
$$

(2.7)

where $R(P^\infty, \varphi)$ is a subgroup of representation ring of $P^\infty$ generated by irreducible $P^\infty$-subrepresentations of $\varphi$.

The argument used in the proof of [8, Prop.1.5] shows that this isomorphism is functorial with respect to $\mathcal{R}_p(L)$. Thus $\mathbb{Z}_p[\mathcal{R}_p(L)]$-modules $\Pi_i^\varphi$ and $\mathbb{Z}_p \otimes R(-, \varphi)$ are isomorphic.

As a consequence we obtain the following

**Proposition 2.8.** Let $\varphi : N \to U(n)$ be a homotopy character of $L$. Assume that

- $H^3(\mathcal{R}_p(L); \Pi_i^\varphi) = 0$
- $H^i(\mathcal{R}_p(L); \hat{M}) = 0$ for any $i > 4$ and any $\mathbb{Z}_p[\mathcal{R}_p(L)]$-module $\hat{M}$.

Then $B\varphi^2$ extends to a map $BL \to BU(m)_p$. $\Box$

Cohomology of $\mathcal{R}_p(L)$. Methods of calculating cohomology of categories $\mathcal{R}_p(L)$ were developed in [6,5]. Let $\hat{M}$ be a $\mathbb{Z}_p[\mathcal{R}_p(L)]$-module. If $\hat{M}$ is atomic (i.e. $\hat{M}(L/P) = 0$ for all $P$ non-conjugate to a given $P_0$), then

$$
H^*(\mathcal{R}_p(L); \hat{M}) = A^*(W_L(P_0); \hat{M}(L/P_0))
$$

(2.9)

(following the notation of [6]) depends only on a $\text{Aut}_{\mathcal{R}_p(L)}(L/P_0)$-module structure of $\hat{M}(L/P_0)$. Grodal [5] recognized that groups $A^*$ can be expressed in terms of generalized Steinberg modules. In the general case $H^*(\mathcal{R}_p(L); \hat{M})$ can be calculated using a spectral sequence. Fix a function $ht : \text{Ob}(\mathcal{R}(L)) \to \mathbb{Z}$ such that for every morphism $L/Q \to L/P$ holds $ht(L/P) \leq ht(L/Q)$ and equity is allowed only for isomorphisms. By [5, 1.3] there is a spectral sequence $E^{*,*}_s := E(\mathcal{R}_p(L), \hat{M})^{*,*}_s$ with the first term

$$
E^{i,t}_1 := \bigoplus_{ht(L/P) = s} A^{i+t}(W_L(P); \hat{M}(L/P))
$$

(2.10)

which converges to $H^*(\mathcal{R}_p(L); \hat{M})$ (Grodal uses a slightly different indexing).

Differentials. For a set $X \subseteq \text{Ob}(\mathcal{R})$ define $\mathbb{Z}_p[\mathcal{R}]$-module $M|_X$ by

$$
M|_X(L/P) := \begin{cases} M(L/P) & \text{if } L/P \text{ is isomorphic to an element of } X \\ 0 & \text{otherwise} \end{cases}
$$

(2.11)
and let $M|_X(L/P \to L/Q) = M(L/P \to L/Q)$ iff both $L/P$ and $L/Q$ are isomorphic to elements of $X$; otherwise $M|_X(L/P \to L/Q) = 0$ (for some sets $X$ it is not a well-defined module). The differential $d^{s,t}_1 : E^{s,t}_1 \to E^{s+1,t}_1$ is given by the formula

$$d^{s,t}_1 = \bigoplus_{\{P : \text{ht}(P) = s\}} \bigoplus_{\{Q : h(Q) = s+1\}} d'(P, Q),$$

(2.12)

where $d'(P, Q) : A^{s+t}(\text{Aut}_R(L/P); M(L/P)) \to A^{s+t+1}(\text{Aut}_R(L/Q); M(L/Q))$ is the differential of the long exact sequence associated to the short exact sequence

$$0 \to M_{L/Q} \to M_{L/P \cdot L/Q} \to M_{L/P} \to 0.$$  

(2.13)

### 3. Stubborn subgroups of $G$ and $\tilde{G}$

In this section we apply the procedure described in the previous section to the special case when $p = 2$ and $L = G$ or $L = \tilde{G}$. First we describe the decomposition categories $R_2(G)$ and $R_2(\tilde{G})$. By [10, Prop. 11], [10, Th. 12] and [6, Prop. 1.6.(i)] the functors

$$R_2(O(7)) \ni O(7)/P \mapsto G/(P \cap G) \in R_2(G)$$

(3.1)

$$R_2(G) \ni G/P \mapsto \tilde{G}/\tilde{P} = \tilde{G}/\pi^{-1}(P) \in R_2(\tilde{G})$$

are natural equivalences. Thus it is sufficient to give a description of the category $R_2(O(7))$.

**Stubborn subgroups of orthogonal groups.** Denote

$$A := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

(3.2)

and for $i < n$ define matrices $A^n_i$, $B^n_i \in O(2^n)$ by

$$A^n_i := I_{2^n} \otimes A \otimes I_{2^{n-i-1}}, \quad B^n_i := I_{2^n} \otimes B \otimes I_{2^{n-i-1}}.$$  

(3.3)

Following [10] define subgroups $\Gamma_{2^n}$, $\tilde{\Gamma}_{2^n} \in O(2^n)$ by setting

$$\Gamma_{2^n} := \langle -I_{2^n}, A^n_0, \ldots, A^n_{n-1}, B^n_0, \ldots, B^n_{n-1} \rangle \subseteq O(2^n)$$

(3.4)

$$\tilde{\Gamma}_{2^n} := \langle \{X \otimes I_{2^{n-i-1}} \mid X \in O(2)\}, A^n_0, \ldots, A^n_{n-1}, B^n_0, \ldots, B^n_{n-1} \rangle \subseteq O(2^n).$$

(3.5)

(these groups are denoted in [10] by $\Gamma_{2^n}$ and $\tilde{\Gamma}_{2^n}$ respectively; notice that $\Gamma_1 = O(1) = \{\pm 1\}$ and $\tilde{\Gamma}_2 = O(2)$). Using [10, Thm.6] we obtain the following list of stubborn subgroups of $O(7)$:

<table>
<thead>
<tr>
<th>$P$</th>
<th>$W_{O(7)}(P)$</th>
<th>$\text{ht}(P)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_2 \times C_2 \times \tilde{\Gamma}_2 \times \Gamma_1$</td>
<td>$1$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\tilde{\Gamma}_2 \times \Gamma_1$</td>
<td>$\Sigma_3$</td>
<td>$1$</td>
</tr>
<tr>
<td>$\tilde{\Gamma}_2 \times C_2 \times \Gamma_1^3$</td>
<td>$\Sigma_3$</td>
<td>$1$</td>
</tr>
<tr>
<td>$\Gamma_1 \times C_2^4 \times \tilde{\Gamma}_2 \times \Gamma_1$</td>
<td>$\Sigma_3$</td>
<td>$1$</td>
</tr>
<tr>
<td>$\tilde{\Gamma}_4 \times \tilde{\Gamma}_2 \times C_2 \times \Gamma_1$</td>
<td>$\Sigma_3$</td>
<td>$1$</td>
</tr>
<tr>
<td>$\tilde{\Gamma}_2 \times \Gamma_1^5$</td>
<td>$\Sigma_5$</td>
<td>$2$</td>
</tr>
<tr>
<td>$\Gamma_1 \times C_2^2 \times \Gamma_1^5$</td>
<td>$\Sigma_3 \times \Sigma_3$</td>
<td>$2$</td>
</tr>
<tr>
<td>$\tilde{\Gamma}_4 \times \Gamma_1^3$</td>
<td>$\Sigma_3 \times \Sigma_3$</td>
<td>$2$</td>
</tr>
<tr>
<td>$\Gamma_4 \times \tilde{\Gamma}_2 \times \Gamma_1$</td>
<td>$\Sigma_3 \otimes \Sigma_2$</td>
<td>$2$</td>
</tr>
<tr>
<td>$\Gamma_1^7$</td>
<td>$\Sigma_7$</td>
<td>$3$</td>
</tr>
<tr>
<td>$\Gamma_4 \times \Gamma_1^3$</td>
<td>$\Sigma_3 \otimes \Sigma_2 \times \Sigma_3$</td>
<td>$3$</td>
</tr>
</tbody>
</table>
It is easy to check that the function $ht$ satisfies condition demanded in the definition of the spectral sequence (2.10). To keep the notation simple and to avoid some problems related to the fact that $R_2(O(7))$ is not a finite category define $\mathfrak{R}$ to be the category whose objects are groups $P$ in the Eq. (3.6) and morphisms are given by

$$\text{Mor}_{\mathfrak{R}}(P, Q) := N_{O(7)}(P, Q)/Q = \{g \in O(7) : g^{-1}Pg \subseteq Q\}/Q.$$ (3.7)

Obviously $\mathfrak{R}$ is a finite category equivalent to both $R_2(G)$ and $R_2(\tilde{G})$.

Cohomology of $\mathfrak{R}$.

**Proposition 3.8.** Fix $P \in \text{Ob}(\mathfrak{R})$. For any $\mathbb{Z}_2^+[\text{Aut}_{\mathfrak{R}}(P)]$-module $M$ we have

$$A^i(\text{Aut}_{\mathfrak{R}}(P); M) = 0$$

for any $i > ht(P)$.

**Proof.** For any finite group $H$ let $\text{Adim}_2(H)$ be the smallest integer $d$ such that $A^i(H; M) = 0$ for any $i > d$ and any $\mathbb{Z}_2^+[H]$-module $M$. Application of the spectral sequence (2.10) shows that if $H$ has no non-trivial normal 2-subgroups (i.e. 1 is stubborn in $H$), then

$$\text{Adim}_2(H) \leq 1 + \sup_{H \neq K \in \text{R}_2(H)} \text{Adim}_2(W_H(K)).$$

Moreover, $\text{Adim}_2(H \times H^\prime) = \text{Adim}_2(H) + \text{Adim}_2(H^\prime)$ (by [6, 6.1.v]). Thus we successively obtain $\text{Adim}_2(\Sigma_3) = 1$, $\text{Adim}_2(H) \leq 2$ for $H = \Sigma_3$, $\Sigma_3 \times \Sigma_3$, $\Sigma_3 \wr C_2$ and finally $\text{Adim}_2(\Sigma_7) \leq 3$. □

**Corollary 3.9.** Let $M$ be an $\mathbb{Z}_2^+[\mathfrak{R}]$-module and let $E^{*,*} = E(\mathfrak{R}, M)^{*,*}$ be the spectral sequence (2.10). Then $E^{i,t}_1 = 0$ if $s > 3$ or if $t > 0$. In particular, the only possible non-trivial entry having the total rank 3 is $E^{3,0}_1$.

**Corollary 3.10.** If $M$ is an $\mathbb{Z}_2^+[\mathfrak{R}]$-module, then $H^1(\mathfrak{R}; M) = 0$ for each $i > 3$.

4. The proof of Main Theorem

Case $L = G$. Let $N \subseteq G$ be a maximal 2-toral subgroup of $G$. Assume that $\varphi$ is a homotopy character of $G$ and set $E^{*,*}_0 := E(\mathfrak{R}, \Pi_2^\varphi)^{*,*}$. By 3.9 the only non-zero entry at the first term having the total dimension 3 is

$$E^{1,0}_1 = A^3(\Sigma_7; \Pi_2^\varphi(\Gamma_1^7)) \oplus A^3(\Sigma \varepsilon_3; \Sigma_3 \times \Sigma_3; \Pi_2^\varphi(\Gamma_4 \times \Gamma_1^3)).$$ (4.1)

(cf. (3.6)). Our goal is to prove that the differential $d_1^{2,0} : E^{2,0}_1 \rightarrow E^{3,0}_1$ is an epimorphism.

The following facts about differentials $d^i(P, Q)$ (cf. (2.12)) will be needed:

**Proposition 4.2 (Lemma A).** A homomorphism

$$d^0(\tilde{\Gamma}_2 \times \Gamma_1^3; \Gamma_1^7) \oplus A^2(\Sigma_3; \Pi_2^\varphi(\tilde{\Gamma}_2 \times \Gamma_1^3)) \rightarrow A^3(\Sigma_7; \Pi_2^\varphi(\Gamma_1^7)).$$

is an epimorphism.

**Proposition 4.3 (Lemma B).** A homomorphism

$$d(\Gamma_4 \times \tilde{\Gamma}_2 \times \Gamma_1, \Gamma_4 \times \tilde{\Gamma}_2 \times \Gamma_1^3) : A^2(\Sigma_3 \times \Sigma_2; \Pi_2^\varphi(\Gamma_4 \times \tilde{\Gamma}_2 \times \Gamma_1)) \rightarrow A^3(\Sigma_3; \Sigma_2 \times \Sigma_3; \Pi_2^\varphi(\Gamma_4 \times \Gamma_1^3))$$

is an epimorphism.

Proofs of these crucial lemmas will be given later. Now we are ready to prove the main theorem for $L = G$:

**Proof of Main Theorem, $L = G$.** We have to check that the assumptions of 2.8 are satisfied. The first one follows from (4.1) and Lemmas A & B, and the second one is Corollary 3.10.
Case $L = \tilde{G}$.

**Definition 4.4.** Let $H$ be any 2-toral subgroup of $G$ and let $\tilde{H} := \pi^{-1}(H) \subseteq \tilde{G}$. An irreducible quasirepresentation $\alpha$ of $\tilde{H}$ is **even** if it is a restriction of some representation of $\tilde{H}$; otherwise it is called **odd**.

Let $u \in \tilde{G}$ be a non-trivial lift of unity. Notice that $\chi_\alpha(u) = \dim \alpha$ if $\alpha$ is even and $\chi_\alpha(u) = -\dim \alpha$ if $\alpha$ is odd. Moreover, all morphisms of $\mathcal{R}$ preserve parity of representations (since morphisms are induced by conjugations and $u$ is central). As a consequence, any homotopy character $\tilde{\varphi}$ of $\tilde{G}$ splits into the sum of the even part $\tilde{\varphi}_{\text{ev}} := \pi^* \varphi$ and the odd part $\tilde{\varphi}_{\text{od}}$ (where $\varphi$ is a homotopy character of $G$). Thus

$$\Pi^\tilde{\varphi}_2 \cong \mathbb{Z}_2^\wedge \otimes R(-, \tilde{\varphi}) \cong \mathbb{Z}_2^\wedge \otimes (R(-, \varphi) \oplus R(-, \tilde{\varphi}_{\text{od}})) \cong \Pi^\varphi_2 \oplus \Pi^\varphi_{\text{od}}. \quad (4.5)$$

Therefore we may consider the even case and the odd case separately (and the first one reduces to the $G$-case). Fix any homotopy character $\tilde{\varphi}$ of $\tilde{G}$ and set $E_{\tilde{\varphi}}^{\text{ev}} = E(\mathcal{R}; \tilde{\varphi})^{\text{ev}}$. The main technical step in the proof of the main theorem for $\tilde{G}$ is

**Proposition 4.6 (Lemma C).**

$$A^3(\Sigma_7; \Pi^\varphi_2(\Gamma^7_1)) = A^3(\Sigma_3 \times \Sigma_2 \times \Sigma_3; \Pi^\varphi_2(\Gamma_4 \times \Gamma^3_1)) = 0.$$  

Again, the proof is postponed till Section 6.

**Proof of Main Theorem, $L = \tilde{G}$.** Again, we check the assumptions of **Proposition 2.8**. The latter one is already proven, so it is sufficient to check that

$$H^3(\mathcal{R}; \Pi^\varphi_2) = H^3(\mathcal{R}; \Pi^\varphi_2) \oplus H^3(\mathcal{R}; \Pi^\varphi_{\text{od}}) = 0. \quad (4.7)$$

But the first summand vanishes by the Main Theorem for $L = G$ and the second one by Lemma C. \qed

5. **Quasirepresentations of stubborn subgroups of $G$**

In this section we classify irreducible quasirepresentations of some subgroups of $G$. These results, as well as the notation introduced here, will be intensively used in the proofs of Lemmas A and B.

Let $\theta$ denote a trivial irreducible representation of any group and let $\iota$ be a non-trivial irreducible representation of an order 2 group. For any 2-toral group $H$ we let $\text{IR}(H)$ (resp. $\text{IR}(H, \omega)\iota$) denote the set of all isomorphism classes of irreducible representations of $H$ (resp. $H$-subrepresentations of $\omega$); similarly let $\text{IQR}(H) := \text{IR}(H^\infty)$ and $\text{IQR}(H, \omega) := \text{IR}(H^\infty, \omega)$ be suitable sets of isomorphism classes of irreducible quasirepresentations. Finally, for any $W \subseteq \text{Out}(H)$ we let $\text{IR}^W(H)$ (resp. $\text{IQR}^W(H)$) denote irreducible $W$-invariant representations (quasirepresentations), i.e. sums if irreducible representations lying in the same orbit of $W$-action on $\text{IR}(H)$ (resp. $\text{IQR}(H)$).

**Quasirepresentations of a torus.** For any 2-adic integer $k$ define

$$\varrho_k : SO(2)^\infty = \mathbb{Z}/2^\infty \ni \frac{n}{2^\ell} \mapsto \exp \left(2\pi i \frac{kn}{2^\ell}\right) \in U(1), \quad k \in \mathbb{Z}_2^\wedge. \quad (5.1)$$

It is easy to check that $\text{IQR}(SO(2)) = \{\varrho_k\}_{k \in \mathbb{Z}_2^\wedge}$.  

**Quasirepresentations of $O(2)$.** Set

$$\alpha_k := \text{ind}_{SO(2)^\infty}^{O(2)^\infty} \varrho_k \quad \text{for} \ k \in \mathbb{Z}_2^\wedge \setminus \{0\}. \quad (5.2)$$

By Mackey’s criterion the representations $\alpha_k$ are irreducible, and

$$\text{ind}_{SO(2)^\infty}^{O(2)^\infty} \theta = \theta \oplus \tau, \quad (5.3)$$

where $\tau$ denotes the determinant. Thus

$$\text{IR}(O(2)^\infty) = \text{IQR}(\tilde{F}_2) = \{\alpha_k\}_{k \in \mathbb{Z}_2^\wedge \setminus \{0\}} \cup \{\theta, \tau\}. \quad (5.4)$$
Representations of $\{\pm1\}^r \cap SO(r)$ for odd $r$. For each sequence $(\mu_1, \ldots, \mu_r) \in \text{IR}(\{\pm1\})^r = \{\theta, \iota\}^r$ define
\[ \tau_{(\mu_1, \ldots, \mu_r)} := \text{res}_{\{\pm1\}^r \cap SO(r)} \mu_1 \otimes \cdots \otimes \mu_r. \] (5.5)

Set $\eta_i^r := \bigoplus \tau_{(\mu_1, \ldots, \mu_i)}$, where the sum is taken over all sequences in which $\iota$ appears exactly $i$ times. Obviously $\eta_i^r \sim \eta_{r-i}^r$. Therefore each $\Sigma_r$-invariant representation of $\{\pm1\}^r \cap SO(r)$ (where $\Sigma_r$ acts by permutations) is isomorphic to a direct sum of $\eta_i^r$'s for $0 \leq i < \ell^r$, or equivalently,
\[ \text{IR}^{\Sigma_r}(\{\pm1\} \cap SO(r)) = \{\eta_i^r \}_{i=0}^{(r-1)/2}. \] (5.6)

Further we will need the following definition: for any $i, j, k$ set $\eta_{i,j}^k := \bigoplus \tau_{(\mu_1, \ldots, \mu_k)}$, where the sum runs over all sequences $(\mu_1, \ldots, \mu_k)$ such that exactly $j$ of the representations $\mu_1, \ldots, \mu_k$ are isomorphic to $\iota$ and exactly $i$ of the representations $\mu_1, \ldots, \mu_k$ are isomorphic to $\iota$. Obviously for each $i, j, k, r$ we have
\[ \eta_i^r = \bigoplus_j \eta_{i,j}^k. \]

Representations of $\{\pm1\} : C_2^4$. Set $K := \{\pm1\} : C_2^4$. All irreducible representations of $K$ are subrepresentations of $\text{ind}_{\{\pm1\}^4}^K (\bigotimes_{a \in C_2^4} \mu_a)$ for $\mu_a \in \text{IR}(\{\pm1\}) = \{\theta, \iota\}$. By Mackey’s criterion the representations
\[ \gamma_1 := \text{ind}_{\{\pm1\}^4}^K \theta \otimes \theta \otimes \theta \otimes \iota \]
\[ \gamma_3 := \text{ind}_{\{\pm1\}^4}^K \theta \otimes \iota \otimes \iota \otimes \iota \]
are irreducible. Furthermore, for any $\mu, \nu \in \text{IR}(C_2) = \{\theta, \iota\}$ set
\[ \gamma_0^{\mu,\nu} := \text{res}_{K}^{C_2^4} (\mu \otimes \nu), \quad \gamma_4^{\mu,\nu} := \text{det} \otimes \text{res}_{K}^{C_2^4} (\mu \otimes \nu). \]

There are decompositions
\[ \text{ind}_{\{\pm1\}^4}^{C_2^4} \theta \otimes \theta \otimes \theta \otimes \theta \simeq \bigoplus_{\mu, \nu \in \text{IR}(C_2)} \gamma_0^{\mu,\nu}, \]
\[ \text{ind}_{\{\pm1\}^4}^{C_2^4} \iota \otimes \iota \otimes \iota \otimes \iota \simeq \bigoplus_{\mu, \nu \in \text{IR}(C_2)} \gamma_4^{\mu,\nu}. \]

Next, for any $a \in C_2^4 \setminus \{0, 0\}$ set $\zeta_a := \theta \otimes \theta \otimes \iota \otimes \iota \in \text{IR}(\{\pm1\}^4) \cong \text{IR}(\{\pm1\}^2)$, where $a$ is the difference between coordinates with the same isomorphism class of representation. Following [12, Section 8.2] we see that $\text{ind}_{\{\pm1\}^4}^{\{\pm1\}^4} \zeta_a$ splits onto the sum of non-isomorphic one-dimensional representations $\zeta_+^a$ and $\zeta_-^a$. Moreover, for any $\epsilon \in \{+, -\}$ the representations $\gamma_2^{\epsilon,a} := \text{ind}_{\{\pm1\}^4}^{\{\pm1\}^4} \zeta_a \simeq \gamma_2^{\epsilon,a} \oplus \gamma_2^{\epsilon,a}$. As a consequence we obtain
\[ \text{IR}(K) = \{\gamma_1, \gamma_3\} \cup \{\gamma_0^{\mu,\nu}, \gamma_4^{\mu,\nu}\}_{\mu, \nu \in \text{IR}(C_2)} \cup \{\gamma_2^{\epsilon,a}\}_{a \in C_2^4 \setminus \{0, 0\}}, \] (5.7)

Since the action of $W_{O(4)}(K) \cong \text{GL}_2(\mathbb{F}_2) \cong S_3$ on $K$ is the obvious one, we have
\[ \text{IR}^{\Sigma_3}(K) = \{\gamma_i^{\theta,\theta}, \gamma_i^{\theta,\iota} \oplus \gamma_i^{\iota,\iota} \}_{i=0,4} \cup \{\gamma_1, \gamma_3\} \]
\[ \cup \{\gamma_2^{\epsilon}\} := \gamma_2^{\epsilon,(0,1)} \oplus \gamma_2^{\epsilon,(1,0)} \oplus \gamma_2^{\epsilon,(1,1)} \}_{\epsilon \in \{+, -\}}. \] (5.8)
Quasirepresentations of some stubborn subgroups of $G$. Stubborn subgroups of $G$ have no natural presentation as products which leads to some notational problems. To solve them, fix the following isomorphisms:

$$O(2) \times ((\pm 1)^5 \cap SO(5)) \ni (g, h) \mapsto (g, h \cdot \det g) \in (\tilde{T}_2 \times I_1^3) \cap G$$

$$K \times ((\pm 1)^3 \cap SO(3)) \ni (g, h) \mapsto (g, h \cdot \det g) \in (I_1 \cdot C_2^2 \times I_1^3) \cap G$$

$$\Gamma_4 \times O(2) \ni (g, h) \mapsto (g, h, \det(g) \det(h)) \in (\Gamma_4 \times \tilde{T}_2 \times I_1) \cap G$$

$$\Gamma_4 \times ((\pm 1)^3 \cap SO(3)) \ni (g, h) \mapsto (g, h) \in (\Gamma_4 \times I_1^3) \cap G.$$ (5.9)

They allow for identification of quasirepresentations of the right-hand groups with quasirepresentations of the corresponding left-hand groups, (which are products of groups whose representations are classified above).

Elementary calculations allow us to obtain the following table which contains the classification of $\text{Aut}_\Omega(P)$-equivariant quasirepresentations of $P \cap G$ for some $P \in \text{Ob}(\Omega)$:

<table>
<thead>
<tr>
<th>$P$</th>
<th>$\text{Aut}_\Omega(P)$</th>
<th>$\text{IQR}^{\text{Aut}_\Omega{P}}(P \cap G)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_1$</td>
<td>$\Sigma_7$</td>
<td>$\eta^7_i$, $0 \leq i \leq 3$</td>
</tr>
<tr>
<td>$\tilde{T}_2 \times I_1^3$</td>
<td>$\Sigma_3 \times \Sigma_3$</td>
<td>$\omega \otimes \eta^5_1$, $\omega \in \text{IQR}(O(2))$, $0 \leq i \leq 2$</td>
</tr>
<tr>
<td>$\Gamma_1 \cdot C_2^2 \times I_1^3$</td>
<td>$\Sigma_3 \times \Sigma_3$</td>
<td>$\omega \otimes \eta^3_3$, $\omega \in \text{IQR}(\Omega_3(K))$, $i = 0, 1$</td>
</tr>
<tr>
<td>$\Gamma_4 \times \tilde{T}_2 \times I_1$</td>
<td>$\Sigma_3 \times C_2$</td>
<td>$\omega \otimes \psi$, $\psi \in \text{IQR}(\Omega_3C_2(\Gamma_4))$, $\psi \in \text{IQR}(O(2))$</td>
</tr>
<tr>
<td>$\Gamma_4 \times I_1^3$</td>
<td>$\Sigma_3 \times C_2 \times \Sigma_3$</td>
<td>$\omega \otimes \eta^3_3$, $\omega \in \text{IQR}(\Omega_3C_2(\Gamma_4))$, $i = 0, 1$</td>
</tr>
</tbody>
</table>

Next, we need to gather some information about restrictions of these representations. The following propositions can be easily proved by the inspection of characters.

**Proposition 5.11.** $\Sigma_3$-invariant quasirepresentations of $(\tilde{T}_2 \times I_1^3) \cap G$ restrict to $\Gamma_1^7 \cap G$ as follows:

$$\alpha_k \otimes \eta^5_1 \mapsto \eta^7_{1[1,2]}, \quad \theta \otimes \eta^0_0 \mapsto \eta^7_0, \quad \tau \otimes \eta^0_0 \mapsto \eta^7_{2[2,2]},$$

$$\alpha_k \otimes \eta^0_0 \mapsto \eta^7_{2[1,2]}, \quad \theta \otimes \eta^0_0 \mapsto \eta^7_{3[2,2]}, \quad \tau \otimes \eta^0_0 \mapsto \eta^7_{1[0,2]},$$

$$\alpha_k \otimes \eta^0_0 \mapsto \eta^7_{3[1,2]}, \quad \theta \otimes \eta^0_0 \mapsto \eta^7_{2[0,2]}, \quad \tau \otimes \eta^0_0 \mapsto \eta^7_{3[0,2]},$$

where $k \in \mathbb{Z}^*_5$ is odd. If $k$ is even, then we have

$$\text{res}_{\Gamma_1^7 \cap G}^{(\tilde{T}_2 \times I_1^3) \cap G} \alpha_k \otimes \eta^5_1 \simeq \text{res}_{\Gamma_1^7 \cap G}^{(\tilde{T}_2 \times I_1^3) \cap G} (\theta \oplus \tau) \otimes \eta^5_1.$$

**Proposition 5.12.** Representations of $(\Gamma_1 \cdot C_2^2 \times I_1^3) \cap G$ restrict to $\Gamma_1^7 \cap G$ as follows:

$$\gamma^\mu \otimes \eta^0_0 \mapsto \eta^7_0, \quad \gamma^\mu \otimes \eta^0_0 \mapsto \eta^7_{2[0,4]}, \quad \gamma^\mu \otimes \eta^0_0 \mapsto \eta^7_{3[0,4]}, \quad \gamma^\mu \otimes \eta^0_0 \mapsto \eta^7_{3[3,4]}, \quad \gamma^\mu \otimes \eta^0_0 \mapsto \eta^7_{2[4,2]}$$

for any $\mu, \nu \in \text{IQR}((\pm 1))$, $a \in C_2^2 \setminus \{0, 0\}$, $e \in \{+, -\}$.

**Proposition 5.13.** For any $\omega \in \text{IQR}(\Sigma_3C_2(\Gamma_4))$ we have

$$\text{res}_{\Gamma_1^7 \cap G}^{(\Gamma_4 \times \Gamma_1 \cap G)} (\omega \otimes \psi) = \begin{cases} \omega \otimes \eta^0_0 & \text{for } \psi = \theta \\ \omega \otimes \eta^0_0 & \text{for } \psi = \tau \\ \omega \otimes \eta^0_0 & \text{for } \psi = \alpha_k, k \text{ is odd} \\ \omega \otimes \eta^3_3 & \text{for } \psi = \alpha_k, k \text{ is even.} \end{cases}$$

6. The proofs of the technical lemmas

Before proceeding any further steps we need to introduce some notation. For any integers $0 \leq i \leq r$ define $\mathbb{Z}^*_2[\Sigma_r]$-module $M(r, i)$ to be a free $\mathbb{Z}^*_2$-module with basis $x_{j_1 \ldots j_i}$, where $\{j_1, \ldots, j_i\}$ runs over all subsets of $\{1, \ldots, r\}$ having
i elements, with an obvious action of $\Sigma_r$. Obviously $M(r, i) \simeq M(r, r - i)$ and $M(r, 0) \simeq \mathbb{Z}_2^\wedge$. Notice also that there is a natural isomorphism

$$M(r, i) \cong \mathbb{Z}_2^\wedge \otimes R((\pm 1)^r \cap SO(r); \eta_i^r). \quad (6.1)$$

Next we need to recall some calculations involving $A^*$-functors.

**Proposition 6.2.** (1) Let $M$ be an $\mathbb{Z}_2^\wedge[\Sigma_3]$-module. Then

$$A^i(\Sigma_3; M) = \begin{cases} M^{\Sigma_2}/M^{\Sigma_3} & \text{for } i = 1 \\ 0 & \text{for } i \neq 1. \end{cases}$$

(2) Let $M$ be an $\mathbb{Z}_2^\wedge[\Sigma_3]$-module. Then

$$A^i(\Sigma_5; M) = \begin{cases} M^{\Sigma_2 \times \Sigma_2 \times \Sigma_3} / (M^{\Sigma_2 \times \Sigma_3} + M^{\Sigma_4 \times 1}) & \text{for } i = 2 \\ 0 & \text{for } i \neq 2. \end{cases}$$

In particular $A^2(\Sigma_5; M(5, i)) = 0$ for $i = 0, 1$, and $A^2(\Sigma_5; M(5, 2)) \neq 0$.

(3) $A^3(\Sigma_7; M(7, i)) = 0$ for $i = 0, 1, 2$, and $A^3(\Sigma_7; M(7, 3)) \neq 0$.

**Proof.** Part (1) is a consequence of [6, 6.2.(ii)]. The formula (2) can be easily deduced with using techniques of [6, Section 6]. Finally, part (3) is a consequence of calculations in [2]. \qed

Now we are ready to prove Lemma A. Let $\varphi$ be a homotopy character of $G$. Our goal is to prove that the homomorphism

$$d_1^{1,0}(\bar{\Gamma}_2 \times \Gamma_5^3, \Gamma_1^7) \oplus A^2(\Sigma_5; \Pi_2^\varphi(G/\bar{\Gamma}_2 \times \Gamma_1^5)) \longrightarrow A^3(\Sigma_7; \Pi_2^\varphi(G/\Gamma_1^5)) \quad (6.3)$$

as an epimorphism. This will be done in two steps: First, we will identify the homomorphism (6.3) with a similar differential in the spectral sequence associated to the restriction of $\Pi_2^\varphi$ to a full subcategory isomorphic to $\mathcal{R}_2(\Sigma_7)$. Next, we compare the restriction of $\Pi_2^\varphi$ to some acyclic module (using a homomorphism of spectral sequences).

In Appendix A we construct the full inclusion $I : \mathcal{R}_2(\Sigma_n) \rightarrow \mathcal{R}_2(O(n))$. In the special case when $n = 7$ the values of the functor $I$ are as follows:

<table>
<thead>
<tr>
<th>$P \in \text{Ob}(\mathcal{R}_2(\Sigma_7))$</th>
<th>$I(P) \in \text{Ob}(\mathcal{R}_2)$</th>
<th>$\text{ht}(P)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_2 \cdot C_2 \times C_2$</td>
<td>$\bar{\Gamma}_2 \cdot C_2 \times \bar{\Gamma}_2 \times \Gamma_1$</td>
<td>0</td>
</tr>
<tr>
<td>$C_2^3$</td>
<td>$\bar{\Gamma}_2^3 \times \Gamma_1$</td>
<td>1</td>
</tr>
<tr>
<td>$C_2 \cdot C_2$</td>
<td>$\bar{\Gamma}_2 \cdot C_2 \times \Gamma_1^3$</td>
<td>1</td>
</tr>
<tr>
<td>$1 \cdot C_2^2 \times C_2$</td>
<td>$\Gamma_1 \cdot C_2^2 \cdot \bar{\Gamma}_2 \times \Gamma_1$</td>
<td>1</td>
</tr>
<tr>
<td>$C_2$</td>
<td>$\bar{\Gamma}_2 \times \Gamma_1^5$</td>
<td>2</td>
</tr>
<tr>
<td>$1 \cdot C_2^2$</td>
<td>$\Gamma_1 \cdot C_2^2 \times \Gamma_1^3$</td>
<td>2</td>
</tr>
<tr>
<td>$1$</td>
<td>$\Gamma_1^7$</td>
<td>3</td>
</tr>
</tbody>
</table>

**Proof of Lemma A.** Set $\tilde{E}_x^{*,*} := E_{\Sigma_7}(\mathcal{R}_2, I^* \Pi_2^\varphi)$. There is a morphism of spectral sequences $\tilde{E}_x^{*,*} \rightarrow E_x^{*,*}$ induced by $I$ which provides an identification of the homomorphism (6.3) with the differential $\tilde{d}_1^{1,0} : \tilde{E}^{1,0}_2 \rightarrow \tilde{E}^{3,0}_1$. Since

$$\text{res}_{\bar{\Gamma}_1^7 \cap G}^N \varphi \simeq (\eta_0^7)^{\otimes l_0} \oplus (\eta_1^7)^{\otimes l_1} \oplus (\eta_2^7)^{\otimes l_2} \oplus (\eta_3^7)^{\otimes l_3},$$

(cf. (5.10)) then $\Pi_2^\varphi(\Gamma_1^7)$ is a direct sum of single copies of modules $M(7, i)$, $i = 0, \ldots, 3$, where $M(7, i)$ appears as a summand of $\Pi_x^\varphi(\Gamma_1^7)$ if and only if $l_i > 0$. From Proposition 6.2.(3) follows that $A^3(\Sigma_7; \Pi_2^\varphi(\Gamma_1^7)) = 0$ if $l_3 = 0$, so assume otherwise.
Define $\mathbb{Z}_2^\wedge[\mathcal{R}_2(\Sigma_7)]$-module $F$ by setting

$$F(P) := M(7, 3)^P.$$  \hfill (6.5)

By [6, Prop. 5.2] $F$ is acyclic. There exists a unique homomorphism $\Phi : I^* \Pi_*^\Phi \to F$ of $\mathbb{Z}_2^\wedge[\mathcal{R}_2(\Sigma_7)]$-modules which maps the summand $M(7, 3)$ of $\Pi_*^\Phi(I^5_2)$ isomorphically and sends the summands $M(7, i)$, $i = 0, 1, 2$ into 0. Set $\tilde{E}_k^*: \equiv E(\mathcal{R}_2(\Sigma_7), F_k^*)$. Clearly $\Phi$ induces a homomorphism of spectral sequences $\tilde{E}_k^* \to \tilde{E}_k^*$. In particular, there is a commutative diagram:

$$\begin{array}{ccc}
\Lambda^2(\Sigma_5; \Pi_*^\Phi(I_2 \times I^5_1)) \oplus \Lambda^2(\Sigma_3^2; \Pi_*^\Phi(I_1 \otimes C_2^2 \times I^3_1)) & \rightarrow & \Lambda^2(\Sigma_5; \Pi_*^\Phi(I^5_1)) \\
\Phi_* & & \Phi_* \\
\Lambda^2(\Sigma_5; M(7, 3)^{C_2}) \oplus \Lambda^2(\Sigma_3^2; M(7, 3)^{1C_2}) & \rightarrow & \Lambda(\Sigma_5; M(7, 3)) \\
\end{array}$$  \hfill (6.6)

The right-hand arrow is an isomorphism by Proposition 6.2.3, and the lower arrow is an epimorphism by acyclicity of $F$. What is left to prove is that the left-hand arrow is an epimorphism. We will prove that the cokernels of the maps

$$\Pi_*^\Phi(I_2 \times I^5_1) \cong \mathbb{Z}_2^\wedge \otimes R((I_2 \times I^5_1) \cap G, \varphi) \xrightarrow{\Phi_*} M(7, 3)^{C_2}$$  \hfill (6.7)

$$\Pi_*^\Phi(I_1 \otimes C_2^2 \times I^3_1) \cong \mathbb{Z}_2^\wedge \otimes R((I_1 \otimes C_2^2 \times I^3_1) \cap G, \varphi) \xrightarrow{\Phi_*} M(7, 3)^{1C_2}$$  \hfill (6.8)

have trivial $\Lambda^2$ (it is sufficient since $\text{Adim}_2(\Sigma_5) = \text{Adim}_2(\Sigma_3 \times \Sigma_3) = 2$). We have

$$M(7, 3)^{C_2} \cong M(5, 1)\{(x_{12j})_{j \geq 3} \oplus M(5, 2)\{x_{1kl} + x_{2kl} k,l \geq 3\} \oplus M(5, 2)\{x_{klm} k,l,m \geq 3\}$$  \hfill (6.9)

$$M(7, 3)^{1C_2} \cong (\mathbb{Z}_2^\wedge \otimes \mathbb{Z}_2^\wedge)\{x_{123} + x_{124} + x_{134} + x_{234}\} \oplus (M(3, 1)\otimes M(3, 1))\{x_{12k} + x_{34k}, x_{13} + x_{24k}, x_{14} + x_{23k} k \geq 5\} \oplus (\mathbb{Z}_2^\wedge \otimes M(3, 1))\{x_{1kl} + x_{2kl} + x_{3kl} + x_{4kl} k,l \geq 5\} \{(Z_2^\wedge \otimes \mathbb{Z}_2^\wedge)\{x_{67}\}.$$  \hfill (6.10)

where $M(r, i)X$ denotes a submodule with basis $X$ which is isomorphic to $M(n, k)$. There are two summands of $M(7, 3)^{C_2}$ having non-trivial $\Lambda^2$, namely $M(5, 2)\{x_{1kl} + x_{2kl} k,l \geq 3\}$ and $M(5, 2)\{x_{klm} k,l,m \geq 3\}$. The inspection of 5.11 shows (since $\text{res}^N_{I_1^5 \cap G}$ has a summand $\eta_2^3$) that the restriction of $\varphi$ to $(I_2 \times I_1) \cap G$ contains a subquasirepresentation isomorphic to $\alpha_{2l+1} \otimes \eta_2^3$ and a subquasirepresentation $\beta \otimes \eta_2^3$, where either $\beta \simeq \alpha_{2l}$ ($l \in \mathbb{Z}_2^\wedge$), or $\beta \simeq \tau$. Thus, there exist $\mathbb{Z}_2^\wedge[\Sigma_3]$-submodules

$$\Pi_*^\Phi(I_2 \times I^5_1) \geq A := \mathbb{Z}_2^\wedge \otimes R((I_2 \times I^5_1) \cap G; \alpha_{2l+1} \otimes \eta_2^3)$$  \hfill (6.11)

$$\Pi_*^\Phi(I_2 \times I^5_1) \geq B := \mathbb{Z}_2^\wedge \otimes R((I_2 \times I^5_1) \cap G; \beta \otimes \eta_2^3),$$  \hfill (6.12)

which are both isomorphic to $M(5, 2)$. Using again 5.11 we obtain

$$\Phi_*A = M(5, 2)\{x_{1kl} + x_{2kl} k,l \geq 3\} \subseteq M(7, 3)^{C_2}$$  \hfill (6.13)

$$\Phi_*B = M(5, 2)\{x_{klm} k,l,m \geq 3\} \subseteq M(7, 3)^{C_2}.$$  \hfill (6.14)

Thus the cokernel of (6.7) has a trivial $\Lambda^2$.

Similarly, the inspection of 5.12 shows that $\text{res}^N_{(I_1 \otimes C_2^2 \times I^3_1) \cap G}$ contains a subrepresentation which is isomorphic to $\gamma_2^3 \otimes \eta_1^3$. There exists a submodule

$$\Pi_*^\Phi(I_1 \otimes C_2^2 \times I^3_1) \geq C := (I_1 \otimes C_2^2 \times I^3_1) \cap G; \gamma_2^3 \otimes \eta_1^3)$$  \hfill (6.15)

such that
Acknowledgment

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Appendix A. Stubborn subgroups of symmetric and orthogonal groups

The main goal of this section is to construct a full inclusion
\[ \mathcal{R}_2(\Sigma_n) \rightarrow \mathcal{R}_2(O(n)). \]  
(A.1)

In the proof of the Main Theorem we need only the case \( n = 7 \), although this result seems to be interesting in itself. First we need a description of categories \( \mathcal{R}_2(\Sigma_n) \). Alperin and Fong [1] classified radical subgroups of symmetric groups (radical is the same as stubborn for finite groups) but we will need also the description of morphisms.

2-Radical subgroups of symmetric groups. Note that if \( G \subseteq \Sigma_m, H \subseteq \Sigma_n \), then the product \( G \times H \) is a subgroup of \( \Sigma_{m+n} \) and the wreath product \( G \wr H \) is a subgroup of \( \Sigma_{mn} \).

Definition A.2. For any sequence \( t_1, \ldots, t_s \) of positive integers set
\[ B(t_1, \ldots, t_s) := 1 : C_2^{t_1} \times \cdots \times C_2^{t_s} \subseteq \Sigma_2^r, \]
where \( t = t_1 + \cdots + t_s \) (we treat 1 as a subgroup of \( \Sigma_1 \)). The groups \( B(t_1, \ldots, t_s) \) will be called basic subgroups of \( \Sigma_2^r \). The set of all basic subgroups of \( \Sigma_2^r \) will be denoted by \( \mathcal{B}_{\text{prod}}(2^r) \). Next, let \( \mathcal{B}_{\text{prod}}(n) \) denote the family of all products of basic subgroups in \( \Sigma_n \), i.e.
\[ \mathcal{B}_{\text{prod}}(n) = \{ P_1 \times \cdots \times P_r : P_i \in \mathcal{B}_{\text{prod}}(2^{t_i}), n = 2^{t_1} + \cdots + 2^{t_r} \}. \]

Here follow two propositions which are consequences of \([1, (2B)]\):

Proposition A.3. If \( t = t_1 + \cdots + t_s \), then
\[ W_{\Sigma_2^r}(B(t_1, \ldots, t_s)) \simeq \text{GL}_{t_1}(\mathbb{F}_2) \times \cdots \times \text{GL}_{t_r}(\mathbb{F}_2). \]

Proposition A.4. Let \( P_i \subseteq \Sigma_{2^{t_i}} \), for \( i = 1, \ldots, r \), be a collection of pairwise non-isomorphic basic subgroups, and let \( k = \sum_{i=1}^r 2^{t_i}l_i \). Then
\[ W_{\Sigma_k} \left( \prod_{i=1}^r P_i^{l_i} \right) = \prod_{i=1}^r W_{\Sigma_{2^{t_i}}}(P_i) \wr \Sigma_{l_i}. \]

Theorem A.5. Each 2-stubborn subgroup \( G \subseteq \Sigma_k \) is, up to conjugacy, a product of basic subgroups (i.e. \( G \in \mathcal{B}_{\text{prod}}(k) \)). A group \( P \in \mathcal{B}_{\text{prod}}(k) \) is stubborn if and only if written as a product of basic subgroups \( P = P_1 \times \cdots \times P_r \) there is no factor isomorphic to \( B(1, \ldots, 1) \) which occurs with multiplicity exactly 2 or 4.

Proof. The first statement is a consequence of \([1, (2A)]\). Since \( \text{GL}_{t_1}(\mathbb{F}_2) \) has no non-trivial normal 2-subgroups and \( \Sigma_n \) has a non-trivial normal 2-subgroup if and only if \( n = 2 \) or \( n = 4 \), then the second statement follows immediately from Propositions A.3 and A.4. \( \square \)

Morphisms of \( \mathcal{R}_2(\Sigma_n) \).

Proposition A.6. If groups \( P, Q \in \mathcal{B}_{\text{prod}}(k) \) are conjugate, then there exists a conjugacy between them which permutes its basic factors.

Proof. Since each basic subgroup of a symmetric group acts transitively on the set of letters, basic factors of \( P \) (and, similarly, \( Q \)) are in bijection with the set of \( P \)-orbits (\( Q \)-orbits). The conjugacy between \( P \) and \( Q \) permutes the orbits and therefore it also permutes its basic factors. \( \square \)

Proposition A.7. Fix collections of subgroups \( P_i \subseteq \Sigma_{k_i} \) for \( i = 1, \ldots, r \) and \( H_j \subseteq \Sigma_{l_j} \) for \( j = 1, \ldots, s \). Assume that for each \( i \) the group \( G_i \) acts transitively on the set of letters, and that \( n := k_1 + \cdots + k_r = l_1 + \cdots + l_s \). If
\[ Q := Q_1 \times \cdots \times Q_s \subseteq P := P_1 \times \cdots \times P_r \subseteq \Sigma_n, \]
then \( Q = (Q \cap P_1) \times \cdots \times (Q \cap P_r) \).
**Proof.** For each $i = 1, \ldots, r$ let $O^P_i$ be an orbit of $P_i \subseteq P$. Note that $\{1, \ldots, k\} = \bigcup_{i=1}^r O^P_i$ is a decomposition onto $G$-orbits. Similarly define $Q$-sets $O^Q_j$, for $j = 1, \ldots, s$. Since $Q \subseteq P$, then for each $j$ there exists $i$ such that $O^Q_j \subseteq O^P_i$. Hence $Q_j \subseteq P \cap \Sigma_{O^Q_i} = P_i$ and the conclusion follows. \[ \square \]

For any subgroup $P \subseteq \Sigma_k$ let $\delta(P) \subseteq P$ be the subgroup generated by all elements $g \in P$ which have a fixed point. Obviously $\delta(P \times Q) = P \times Q$ for any $P \subseteq \Sigma_m$, $Q \subseteq \Sigma_n$, and

$$
\delta(B(t_1, \ldots, t_r)) = B(t_1, \ldots, t_{r-1})^{2^{t_r}}.
$$

(A.8)

**Theorem A.9.** Let $P$, $Q \in \mathcal{B}_{\text{prod}}(k)$. Assume that $Q \subseteq P$. Then the inclusion $Q \subseteq P$ is a composite of products of inclusions of the following types:

(a) $B(t_1, \ldots, t_r-1)^{2^{t_r}} \subseteq B(t_1, \ldots, t_r-1, t_r)$,

(b) $B(t_1, \ldots, t_j + t_j + 1, \ldots, t_r) \subseteq B(t_1, \ldots, t_j, t_j + 1, \ldots, t_r)$.

**Proof.** The proof is inductive with respect to the index of $Q$ in $P$. If $Q$ is reducible, then $\delta(Q) = Q$ and by A.7 the inclusion is the product of inclusions $(Q \cap P_i) \subseteq P_i$, where $P_i$'s are irreducible, so assume that $P = B(t_1, \ldots, t_r)$. Thus

$$
Q = \delta(Q) \subseteq \delta(P) = B(t_1, \ldots, t_r-1)^{2^{t_r}} \subseteq B(t_1, \ldots, t_r-1, t_r) = P
$$

and the inclusion is the composition of an inclusion of type (a) with $Q \subseteq \delta(P)$. If $Q$ is irreducible then $Q = B(t_1', \ldots, t_r')$ and we have

$$
\delta(Q) = B(t_1', \ldots, t_r')^{2^{t_r'}} \subseteq B(t_1, \ldots, t_r-1)^{2^{t_r}} = \delta(P).
$$

Since $Q$-orbits are contained in $P$-orbits, then $t'_s \geq t_r$. If $t'_s = t_r$, then we are reduced to the case of smaller inclusion of irreducible subgroups. If $t'_s > t_r$, then the inclusion

$$
B(t_1', \ldots, t'_s-1)^{2^{t'_s-t_r}} \subseteq B(t_1, \ldots, t_r-1)
$$

factors through $B(t_1, \ldots, t_r-2)^{2^{t_r-1}}$. Hence $t'_s - t_r \geq t_r - 1$ and then $t'_s \geq t_r - 1 + t_r$. Finally, we obtain the factorization

$$
Q \subseteq B(t_1, \ldots, t_r-2, t_r-1 + t_r) \subseteq B(t_1, \ldots, t_r-2, t_r-1, t_r) = G. \quad \square
$$

As a consequence we obtain

**Corollary A.10.** Each morphism in $\mathcal{R}_2(O(n))$ is a composition of automorphisms and inclusions enlisted in A.9.

A full inclusion $\mathcal{R}_2(\Sigma_n) \rightarrow \mathcal{R}_2(O(n))$. For any $P \in \mathcal{B}_{\text{prod}}(n)$ let $\bar{P} \subseteq O(n)$ be a 2-stubborn subgroup given by the formula

$$
\bar{P} = \begin{cases}
\{\pm 1\} : C_2^t & \text{for } P = B(t_1, \ldots, t_r), t_1 > 1 \\
O(2) : C_2^t & \text{for } P = B(1, t_2, \ldots, t_r) \\
\bar{P}_1 \times \cdots \times \bar{P}_r & \text{for } P_1 \in \mathcal{B}_{\text{irr}}(k_i).
\end{cases}
$$

(A.11)

**Remark.** For each $P \in \mathcal{B}_{\text{prod}}(n)$ holds

$$
\bar{P} \cap \{\pm 1\} : \Sigma_k = \{\pm 1\} : P.
$$

(A.12)

**Theorem A.13.** The formulae

$$
\mathcal{R}_2(\Sigma_n) \ni \Sigma_n / P \mapsto O(n) / \bar{P} \in \mathcal{R}_2(O(n))
$$

$$
\text{Mor}_{\mathcal{R}_2(\Sigma_n)}(Q, P) \ni gP \mapsto g\bar{P} \in \text{Mor}_{\mathcal{R}_2(O(n))}(\bar{Q}, \bar{P})
$$

define the functor $I : \mathcal{R}_2(\Sigma_n) \rightarrow \mathcal{R}_2(O(n))$ which is an inclusion onto the full subcategory.
There is an ascending sequence of finite groups $\Gamma_b$. It is clear for automorphisms (cf. Propositions A.3 and A.4), so assume that $g = 1$ and $Q \to P$ is a product of the inclusions enlister in Theorem A.9. If the inclusion $Q \subseteq P$ is a non-trivial product of inclusions $Q_1 \subseteq P_1$ and $Q_2 \subseteq P_2$, then $\bar{Q} \subseteq \bar{P}$ if and only if $\bar{Q}_1 \subseteq \bar{P}_1$ and $\bar{Q}_2 \subseteq \bar{P}_2$. Hence we are reduced to the case when the inclusion is of type (a) or type (b) (cf. A.9). If

$$Q = B(t_1, \ldots, t_{r-1})^{2\nu} \subseteq B(t_1, \ldots, t_{r-1}, t_r) = P,$$

then for $t_1 > 1$ we obtain

$$\bar{Q} = ((\pm 1) : C_2^{t_1} \cdots C_2^{t_{r-1}})^{2\nu} \subseteq \{\pm 1\} \cdot C_2^{t_1} \cdots C_2^{t_r} = \bar{P}$$

and for $t_1 = 1$

$$\bar{Q} = (O(2) : C_2^{t_1} \cdots C_2^{t_{r-1}})^{2\nu} \subseteq O(2) : C_2^{t_1} \cdots C_2^{t_r} = \bar{P}.$$

If

$$Q = B(t_1, \ldots, t_j + t_{j+1}, \ldots, t_r) \subseteq B(t_1, \ldots, t_j, t_{j+1}, \ldots, t_r) = P,$$

then for $j > 1$ we obtain the inclusion

$$\bar{Q} = K \cdot C_2^{t_1} \cdots C_2^{t_{j+1}} \cdots C_2^{t_r} \subseteq K \cdot C_2^{t_1} \cdots C_2^{t_{j+1}} \cdots C_2^{t_r} = \bar{P},$$

where $K = O(2)$ if $t_1 = 1$ and $K = \{\pm 1\} \cdot C_2^{t_1}$ otherwise. Similarly if $j = 1$ and $t_1 > 1$, then the inclusion $\bar{Q} \subseteq \bar{P}$ is straightforward. The only non-trivial case appears when $j = t_1 = 1$. Then

$$\bar{Q} = \{\pm 1\} \cdot C_2^{t_1+t_2} \cdots C_2^{t_{r-1}} \subseteq O(2) : C_2^{t_1} \cdots C_2^{t_r}.$$

Hence $I$ is well defined.

The functor $I$ is fully faithful.

By combining [11, Thm. 6], A.3 and A.4 we see that for each subgroup $P \in B_{\prod}(k)$ the homomorphism $I : \text{Aut}_{R_2(\Sigma_k)}(P) \to \text{Aut}_{R_2(O(k))}((P)$ is actually an isomorphism. Fix $Q \neq P \in B_{\prod}(k)$ and choose morphisms $g_1 P, g_2 P : Q \to P$ in the category of $\Sigma_k$-orbits. Consider the compositions

$$Q \xrightarrow{g_1} \bar{Q} \xrightarrow{g_1^{-1}} Q g_1 \xrightarrow{1} P$$

for $i = 1, 2$. By Proposition A.6 $g_1^{-1} Q g_1$ and $g_2^{-1} Q g_2$ differ by conjugation by an element $h$ which permutes irreducible factors. Hence the conjugation by $i(h)$, where $i : \Sigma_k \to O(k)$ is an obvious inclusion, sends the group $g_1^{-1} Q g_1$ onto $g_2^{-1} Q g_2$ and also permutes irreducible factors. By (A.12) $h \in P$ if and only if $i(h) \in \bar{P}$. It shows that $g_1 P$ and $g_2 P$ represent the same morphism $Q \to P$ in $R_2(\Sigma_k)$ if and only if they represent the same morphism in $R_2(\Sigma_k)$. As a consequence we get that $I$ is an isomorphism on sets of morphisms. \qed

Appendix B. Representations of countable locally finite groups

A group $\Gamma$ is countable locally finite if it satisfies the following equivalent conditions:

- $\Gamma$ is countable and every finitely generated subgroup of $\Gamma$ is finite.
- There is an ascending sequence of finite groups

$$\{1\} = \Gamma^{(0)} \subseteq \Gamma^{(1)} \subseteq \Gamma^{(2)} \subseteq \cdots \subseteq \Gamma$$

such that $\bigcup \Gamma^{(k)} = \Gamma$. (B.1)

Discrete approximations of $p$-toral groups are countable locally finite [4, 6.20]. The main goal of this section is to prove that the representation theory of countable locally finite groups is similar to the representation theory of finite groups. Namely, we prove Schur’s Proposition B.3, semi-simplicity of representations (B.4) and that they are distinguished by characters (B.6). Then we classify representations of products (B.7) and show that each linear representation has a unique unitary structure (B.8). All of these results are elementary but could not be found in the literature.
Throughout this section \( \Gamma \) denotes any countable locally finite group and \( \Gamma^{(s)} \) is a fixed sequence (B.1). Let \( \text{Rep}(\Gamma) := \bigcup_n \text{Rep}(\Gamma, \text{GL}(\mathbb{C}^n)) \) and let \( \text{IR}(\Gamma) \subset \text{Rep}(\Gamma) \) be the set of isomorphism classes of irreducible representations of \( \Gamma \). The following proposition provides a main reason for the similarity of the representation theory of locally finite groups with the finite case.

**Proposition B.2.** If \( \varphi : \Gamma \to \text{GL}(V) \) and \( \psi : \Gamma \to \text{GL}(W) \) are representations, then there is \( s < \infty \) such that
\[
\text{Hom}_\Gamma(V, W) = \text{Hom}_{\Gamma^{(s)}}(V, W).
\]

**Proof.** A sequence
\[
\text{Hom}_{\Gamma^{(0)}}(V, W) \supseteq \text{Hom}_{\Gamma^{(1)}}(V, W) \supseteq \text{Hom}_{\Gamma^{(2)}}(V, W) \supseteq \cdots \supseteq \text{Hom}_\Gamma(V, W)
\]
eventually stabilizes since the dimensions of these spaces do. Moreover, we have \( \text{Hom}_\Gamma(V, W) = \bigcap_{i=0}^\infty \text{Hom}_{\Gamma^{(i)}}(V, W) \). Thus \( \text{Hom}_\Gamma(V, W) = \text{Hom}_{\Gamma^{(s)}}(V, W) \) for large enough \( s \). \( \square \)

Here follow immediate corollaries of Proposition B.2:

**Proposition B.3.** Fix representations \( \varphi : \Gamma \to \text{GL}(V) \) and \( \psi : \Gamma \to \text{GL}(W) \).

1. If \( \varphi \) is irreducible, then for some \( s \) \( \text{res}_{\Gamma^{(s)}} \varphi \) is irreducible.
2. If for all \( s \) \( \text{res}_{\Gamma^{(s)}} \varphi \) and \( \text{res}_{\Gamma^{(s)}} \psi \) are isomorphic, then \( \varphi \) and \( \psi \) are isomorphic.
3. (Schur lemma) If \( \varphi \) is irreducible, then \( \text{End}_\Gamma(V) \) contains only homoteties. If additionally \( \psi \) is irreducible and non-isomorphic to \( \varphi \), then \( \text{Hom}_\Gamma(V, W) = 0 \). \( \square \)

**Proof.** Fix \( s \) such that \( \text{End}_\Gamma(V) = \text{End}_{\Gamma^{(s)}}(V) \) and \( \text{Hom}_\Gamma(V, W) = \text{Hom}_{\Gamma^{(s)}}(V, W) \)

1. If \( \text{res}_{\Gamma^{(s)}} \varphi \) is reducible, then there is a projection \( f \in \text{End}_\Gamma(V) \) onto one of its irreducible summands. Since \( f \in \text{End}_\Gamma(V) \) it denies irreducibility of \( \varphi \).
2. If \( \text{Hom}_{\Gamma^{(s)}}(V, W) \) contains an isomorphism, then \( \text{Hom}_\Gamma(V, W) \) also does.
3. By (1) and (2) it follows from the Schur lemma for finite groups. \( \square \)

**Proposition B.4** ([13]). Every representation of \( \varphi : \Gamma \to \text{GL}(V) \) is semi-simple, and its decomposition into irreducible summands is unique up to permutation of summands.

**Proof.** Fix \( s \) such that \( \text{End}_{\Gamma^{(s)}} V = \text{End}_\Gamma(V) \). Let \( \bigoplus_i W_i \) be a decomposition of \( \text{res}_{\Gamma^{(s)}} \varphi \) onto irreducible summands. All \( W_i \)'s are irreducible \( \Gamma \)-subrepresentations (since any \( \Gamma^{(s)} \)-projection \( V \to W_i \) is also \( \Gamma \)-homomorphism). Then \( \bigoplus_i W_i \) is a decomposition of \( \varphi \). If \( \bigoplus_i W_i \) and \( \bigoplus_j X_j \) are decompositions of \( \varphi \), then there is a \( \Gamma^{(s)} \)-isomorphism
\[
f : V = \bigoplus_i W_i \longrightarrow \bigoplus_j X_j = V
\]
permuting irreducible \( \Gamma^{(s)} \)-summands. But \( f \) is also \( \Gamma \)-isomorphism. \( \square \)

Let \( Ch(\Gamma) \subseteq \mathbb{C}^\Gamma \) be a vector subspace spanned by all characters of representations (it does not contain all class functions).

**Proposition B.5.** If \( \chi, \chi' \in Ch(\Gamma) \), then the sequence \( (\chi|\chi')_s := (\chi|\chi')_{\Gamma^{(s)}} \) stabilizes. In particular,
\[
(\chi|\chi') := \lim_{s \to \infty} (\chi|\chi')_s
\]
is a hermitian product on \( Ch(\Gamma) \) and characters of irreducible representations form an orthonormal basis of \( Ch(\Gamma) \).

**Proof.** By Proposition B.4 it is enough to prove that the sequence \( (\chi|\chi')_s \) stabilizes for \( \varphi = \chi, \chi' = \chi' \), where \( \varphi, \psi \in \text{IR}(\Gamma) \). By B.3.(1) for large enough \( s \) both \( \text{res}_{\Gamma^{(s)}} \varphi \) and \( \text{res}_{\Gamma^{(s)}} \psi \) are irreducible. Then if \( \varphi \) and \( \psi \) are isomorphic, then \( (\chi\varphi|\chi\psi)_s = 1 \); if not \( (\chi\varphi|\chi\psi)_s = 0 \) (cf. B.3.(2)).

From Propositions B.4 and B.5 we obtain
Corollary B.6. Two representations of a given locally finite group are isomorphic iff their characters are equal.

The next proposition states that representations of products behave similarly to the finite case:

Proposition B.7. Let $\Gamma$ and $\Delta$ be countable locally finite groups. The map
$$\circledast : \text{IR}(\Gamma) \times \text{IR}(\Delta) \ni (\varphi, \psi) \mapsto \text{res}^{\Gamma \times \Delta}_{\Gamma (s) \times \Delta (s)} \varphi \otimes \text{res}^{\Gamma \times \Delta}_{\Gamma (s) \times \Delta (s)} \psi \in \text{IR}(\Gamma \times \Delta)$$
is a bijection.

Proof. If $\varphi \in \text{IR}(\Gamma)$, $\psi \in \text{IR}(\Delta)$ then for some $s$ both representations $\text{res}^{\Gamma}_{\Gamma (s)} \varphi$ and $\text{res}^{\Delta}_{\Delta (s)} \psi$ are irreducible. Hence $\text{res}^{\Gamma \times \Delta}_{\Gamma (s) \times \Delta (s)} \varphi \otimes \psi$ is irreducible and so is $\varphi \circledast \psi$. Then the map $\circledast$ is well defined.

Now let $\omega \in \text{IR}(\Gamma \times \Delta)$. There is $r$ such that for $s \geq r$ $\text{res}^{\Gamma\times \Delta}_{\Gamma (s)\times \Delta (s)} \omega$ is irreducible and then it is isomorphic to a tensor product of irreducible representations of factors. Then both $\text{res}^{\Gamma\times \Delta}_{\Gamma (s)\times \{1\}} \omega$ and $\text{res}^{\Gamma\times \Delta}_{\{1\}\times \Delta (s)} \omega$ are sums of pairwise isomorphic irreducible representations. As a consequence of Proposition B.3.(2) we have
$$\text{res}^{\Gamma\times \Delta}_{\Gamma (s)\times \{1\}} \omega \simeq \varphi \otimes \dim \varphi, \quad \text{res}^{\Gamma\times \Delta}_{\{1\}\times \Delta (s)} \omega \simeq \psi \otimes \dim \psi$$
for some $\varphi \in \text{IR}(\Gamma)$, $\psi \in \text{IR}(\Delta)$. For each $s \geq r$ the characters of $\omega$ and $\varphi \circledast \psi$ are equal on $\Gamma^{(s)} \times \Delta^{(s)}$ and hence they are equal on $\Gamma \times \Delta$. Now Corollary B.6 implies that $\omega$ is isomorphic to $\varphi \circledast \psi$. □

Up to this point we have considered linear representations when Dwyer–Zabrodsky theorem requires unitary ones. The following proposition states that any complex representation admits a unique unitary structure.

Proposition B.8. Let $V$ be a complex linear space with a hermitian scalar product. Then the map $\text{Rep}(\Gamma, U(V)) \rightarrow \text{Rep}(\Gamma, \text{GL}(V))$ is a bijection.

Proof. Consider the following (obviously commutative) diagram (lim stands for an inverse limit)
$$\text{Rep}(\Gamma, U(V)) \twoheadrightarrow \text{Rep}(\Gamma, \text{GL}(V)) \quad \text{lim}_{s \rightarrow \infty} \text{Rep}(\Gamma^{(s)}, U(V)) \twoheadrightarrow \text{lim}_{s \rightarrow \infty} \text{Rep}(\Gamma^{(s)}, \text{GL}(V))$$

where the vertical maps assign to any representation the collection of its restrictions to subgroups $\Gamma^{(s)}$. The lower horizontal arrow is bijective since groups $\Gamma^{(s)}$ are finite. For $L = U(V)$ or $L = \text{GL}(V)$ let $\{\varphi_s\} \in \lim_{s \rightarrow \infty} \text{Hom}(\Gamma^{(s)}, L)$. Define homomorphisms $\psi_s \in \text{Hom}(\Gamma^{(s)}, L)$ by induction. For $s > 0$ let $\psi_s (g) = h^{-1} \varphi_s (g) h$ where $h$ is any element such that $\psi_{s-1} = h^{-1} \varphi_{s-1} h$. Then $\psi = \bigcup_s \psi_s$ is a well-defined homomorphism $\Gamma \rightarrow L$ such that $\psi|_{\Gamma^{(s)}}$ is conjugate to $\varphi_s$. This implies the surjectivity of both vertical maps. Let $\varphi, \psi \in \text{Hom}(\Gamma, U(V))$ be non-conjugate homomorphisms and put $G_s = \{g \in U(V) : g^{-1} \varphi g = \psi\}$. $G_s$ is a non-increasing sequence of closed subsets of $U(V)$. Since $\bigcap_s G_s = \emptyset$ and $U(V)$ is compact, then there is $s$ such that $G_s = \emptyset$. Then $\varphi|_{\Gamma^{(s)}}$ is not conjugate to $\psi|_{\Gamma^{(s)}}$ and the map $\text{Rep}(\Gamma, U(V)) \rightarrow \lim_{s \rightarrow \infty} \text{Rep}(\Gamma^{(s)}, U(V))$ is injective. Injectivity of the right vertical arrow follows from Corollary B.6. Now the conclusion follows. □

References


