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$\mathcal{C}$-filtered modules and proper costratifying systems

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ABSTRACT

In this paper we define and study the notion of a proper costratifying system, which is a generalization of the so-called proper costandard modules to the context of stratifying systems. The proper costandard modules were defined by V. Dlab in his study of quasi-hereditary algebras (see Dlab, 1996 [D1]).

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Introduction

Standard modules over artin algebras were defined by C.M. Ringel (see [R]) in connection with the study of quasi-hereditary algebras, where the category of modules filtered by them plays an essential role. Let $P(1), \dots, P(n)$ be an ordered sequence of the non-isomorphic indecomposable projective modules over an artin algebra Λ. By definition, the standard module ${}_\Lambda\Delta(i)$ is the largest factor module of $P(i)$ with composition factors only amongst $S(1), \dots, S(i)$, where $S(j)$ is the simple top of $P(j)$. Let $\operatorname{mod}(\Lambda)$ denote the category of finitely generated left Λ-modules. Denote by $\mathcal{F}({}_\Lambda\Delta)$ the subcategory of $\operatorname{mod}(\Lambda)$ consisting of the Λ-modules having a filtration with factors isomorphic to standard modules. The algebra Λ is said to be standardly stratified if all projective Λ-modules belong to $\mathcal{F}({}_\Lambda\Delta)$. This class of algebras was originally defined by Cline, Parshall and Scott in [CPS], and was widely studied by different mathematicians (see [ADL,AHLU,W,ES,PR,Xi]).

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Erdmann and Sáenz extended the notion of standard modules and defined the stratifying systems in [ES] with respect to a finite linear ordered set. They proved that, for a stratifying system Θ, the category of modules filtered by Θ is equivalent to the category of modules filtered by the standard modules over an appropriate standardly stratified algebra. The same was done in [MSXi] for stratifying systems defined on a finite pre-ordered set. Important work in this direction was done in the earlier paper [W] of Webb, though stratifying systems were not defined there.

In contrast with the situation for quasi-hereditary algebras, if Λ is a standardly stratified algebra then Λ^{op} need not also be standardly stratified. However, Dlab defined a new class of modules, the proper standard modules (see [D1]), with the property that Λ is a standardly stratified algebra, that is, Λ is filtered by the standard modules, if and only if Λ^{op} is filtered by the proper standard modules. This motivates the study of the category of modules filtered by the proper standard modules (see [AHLU,L]).

In this paper we define and study the notion of a proper costratifying system, which is a generalization of the so-called proper costandard modules to the context of stratifying systems.

One of our main results states that the category of modules filtered by a proper costratifying system is dual to the category of modules filtered by the proper costandard modules over a certain standardly stratified algebra.

Although stratifying systems and proper costratifying systems are quite different, they have similar features, and they can be studied under a common frame. This comes from the observation that, in either case, there is a module M in $\mathrm{mod}(\Lambda)$ such that the category $\mathcal{F}$ of modules filtered by the corresponding system satisfies the following property:

> For each X in $\mathcal{F}$ there is an exact sequence $M_2 \to M_1 \to M_0 \to X \to 0$ with M_0, M_1, M_2 in $\mathrm{add}(M)$, which remains exact under the functor $F = \mathrm{Hom}_\Lambda(M, -)$. Here $\mathrm{add}(M)$ denotes the full subcategory of $\mathrm{mod}(\Lambda)$ consisting of the direct sums of direct summands of M,

as proved in Lemma 2.12 and Proposition 4.1.

For a Λ-module M, the category consisting of the Λ-modules admitting a sequence as above, is denoted by C_2^M, and was studied by Platzeck and Pratti in [PP1]. In both cases M is Ext-projective in $\mathcal{F}$. More precisely, M is the sum of the non-isomorphic Ext-projective indecomposable modules in $\mathcal{F}$. Using this fact, results in [PP1] can be applied to prove properties of the category $\mathcal{F}$.

The paper is organized as follows. After a brief section of preliminaries, we devote Section 2 to the study of C_2^M categories and show how these results apply to give a new proof of the above mentioned theorem of Erdmann and Sáenz concerning stratifying systems. In Section 3 we introduce the notion of a stratifying system and study their properties. Finally, in Section 4 we prove our main results about proper costratifying systems.

1. Preliminaries

Throughout this paper algebra means *artin R-algebra*, where R is a commutative artinian ring. When Λ is an algebra the term 'Λ-module' will mean *finitely generated left Λ-module*. The category of finitely generated left Λ-modules is denoted by $\mathrm{mod}(\Lambda)$ and the full subcategory of finitely generated projective Λ-modules by $\mathrm{proj}(\Lambda)$. For Λ-modules M and N, $\mathrm{Tr}_M(N)$ is the *trace* of M in N, that is, $\mathrm{Tr}_M(N)$ is the Λ-submodule of N generated by the images of all morphisms from M to N. Let $D : \mathrm{mod}(\Lambda) \to \mathrm{mod}(\Lambda^{op})$ denote the *usual duality* for artin algebras, and $*$ denote the functor $\mathrm{Hom}_\Lambda(-, \Lambda) : \mathrm{mod}(\Lambda) \to \mathrm{mod}(\Lambda^{op})$. Then $*$ induces a duality from $\mathrm{proj}(\Lambda)$ to $\mathrm{proj}(\Lambda^{op})$. For a given natural number t, we set $[1, t] = \{1, 2, \ldots, t\}$.

Let Λ be an algebra. We next recall the definition (see [R,DR,ADL,D1]) of the following classes of Λ-modules: standard, proper standard, costandard and proper costandard. Let n be the rank of the Grothendieck group $K_0(\Lambda)$. We fix a linear order $\leqslant$ on $[1, n]$ and a representative set ${}_\Lambda P = \{{}_\Lambda P(i)\colon\ i \in [1, n]\}$ containing one module of each iso-class of indecomposable projective Λ-modules. The injective envelope of the simple Λ-module ${}_\Lambda S(i) = \mathrm{top}({}_\Lambda P(i))$ is denoted by ${}_\Lambda I(i)$. For the opposite algebra Λ^{op}, we always consider the representative set ${}_{\Lambda^{op}} P = \{{}_{\Lambda^{op}} P(i)\colon\ i \in [1, n]\}$ of indecomposable projective Λ^{op}-modules, where ${}_{\Lambda^{op}}P(i) = ({}_\Lambda P(i))^*$ for all $i \in [1,n]$. So, with these choices in mind, we introduce now the following classes of modules:

The set of *standard Λ-modules* is ${}_\Lambda\Delta = \{{}_\Lambda\Delta(i)\colon\ i \in [1,n]\}$, where ${}_\Lambda\Delta(i) = {}_\Lambda P(i)/\operatorname{Tr}_{\bigoplus_{j>i} {}_\Lambda P(j)}({}_\Lambda P(i))$. Then, ${}_\Lambda\Delta(i)$ is the largest factor module of ${}_\Lambda P(i)$ with composition factors only amongst ${}_\Lambda S(j)$ for $j \leqslant i$. The set of *costandard Λ-modules* is ${}_\Lambda\nabla = D({}_{\Lambda^{op}}\Delta)$, where the pair $({}_{\Lambda^{op}}P, \leqslant)$ is used to compute ${}_{\Lambda^{op}}\Delta$.

The set of *proper standard Λ-modules* is ${}_\Lambda\overline{\Delta} = \{{}_\Lambda\overline{\Delta}(i)\colon\ i \in [1,n]\}$, where ${}_\Lambda\overline{\Delta}(i) = {}_\Lambda P(i)/\operatorname{Tr}_{\bigoplus_{j\geqslant i} {}_\Lambda P(j)}(\operatorname{rad} {}_\Lambda P(i))$. Then, ${}_\Lambda\overline{\Delta}(i)$ is the largest factor module of ${}_\Lambda\Delta(i)$ satisfying the multiplicity condition $[{}_\Lambda\overline{\Delta}(i) : S(i)] = 1$. The set of *proper costandard Λ-modules* is ${}_\Lambda\overline{\nabla} = D({}_{\Lambda^{op}}\overline{\Delta})$, where the pair $({}_{\Lambda^{op}}P, \leqslant)$ is used to compute ${}_{\Lambda^{op}}\overline{\Delta}$.

Let $\mathcal{F}({}_\Lambda\Delta)$ be the subcategory of $\operatorname{mod}(\Lambda)$ consisting of the Λ-modules having a ${}_\Lambda\Delta$-filtration, that is, a filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_s = M$ with factors M_{i+1}/M_i isomorphic to a module in ${}_\Lambda\Delta$ for all i. The algebra Λ is a *standardly stratified algebra* with respect to the linear order $\leqslant$ on the set $[1,n]$, if $\operatorname{proj}(\Lambda) \subseteq \mathcal{F}({}_\Lambda\Delta)$ (see [ADL,D1,CPS]). The algebra Λ is a *properly stratified algebra* with respect to the linear order $\leqslant$ on the set $[1,n]$, if and only if its regular representation is filtered by standard as well as by proper standard modules. That is, $\operatorname{proj}(\Lambda) \subseteq \mathcal{F}({}_\Lambda\Delta) \cap \mathcal{F}({}_\Lambda\overline{\Delta})$ (see [D2]).

Recall that a morphism $f : C \to M$ in $\operatorname{mod}(\Lambda)$ is *right minimal* if any morphism $g : C \to C$, with $f = fg$, is an automorphism. Moreover, for a given class $\mathcal{C}$ of objects in $\operatorname{mod}(\Lambda)$, $f : C \to M$ is a *right $\mathcal{C}$-approximation* of M if $C \in \mathcal{C}$ and the map $\operatorname{Hom}_\Lambda(C_1, f) : \operatorname{Hom}_\Lambda(C_1, C) \to \operatorname{Hom}_\Lambda(C_1, M)$ is surjective for all $C_1 \in \mathcal{C}$. A *right minimal $\mathcal{C}$-approximation* is a right $\mathcal{C}$-approximation which is right minimal. The notion of *left minimal* morphism and *left minimal $\mathcal{C}$-approximation* are defined dually.

Let Λ be an algebra and $\mathcal{X}$ a class of objects in $\operatorname{mod}(\Lambda)$. For each natural number n, we set ${}^{\perp_n}\mathcal{X} = \{M \in \operatorname{mod}(\Lambda)\colon\ \operatorname{Ext}^n_\Lambda(M, -)|_{\mathcal{X}} = 0\}$ and ${}^{\perp}\mathcal{X} = \bigcap_{n>0} {}^{\perp_n}\mathcal{X}$. Similarly, the notions of $\mathcal{X}^{\perp_n}$ and $\mathcal{X}^{\perp}$ are introduced.

2. C_2^M categories and $\mathcal{C}$-filtered modules

The categories C_n^M, whose definition is recalled in the next paragraph, were introduced by Platzeck and Pratti in [PP1], where particular interest was focused on the case when $C_0^M = C_1^M$. Here, we will apply these ideas in a different context. We will mainly concentrate in the case when $\mathcal{C} \subseteq C_2^M$, and study properties of the category $\mathcal{F}(\mathcal{C})$ of modules filtered by $\mathcal{C}$. These results apply to the category of modules filtered by a stratifying system (Theorem 2.14), as well as to those filtered by a proper costratifying system (Theorem 4.3), since both categories are contained in C_2^Q, for an appropriate Q. Other examples of categories contained in C_2^M are the torsion modules of a tilting module M. This gives a new approach to prove well-known results in tilting theory [PP1,PP2].

Let Λ be an artin R-algebra. For each $M \in \operatorname{mod}(\Lambda)$, we consider the opposite algebra $\Gamma = \operatorname{End}_\Lambda(M)^{op}$ and the R-functors

$$\operatorname{mod}(\Lambda) \xrightarrow{F} \operatorname{mod}(\Gamma) \xrightarrow{G} \operatorname{mod}(\Lambda),$$

where $F = \operatorname{Hom}_\Lambda({}_\Lambda M_\Gamma, -)$ and $G = {}_\Lambda M_\Gamma \otimes_\Gamma -$. Following M.I. Platzeck and N.I. Pratti in Section 2 of [PP1], for $n \geqslant 0$ we denote by C_n^M the full subcategory of $\operatorname{mod}(\Lambda)$ consisting of the Λ-modules X admitting an exact sequence in $\operatorname{mod}(\Lambda)$

$$M_n \to M_{n-1} \to \cdots \to M_1 \to M_0 \to X \to 0$$

with $M_i \in \operatorname{add}(M)$, and such that the induced sequence

$$F(M_n) \to F(M_{n-1}) \to \cdots \to F(M_1) \to F(M_0) \to F(X) \to 0$$

is exact in $\operatorname{mod}(\Gamma)$.

We recall next the following useful result due to M. Auslander in [A].

Theorem 2.1. *For $M \in \operatorname{mod}(\Lambda)$, $\Gamma = \operatorname{End}_\Lambda(M)^{op}$ and $F = \operatorname{Hom}_\Lambda(M, -) : \operatorname{mod}(\Lambda) \to \operatorname{mod}(\Gamma)$, the following statements hold.*

(a) *The restriction $F|_{C_1^M} : C_1^M \to \operatorname{mod}(\Gamma)$ is full and faithful.*
(b) *$F : \operatorname{Hom}_\Lambda(Z, X) \to \operatorname{Hom}_\Gamma(F(Z), F(X))$ is an isomorphism in $\operatorname{mod}(R)$, for all $Z \in \operatorname{add}(M)$, $X \in \operatorname{mod}(\Lambda)$.*
(c) *The restriction $F|_{\operatorname{add}(M)} : \operatorname{add}(M) \to \operatorname{proj}(\Gamma)$ is an equivalence of R-categories.*

Remark 2.2. As a consequence of Theorem 2.1(b) and the left exactness of F, it follows that if $X \in C_2^M$ then there exists an exact sequence in $\operatorname{mod}(\Lambda)$, $0 \to K \to M_0 \to X \to 0$, with $M_0 \in \operatorname{add}(M)$ and $K \in C_1^M$, such that the sequence $0 \to F(K) \to F(M_0) \to F(X) \to 0$ is exact in $\operatorname{mod}(\Gamma)$.

The following result, proven in [PP1], will be very useful in what follows, where $\epsilon : GF \to 1$ is the co-unit of the adjunction $\eta : \operatorname{Hom}_\Lambda(G-, -) \to \operatorname{Hom}_\Gamma(-, F-)$, that is, $\epsilon_X = \eta^{-1}(1_{F(X)}) : GF(X) \to X$.

Proposition 2.3. *Let $M \in \operatorname{mod}(\Lambda)$, $\Gamma = \operatorname{End}_\Lambda(M)^{op}$, $F = \operatorname{Hom}_\Lambda(M, -) : \operatorname{mod}(\Lambda) \to \operatorname{mod}(\Gamma)$ and $G = M \otimes_\Gamma - : \operatorname{mod}(\Gamma) \to \operatorname{mod}(\Lambda)$. Then*

$$C_1^M \subseteq \{X \in \operatorname{mod}(\Lambda) \text{ such that } \epsilon_X : GF(X) \to X \text{ is an isomorphism}\}.$$

Proof. See [PP1, Proposition 2.2]. □

The next propositions show that the modules in C_2^M have nice homological properties.

Proposition 2.4. *Let $M \in \operatorname{mod}(\Lambda)$, $\Gamma = \operatorname{End}_\Lambda(M)^{op}$, $F = \operatorname{Hom}_\Lambda(M, -) : \operatorname{mod}(\Lambda) \to \operatorname{mod}(\Gamma)$ and $\mathcal{Y} \subseteq C_1^M$ be such that $M \in {}^{\perp_1}\mathcal{Y}$. Then, for all $X \in C_2^M$, $Y \in \mathcal{Y}$, the map induced by F*

$$\rho_{X,Y} : \operatorname{Ext}^1_\Lambda(X, Y) \to \operatorname{Ext}^1_\Gamma(F(X), F(Y))$$

is an isomorphism of R-modules.

Proof. Let $X \in C_2^M$ and $Y \in \mathcal{Y}$. By Remark 2.2 there exists an exact sequence

$$\varepsilon : 0 \to K \to M_0 \to X \to 0,$$

with $K \in C_1^M$ and $M_0 \in \operatorname{add}(M)$, such that the sequence

$$F(\varepsilon) : 0 \to F(K) \to F(M_0) \to F(X) \to 0$$

is exact in $\operatorname{mod}(\Gamma)$. Since $M_0 \in \operatorname{add}(M)$ and $M \in {}^{\perp_1}\mathcal{Y}$, we have that $\operatorname{Ext}^1_\Lambda(M_0, Y) = 0 = \operatorname{Ext}^1_\Gamma(F(M_0), F(Y))$ because $F(M_0) \in \operatorname{proj}(\Gamma)$ (see Theorem 2.1(c)). Then, by applying $\operatorname{Hom}_\Lambda(-, Y)$ to ε, and $\operatorname{Hom}_\Gamma(-, F(Y))$ to $F(\varepsilon)$, we get the following exact and commutative diagram

$$\begin{array}{ccccccc}
\operatorname{Hom}_\Lambda(M_0, Y) & \longrightarrow & \operatorname{Hom}_\Lambda(K, Y) & \longrightarrow & \operatorname{Ext}^1_\Lambda(X, Y) & \longrightarrow & 0 \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \rho_{X,Y} & & \\
\operatorname{Hom}_\Gamma(F(M_0), F(Y)) & \longrightarrow & \operatorname{Hom}_\Gamma(F(K), F(Y)) & \longrightarrow & \operatorname{Ext}^1_\Gamma(F(X), F(Y)) & \longrightarrow & 0.
\end{array}$$

By Theorem 2.1, we have that the two first vertical arrows are isomorphisms. Hence $\rho_{X,Y}$ is an isomorphism. □

Proposition 2.5. *Let $M \in \mathrm{mod}(\Lambda)$, $\Gamma = \mathrm{End}_\Lambda(M)^{op}$ and $F = \mathrm{Hom}_\Lambda(M, -) : \mathrm{mod}(\Lambda) \to \mathrm{mod}(\Gamma)$. Then the following statements hold.*

(a) $F(\mathcal{C}_2^M) \subseteq \mathrm{Ker}\,\mathrm{Tor}_1^\Gamma(M, -)$.
(b) *If $\varepsilon : 0 \to F(X) \to Y' \to F(Z) \to 0$ is exact in $\mathrm{mod}(\Gamma)$, with $X, Z \in \mathcal{C}_2^M$, then there is an exact sequence $\eta : 0 \to X \to Y \to Z \to 0$ in $\mathrm{mod}(\Lambda)$ such that $\varepsilon \simeq F(\eta)$.*

Proof. We consider the functor $G = M \otimes_\Gamma - : \mathrm{mod}(\Gamma) \to \mathrm{mod}(\Lambda)$.

(a) The arguments in the proof of [PP1, Theorem 3.7] can be easily adapted to this case.

(b) Let $\varepsilon : 0 \to F(X) \to Y' \xrightarrow{g} F(Z) \to 0$ be exact in $\mathrm{mod}(\Gamma)$, with $X, Z \in \mathcal{C}_2^M$. Applying G to ε, we have the exact sequence

$$\mathrm{Tor}_1^\Gamma\big(M, F(Z)\big) \to GF(X) \to G\big(Y'\big) \to GF(Z) \to 0.$$

Since $\mathcal{C}_2^M \subseteq \mathcal{C}_1^M$, from the last sequence and (a), we get the exact and commutative diagram

$$\begin{array}{ccccccccc}
G(\varepsilon): 0 & \longrightarrow & GF(X) & \longrightarrow & G(Y') & \longrightarrow & GF(Z) & \longrightarrow & 0 \\
 & & \downarrow \simeq & & \| & & \downarrow \simeq & & \\
\eta: 0 & \longrightarrow & X & \longrightarrow & G(Y') & \longrightarrow & Z & \longrightarrow & 0.
\end{array}$$

We will show next that η is the desired sequence, where $Y = G(Y')$. Indeed, by applying the left exact functor F to $G(\varepsilon)$, we obtain the following exact and commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F(X) & \longrightarrow & Y' & \xrightarrow{g} & F(Z) & \longrightarrow & 0 \\
 & & \downarrow \simeq & & \downarrow h & & \theta \downarrow \simeq & & \\
0 & \longrightarrow & FGF(X) & \longrightarrow & FG(Y') & \xrightarrow{FG(g)} & FGF(Z). & &
\end{array}$$

From the equality $\theta g = FG(g)h$ and the fact that θ is an isomorphism, it follows that $FG(g)$ is an epimorphism. Hence h is an isomorphism, and this proves that $\varepsilon \simeq F(\eta)$. □

For a functor $F : \mathcal{A} \to \mathcal{B}$ and a class of objects $\mathcal{X}$ in $\mathcal{A}$, let $F(\mathcal{X}) = \{Z \in \mathcal{B}\colon\ Z \simeq F(X)$ for some $X \in \mathcal{X}\}$.

Corollary 2.6. *Let $M \in \mathrm{mod}(\Lambda)$, $\Gamma = \mathrm{End}_\Lambda(M)^{op}$, $F = \mathrm{Hom}_\Lambda(M, -) : \mathrm{mod}(\Lambda) \to \mathrm{mod}(\Gamma)$ and $\mathcal{X} \subseteq \mathcal{C}_2^M$. If $\mathcal{X}$ is closed under extensions, then $F(\mathcal{X})$ is so.*

Proof. The proof follows immediately from Proposition 2.5(b). □

Let Λ be an algebra and $\mathcal{C}$ be a class of objects in $\mathrm{mod}(\Lambda)$. We denote by $\mathcal{F}(\mathcal{C})$ the class of the Λ-modules having a $\mathcal{C}$-filtration, that is, a filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ of submodules with factors M_{i+1}/M_i isomorphic to a module in $\mathcal{C}$ for all i. Then $\mathcal{F}(\mathcal{C})$ is the smallest class in $\mathrm{mod}(\Lambda)$ which is closed under extensions and contains $\mathcal{C}$. Moreover, it is straightforward to see that ${}^{\perp_1}\mathcal{C} = {}^{\perp_1}\mathcal{F}(\mathcal{C})$.

Corollary 2.7. *Let $M \in \mathrm{mod}(\Lambda)$, $\Gamma = \mathrm{End}_\Lambda(M)^{op}$, $F = \mathrm{Hom}_\Lambda(M, -) : \mathrm{mod}(\Lambda) \to \mathrm{mod}(\Gamma)$ and $\mathcal{C} \subseteq \mathrm{mod}(\Lambda)$. If the restriction $F|_{\mathcal{F}(\mathcal{C})} : \mathcal{F}(\mathcal{C}) \to \mathrm{mod}(\Gamma)$ is an exact functor and $\mathcal{F}(\mathcal{C}) \subseteq \mathrm{C}_2^M$, then $F(\mathcal{F}(\mathcal{C})) = \mathcal{F}(F(\mathcal{C}))$.*

Proof. Since the restriction $F|_{\mathcal{F}(\mathcal{C})} : \mathcal{F}(\mathcal{C}) \to \mathrm{mod}(\Gamma)$ is an exact functor, it follows that $F(\mathcal{F}(\mathcal{C})) \subseteq \mathcal{F}(F(\mathcal{C}))$. On the other hand, the condition $\mathcal{F}(\mathcal{C}) \subseteq \mathrm{C}_2^M$ and Corollary 2.6 give us the other inclusion. $\square$

We recall that a class $\mathcal{X}$ of objects in $\mathrm{mod}(\Lambda)$ is *resolving* if it is closed under extensions, kernels of epimorphisms and $\mathrm{proj}(\Lambda) \subseteq \mathcal{X}$ (see [AR]).

Lemma 2.8. *If $\mathcal{X}$ is a resolving subcategory of $\mathrm{mod}(\Lambda)$, then $\mathrm{proj}(\Lambda) = \mathcal{X} \cap {}^{\perp_1}\mathcal{X}$.*

Proof. Assume that $\mathcal{X}$ is resolving. It is clear that $\mathrm{proj}(\Lambda) \subseteq \mathcal{X} \cap {}^{\perp_1}\mathcal{X}$. The proof is completed by showing the other inclusion. Let $X \in \mathcal{X} \cap {}^{\perp_1}\mathcal{X}$, and consider the exact sequence $\varepsilon : 0 \to K \to P_0(X) \to X \to 0$ in $\mathrm{mod}(\Lambda)$, where $P_0(X)$ is the projective cover of X. Since $\mathcal{X}$ is resolving, we conclude that $K \in \mathcal{X}$, and hence ε splits, because $X \in {}^{\perp_1}\mathcal{X}$. Thus $X \in \mathrm{proj}(\Lambda)$. $\square$

The following lemma will be useful in the sequel.

Lemma 2.9. *Let $M \in \mathrm{mod}(\Lambda)$, $\Gamma = \mathrm{End}_\Lambda(M)^{op}$, $F = \mathrm{Hom}_\Lambda(M, -) : \mathrm{mod}(\Lambda) \to \mathrm{mod}(\Gamma)$ and $G = M \otimes_\Gamma - : \mathrm{mod}(\Gamma) \to \mathrm{mod}(\Lambda)$. Let $\mathcal{A}$ and $\mathcal{B}$ be full subcategories of $\mathrm{mod}(\Lambda)$ and $\mathrm{mod}(\Gamma)$ respectively, closed under isomorphisms and such that the restriction $F|_{\mathcal{A}} : \mathcal{A} \to \mathcal{B}$ is an equivalence of categories. If $\epsilon_A : GF(A) \to A$ is an isomorphism for all $A \in \mathcal{A}$, then the restriction $G|_{\mathcal{B}} : \mathcal{B} \to \mathcal{A}$ is a quasi-inverse of $F|_{\mathcal{A}}$.*

Proof. Let $B \in \mathcal{B}$. First, we prove that $G(B) \in \mathcal{A}$. Indeed, since $B \in \mathcal{B}$ and $F|_{\mathcal{A}}$ is dense, there exists an isomorphism $\rho : B \to F(A)$ in $\mathcal{B}$ for some $A \in \mathcal{A}$. Therefore $G(B) \simeq GF(A) \simeq A$ and hence $G(B) \in \mathcal{A}$.

Let now $\mu : 1 \to FG$ denote the unit of the adjunction $\eta : \mathrm{Hom}_\Lambda(G-, -) \to \mathrm{Hom}_\Gamma(-, F-)$, that is, $\mu_Y = \eta(1_{G(Y)}) : Y \to FG(Y)$. We next prove that the natural transformation $\mu_B : B \to FG(B)$ is an isomorphism for all $B \in \mathcal{B}$. To do so, we consider the following commutative diagram

$$\begin{array}{ccccc} B & \xrightarrow{\mu_B} & FG(B) & & \\ \rho \downarrow \simeq & & FG(\rho) \downarrow \simeq & & \\ F(A) & \xrightarrow[\mu_{F(A)}]{} & FGF(A) & \xrightarrow[F(\epsilon_A)]{} & F(A). \end{array}$$

Observe that $F(\epsilon_A)$ is an isomorphism since ϵ_A is so. From this and the fact that $F(\epsilon_A)\mu_{F(A)} = 1_{F(A)}$, we conclude that $\mu_{F(A)}$ is an isomorphism. Hence μ_B is an isomorphism and this proves the lemma. $\square$

We are in a position to prove the main result of this section, which we state in the next theorem.

Theorem 2.10. *Let $\mathcal{C}$ be a class of objects in $\mathrm{mod}(\Lambda)$, $M \in {}^{\perp_1}\mathcal{C}$, $\Gamma = \mathrm{End}_\Lambda(M)^{op}$, $F = \mathrm{Hom}_\Lambda(M, -) : \mathrm{mod}(\Lambda) \to \mathrm{mod}(\Gamma)$ and $G = M \otimes_\Gamma - : \mathrm{mod}(\Gamma) \to \mathrm{mod}(\Lambda)$. If $\mathcal{F}(\mathcal{C}) \subseteq \mathrm{C}_2^M$, then the following statements hold.*

(a) *$F|_{\mathcal{F}(\mathcal{C})} : \mathcal{F}(\mathcal{C}) \to \mathcal{F}(F(\mathcal{C}))$ is an exact equivalence of categories and $G|_{\mathcal{F}(F(\mathcal{C}))} : \mathcal{F}(F(\mathcal{C})) \to \mathcal{F}(\mathcal{C})$ is a quasi-inverse of $F|_{\mathcal{F}(\mathcal{C})}$.*
(b) *If $\mathrm{add}(M) \subseteq \mathcal{F}(\mathcal{C})$ and $\mathcal{F}(F(\mathcal{C}))$ is closed under kernels of epimorphisms, then $\mathcal{F}(F(\mathcal{C}))$ is resolving and $\mathrm{add}(M) = \mathcal{F}(\mathcal{C}) \cap {}^{\perp_1}\mathcal{F}(\mathcal{C})$.*

Proof. Let $\mathcal{F}(\mathcal{C}) \subseteq C_2^M$ and recall that ${}^{\perp_1}\mathcal{C} = {}^{\perp_1}\mathcal{F}(\mathcal{C})$.

(a) By Theorem 2.1(a), we have that $F|_{\mathcal{F}(\mathcal{C})} : \mathcal{F}(\mathcal{C}) \to F(\mathcal{F}(\mathcal{C}))$ is an equivalence of categories. Furthermore, since $M \in {}^{\perp_1}\mathcal{F}(\mathcal{C})$, it follows that $F|_{\mathcal{F}(\mathcal{C})}$ is exact. Then, by Corollary 2.7, we get that $F(\mathcal{F}(\mathcal{C})) = \mathcal{F}(F(\mathcal{C}))$ and this proves the first claim in (a). The rest of the proof of (a) follows immediately from Proposition 2.3 and Lemma 2.9.

(b) Let $\operatorname{add}(M) \subseteq \mathcal{F}(\mathcal{C})$ and let $\mathcal{F}(F(\mathcal{C}))$ be closed under kernels of epimorphisms. By Theorem 2.1(c), we have that $F|_{\operatorname{add}(M)} : \operatorname{add}(M) \to \operatorname{proj}(\Gamma)$ is an equivalence and therefore $\operatorname{proj}(\Gamma) \subseteq \mathcal{F}(F(\mathcal{C}))$. Then, by Lemma 2.8, $\operatorname{proj}(\Gamma) = \mathcal{F}(F(\mathcal{C})) \cap {}^{\perp_1}\mathcal{F}(F(\mathcal{C}))$.

The hypotheses imply that $\operatorname{add}(M) \subseteq \mathcal{F}(\mathcal{C}) \cap {}^{\perp_1}\mathcal{F}(\mathcal{C})$. Let $A \in \mathcal{F}(\mathcal{C}) \cap {}^{\perp_1}\mathcal{F}(\mathcal{C})$. Then $F(A) \in \mathcal{F}(F(\mathcal{C}))$ and, since $\mathcal{F}(F(\mathcal{C}))$ is resolving, there exists an exact sequence in $\mathcal{F}(F(\mathcal{C}))$

$$\varepsilon : 0 \to Z' \to P \to F(A) \to 0$$

with $P \in \operatorname{proj}(\Gamma)$. By (a), we have that $Z' \simeq F(Z)$ for some $Z \in \mathcal{F}(\mathcal{C})$. Hence, by Proposition 2.5(b), there exists an exact sequence in $\mathcal{F}(\mathcal{C})$

$$\eta : 0 \to Z \to Q \to A \to 0$$

such that $F(\eta) \simeq \varepsilon$. Since $A \in {}^{\perp_1}\mathcal{F}(\mathcal{C})$, then η splits, and so does ε. Thus $F(A) \in \operatorname{proj}(\Gamma) = F(\operatorname{add}(M))$. Consequently, $A \in \operatorname{add}(M)$. □

The above results can be applied to the study of proper costratifying systems, which will be introduced in the next section. We will show next that they can also be used to obtain, in a unified way, known results about Ext-projective stratifying systems. To do so, we start with two lemmas. The first one states a result proven in [MMS], which is fundamental for our considerations, and the second one is a useful technical lemma. To start with, we recall firstly the definition of Ext-projective stratifying system.

Definition 2.11. (See [MMS].) Let Λ be an artin R-algebra. An Ext-projective stratifying system $(\Theta, \underline{Q}, \leqslant)$, of size t in $\operatorname{mod}(\Lambda)$, consists of two families of non-zero Λ-modules $\Theta = \{\Theta(i)\}_{i=1}^t$ and $\underline{Q} = \{Q(i)\}_{i=1}^t$, with $Q(i)$ indecomposable for all i, and a linear order $\leqslant$ on the set $[1, t]$, satisfying the following conditions.

(a) $\operatorname{Hom}_\Lambda(\Theta(i), \Theta(j)) = 0$ if $i > j$.
(b) For each $i \in [1, t]$, there is an exact sequence

$$\varepsilon_i : 0 \to K(i) \to Q(i) \xrightarrow{\beta_i} \Theta(i) \to 0,$$

with $K(i) \in \mathcal{F}(\{\Theta(j) \colon\ j > i\})$.
(c) $\underline{Q} \subseteq {}^{\perp_1}\Theta$, that is, $\operatorname{Ext}^1_\Lambda(Q(i), -)|_\Theta = 0$ for any $i \in [1, n]$.

Lemma 2.12. *Let $(\Theta, \underline{Q}, \leqslant)$ be an Ext-projective stratifying system in* $\operatorname{mod}(\Lambda)$ *of size t. Then, for each $M \in \mathcal{F}(\{\Theta(j) \colon\ j \geqslant i\})$, there exists an exact sequence in $\mathcal{F}(\Theta)$*

$$0 \to N \to Q_0(M) \to M \to 0$$

such that $Q_0(M) \in \operatorname{add}(\bigoplus_{j \geqslant i} Q(j))$ and $N \in \mathcal{F}(\{\Theta(j) \colon\ j > i\})$. Moreover, for $Q = \bigoplus_{i=1}^t Q(i)$, $\mathcal{F}(\Theta) \subseteq C_m^Q$ for all $m \geqslant 1$.

Proof. See in [MMS, Proposition 2.10]. The last statement of the lemma is proved using Definition 2.11(c). □

For the following lemma, we consider a set $\{M_1, \dots, M_n\}$ of indecomposable Λ-modules which are pairwise not isomorphic, $M = \bigoplus_{i=1}^{t} M_i$, $\Gamma = \operatorname{End}_\Lambda(M)^{op}$ and $F = \operatorname{Hom}_\Lambda(M, -) : \operatorname{mod}(\Lambda) \to \operatorname{mod}(\Gamma)$.

Lemma 2.13. *Let $M' \xrightarrow{\alpha} M_i \xrightarrow{\beta} X \to 0$ be an exact sequence in $\operatorname{mod}(\Lambda)$, with $J \subseteq [1, t]$, $M' \in \operatorname{add}(\bigoplus_{j\in J} M_j)$ and $\beta \neq 0$, such that the induced sequence*

$$F(M') \xrightarrow{F(\alpha)} F(M_i) \xrightarrow{F(\beta)} F(X) \to 0$$

is exact in $\operatorname{mod}(\Gamma)$. Then the following statements hold.

(a) $\operatorname{Im}(F(\alpha)) \subseteq \operatorname{Tr}_{\bigoplus_{j\in J} F(M_j)}(\operatorname{rad} F(M_i))$.
(b) *If $\operatorname{Hom}_\Lambda(M_j, X) = 0$ for all $j \in J$, then*

$$\operatorname{Im}(F(\alpha)) = \operatorname{Tr}_{\bigoplus_{j\in J} F(M_j)}(F(M_i)).$$

Proof. The proof is a straightforward consequence of Theorem 2.1. □

We are now in a position to give a different proof of the following known result ([ES,MMS]; see also [W] for related results).

Theorem 2.14. *Let $(\Theta, \underline{Q}, \leqslant)$ be an Ext-projective stratifying system of size t in $\operatorname{mod}(\Lambda)$, $Q = \bigoplus_{i=1}^{t} Q(i)$, $\Gamma = \operatorname{End}_\Lambda(Q)^{op}$, $F = \operatorname{Hom}_\Lambda(Q, -) : \operatorname{mod}(\Lambda) \to \operatorname{mod}(\Gamma)$ and $G = Q \otimes_\Gamma - : \operatorname{mod}(\Gamma) \to \operatorname{mod}(\Lambda)$. Then, the following statements hold.*

(a) *The family ${}_\Gamma P = \{F(Q(i)) \colon i \in [1, t]\}$ is a representative set of the indecomposable projective Γ-modules. In particular, Γ is a basic algebra and $\operatorname{rk} K_0(\Gamma) = t$.*
(b) *$(\Gamma, \leqslant)$ is a standardly stratified algebra, that is, $\operatorname{proj}(\Gamma) \subseteq \mathcal{F}({}_\Gamma\Delta)$.*
(c) *The restriction $F|_{\mathcal{F}(\Theta)} : \mathcal{F}(\Theta) \to \mathcal{F}({}_\Gamma\Delta)$ is an exact equivalence of categories and $G|_{\mathcal{F}({}_\Gamma\Delta)} : \mathcal{F}({}_\Gamma\Delta) \to \mathcal{F}(\Theta)$ is a quasi-inverse of $F|_{\mathcal{F}(\Theta)}$.*
(d) *$F(\Theta(i)) \simeq {}_\Gamma\Delta(i)$, for all $i \in [1, t]$.*
(e) *$\operatorname{add}(Q) = \mathcal{F}(\Theta) \cap {}^{\perp_1}\mathcal{F}(\Theta)$.*

Proof. (a) follows from the fact that $Q = \{Q(i)\}_{i=1}^{t}$ is a family of indecomposable and pairwise not isomorphic Λ-modules (see [ARS, II Proposition 2.1]).

On the other hand, by Lemma 2.12 we know that $\mathcal{F}(\Theta) \subseteq C_2^Q$, so the hypotheses of Theorem 2.10 are satisfied for $\mathcal{C} = \Theta$ and $M = Q$. Furthermore, the same lemma implies, for each $i \in [1, t]$, the existence of a presentation

$$Q' \xrightarrow{\alpha_i} Q(i) \xrightarrow{\beta_i} \Theta(i) \to 0$$

with $Q' \in \operatorname{add}(\bigoplus_{j>i} Q(j))$ and such that the induced sequence

$$F(Q') \xrightarrow{F(\alpha_i)} F(Q(i)) \xrightarrow{F(\beta_i)} F(\Theta(i)) \to 0$$

is exact in $\operatorname{mod}(\Gamma)$. Since $\operatorname{Hom}_\Lambda(Q(j), \Theta(i)) = 0$ for all $j > i$ (see [MMS, Lemma 2.6(b)]), we conclude from Lemma 2.13(b) that

$$\operatorname{Im}(F(\alpha_i)) = \operatorname{Tr}_{\bigoplus_{j>i} F(Q(j))} F(Q(i)).$$

But, according with (a), the standard Γ-modules are the factors ${}_\Gamma\Delta(i) = F(Q(i))/\mathrm{Tr}_{\bigoplus_{j>i} F(Q(j))} F(Q(i))$. Hence ${}_\Gamma\Delta(i) = F(Q(i))/\mathrm{Im}(F(\alpha_i)) \simeq F(\Theta(i))$ for all $i \in [1,t]$. Items (c) and (d) follow now from Theorem 2.10(a).

On the other hand, since $\mathrm{add}(Q) \subseteq \mathcal{F}(\Theta)$ and $F(\mathcal{F}(\Theta)) = \mathcal{F}({}_\Gamma\Delta)$ is closed under kernels of epimorphisms (see [DR,Xi]), we can apply Theorem 2.10 and obtain that (b) and (e) hold. □

3. Proper costratifying systems

In this section we introduce the notion of a proper costratifying system $(\Psi, \mathbf{Q}, \leqslant)$ and illustrate it with some examples. We also show that the notions of Ψ-length and Ψ-multiplicity are well defined.

Definition 3.1. Let Λ be an artin R-algebra. A proper costratifying system $(\Psi, \mathbf{Q}, \leqslant)$, of size t in $\mathrm{mod}(\Lambda)$, consists of two families of Λ-modules $\Psi = \{\Psi(i)\}_{i=1}^t$ and $\mathbf{Q} = \{Q(i)\}_{i=1}^t$, with $Q(i)$ indecomposable for all i, and a linear order $\leqslant$ on the set $[1,t]$, satisfying the following conditions.

(a) $\mathrm{End}_\Lambda(\Psi(i))$ is a division ring for all $i \in [1,t]$.
(b) $\mathrm{Hom}_\Lambda(\Psi(i), \Psi(j)) = 0$ if $i < j$.
(c) For each $i \in [1,t]$, there is an exact sequence

$$\varepsilon_i : 0 \to Z(i) \to Q(i) \xrightarrow{\beta_i} \Psi(i) \to 0,$$

with $Z(i) \in \mathcal{F}(\{\Psi(j)\colon\ j \leqslant i\})$.
(d) $\mathbf{Q} \subseteq {}^{\perp_1}\Psi$, that is, $\mathrm{Ext}^1_\Lambda(Q(i), -)|_\Psi = 0$ for any $i \in [1,n]$.

We will denote by Q the Λ-module $\bigoplus_{i=1}^t Q(i)$.

The notion of a *proper stratifying system* is defined dually.

Remark 3.2. Let Λ be an artin R-algebra and $(\Psi, \mathbf{Q}, \leqslant)$ be a proper costratifying system of size t in $\mathrm{mod}(\Lambda)$. Then:

(a) For any $i \in [1,t]$, the map $\beta_i : Q(i) \to \Psi(i)$ is a right-minimal $\mathrm{add}\,Q$-approximation of $\Psi(i)$. Indeed, this follows from the fact that $Q(i)$ is indecomposable and $\mathbf{Q} \subseteq {}^{\perp_1}\Psi = {}^{\perp_1}\mathcal{F}(\Psi)$.
(b) Let $(\Psi', \mathbf{Q}', \leqslant)$ be another proper costratifying system of size t in $\mathrm{mod}(\Lambda)$. If $\Psi(i) \simeq \Psi'(i)$ for all $i \in [1,t]$, then there is an exact and commutative diagram in $\mathcal{F}(\Psi)$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Z(i) & \longrightarrow & Q(i) & \xrightarrow{\beta_i} & \Psi(i) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Z'(i) & \longrightarrow & Q'(i) & \xrightarrow{\beta'_i} & \Psi'(i) & \longrightarrow & 0, \end{array}$$

where the vertical arrows are isomorphisms. This statement follows from the item (a), since $\mathcal{F}(\Psi) = \mathcal{F}(\Psi')$.

Example 3.3. Let $(\Theta, \underline{Q}, \leqslant)$ be an Ext-projective stratifying system of size t in $\mathrm{mod}(\Lambda)$. If $\mathrm{End}_\Lambda(\Theta(i))$ is a division ring for all $i \in [1,t]$, then $(\Psi = \Theta, \mathbf{Q} = \underline{Q}, \leqslant^{op})$ is a proper costratifying system of size t.

Example 3.4. Let $(\Psi, \mathbf{Q}, \leqslant)$ be a proper costratifying system of size t in $\mathrm{mod}(\Lambda)$. For $i \in [1,t]$, consider the families of Λ-modules $\Psi_i = \{\Psi(j)\colon\ j \leqslant i\}$ and $\mathbf{Q}_i = \{Q(j)\colon\ j \leqslant i\}$. Then, $(\Psi_i, \mathbf{Q}_i, \leqslant)$ is a proper costratifying system in $\mathrm{mod}(\Lambda)$, with size less or equal than t.

Example 3.5. Let $(\Lambda, \leqslant)$ be a standardly stratified algebra and $T = \bigoplus_{i=1}^{n} T(i)$ be the characteristic tilting module. We consider $\mathbf{Q} = \{T(1), \dots, T(n)\}$ and $\Psi = {}_{\Lambda}\overline{\nabla}$ the proper costandard modules. Then, by [AHLU, Lemma 1.2(iii), Theorem 2.1 and Lemma 2.5(iii)], it follows that $(\Psi, \mathbf{Q}, \leqslant)$ is a proper costratifying system of size n in $\mathrm{mod}(\Lambda)$. We say that $({}_{\Lambda}\overline{\nabla}, \{T(i)\}_{i=1}^{n}, \leqslant)$ is the canonical proper costratifying system associated to the standardly stratified algebra $(\Lambda, \leqslant)$.

Example 3.6. The following is an example of a proper costratifying system $(\Psi, \mathbf{Q}, \leqslant)$ such that $\Psi \neq {}_{\Lambda}\overline{\nabla}$ and $(\Lambda, \leqslant)$ is a standardly stratified algebra. Let Λ be the path algebra of the quiver

$$\underset{1}{\circ} \rightarrow \underset{2}{\circ} \rightarrow \underset{3}{\circ}.$$

Consider the natural order on $\{1, 2, 3\}$. The proper costandard Λ-modules can be described as follows

$${}_{\Lambda}\overline{\nabla}(1) = 1, \qquad {}_{\Lambda}\overline{\nabla}(2) = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}, \qquad {}_{\Lambda}\overline{\nabla}(3) = \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}.$$

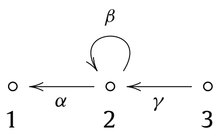
Now, consider $\Psi = \{\Psi(1) = 3,\ \Psi(2) = 1,\ \Psi(3) = \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\}$ and

$$\mathbf{Q} = \left\{ Q(1) = 3,\ Q(2) = 1,\ Q(3) = \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \right\}.$$

Then $(\Psi, \mathbf{Q}, \leqslant)$ is a proper costratifying system of size 3 in $\mathrm{mod}(\Lambda)$, which is not the canonical one.

Example 3.7. The following is an example of a proper costratifying system $(\Psi, \mathbf{Q}, \leqslant)$ such that $\Psi \neq {}_{\Lambda}\overline{\nabla}$ and $(\Lambda, \leqslant)$ is not a standardly stratified algebra.

Let Λ be given by the quiver



with the relations $\beta^2 = 0$, $\beta\alpha = 0$ and $\gamma\beta = 0$. Consider the natural order $\leqslant$ on $\{1, 2, 3\}$, and the sets

$$\Psi = \left\{ \Psi(1) = 2,\ \Psi(2) = \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix},\ \Psi(3) = \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \right\}$$

and

$$\mathbf{Q} = \left\{ Q(1) = \begin{smallmatrix} 2 \\ 2 \end{smallmatrix},\ Q(2) = \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix},\ Q(3) = \begin{smallmatrix} & 2 & \\ 1 & & 2 \end{smallmatrix} \right\}.$$

Then $(\Psi, \mathbf{Q}, \leqslant)$ is a proper costratifying system of size 3 in $\mathrm{mod}(\Lambda)$, and $\Psi \neq {}_{\Lambda}\overline{\nabla}$.

Lemma 3.8. *Let $(\Psi, \mathbf{Q}, \leqslant)$ be a proper costratifying system of size t in* $\mathrm{mod}(\Lambda)$*. If $i < j$ then*

$$\mathrm{Hom}_{\Lambda}\big(Q(i), \Psi(j)\big) = 0 = \mathrm{Hom}_{\Lambda}\big(Z(i), \Psi(j)\big) \quad \textit{and} \quad \mathrm{Ext}^{1}_{\Lambda}\big(\Psi(i), \Psi(j)\big) = 0.$$

Proof. Let $i < j$. By Definition 3.1(c), there is an exact sequence in $\mathcal{F}(\Psi)$

$$\varepsilon_i : 0 \to Z(i) \to Q(i) \to \Psi(i) \to 0.$$

Applying the functor $\mathrm{Hom}_\Lambda(-, \Psi(j))$ to ε_i, we get the exact sequence

$$0 \to \mathrm{Hom}_\Lambda\big(\Psi(i), \Psi(j)\big) \to \mathrm{Hom}_\Lambda\big(Q(i), \Psi(j)\big) \to \mathrm{Hom}_\Lambda\big(Z(i), \Psi(j)\big) \to \mathrm{Ext}^1_\Lambda\big(\Psi(i), \Psi(j)\big) \to 0.$$

We know that $Z(i) \in \mathcal{F}(\{\Psi(\lambda)\colon \lambda \leqslant i\})$ and, since $\lambda \leqslant i < j$, $\mathrm{Hom}_\Lambda(\Psi(\lambda), \Psi(j)) = 0$ (see Definition 3.1(b), (c)). Then, it is easy to see that $\mathrm{Hom}_\Lambda(Z(i), \Psi(j)) = 0$. Finally, the lemma follows from the last sequence. □

K. Erdmann and C. Saenz proved in [ES] that the filtration multiplicity $[M : \Theta(i)]$ of $\Theta(i)$ in a Θ-filtered Λ-module M is well defined, for the relative simple module $\Theta(i)$ associated to a stratifying system $(\Theta, \leqslant)$. The same result holds for the relative simple module $\Psi(i)$ of a proper costratifying system, as we state in the following lemma.

Lemma 3.9. *Let $(\Psi, \mathbf{Q}, \leqslant)$ be a proper costratifying system of size t in* $\mathrm{mod}(\Lambda)$. *Then the following statements hold.*

(a) *For any $M \in \mathcal{F}(\Psi)$, the filtration multiplicity $[M : \Psi(i)]_\xi$ of $\Psi(i)$ in M does not depend on the given Ψ-filtration ξ of M.*
(b) *$Q(i) \not\simeq Q(j)$ if $i \neq j$.*

Proof. (a) The proof is dual to the one given in [ES, Lemma 1.4] (see also [MMS, Lemma 2.6]), which can be adapted by using Lemma 3.8 and length instead of dimension.

(b) Suppose that $Q(i) \simeq Q(j)$ and $i < j$. By (a) and Definition 3.1, we know that $[Q(i) : \Psi(j)] = 0$ and $[Q(j) : \Psi(j)] > 0$, contradicting our first assumption. □

Given a proper costratifying system $(\Psi, \mathbf{Q}, \leqslant)$ of size t in $\mathrm{mod}(\Lambda)$, the above lemma shows that the filtration multiplicity is well defined. Thus we can define the function *Ψ-length* $\ell_\Psi : \mathcal{F}(\Psi) \to \mathbb{N}$ as follows, $\ell_\Psi(M) = \sum_{i=1}^t [M : \Psi(i)]$. It can be seen that the Ψ-length is an additive function, that is, for any exact sequence $0 \to N \to E \to M \to 0$ in $\mathcal{F}(\Psi)$, we have that $\ell_\Psi(E) = \ell_\Psi(N) + \ell_\Psi(M)$.

Lemma 3.10. *Let $\Psi = \{\Psi(i)\}_{i=1}^t$ be a family of Λ-modules satisfying that $\mathrm{Ext}^1_\Lambda(\Psi(i), \Psi(j)) = 0$ for $i < j$. Then, for all $M \in \mathcal{F}(\Psi)$, any Ψ-filtration of M can be rearranged, with the same Ψ-composition factors, to get a Ψ-filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_s = M$ such that $M_i/M_{i-1} \simeq \Psi(k_i)$, with $k_1 \leqslant \cdots \leqslant k_s$.*

Proof. The proof is based on the following observation. Let $Z \subseteq Y \subseteq X$ be a chain of Λ-submodules such that $X/Y \simeq A$ and $Y/Z \simeq B$. If $\mathrm{Ext}^1_\Lambda(A, B) = 0$ then there exists a Λ-submodule W such that $Z \subseteq W \subseteq X$ with $X/W \simeq B$ and $W/Z \simeq A$. □

The following result is the straightforward generalization of [AHLU, Lemma 1.7] to the context of proper costratifying systems $(\Psi, \mathbf{Q}, \leqslant)$. This lemma shows, in particular, that the $\Psi(i)$'s behave in some sense as simple objects in $\mathcal{F}(\Psi)$, since non-zero morphisms into them are surjective, and it is fundamental in all that follows.

Lemma 3.11. *Let $(\Psi, \mathbf{Q}, \leqslant)$ be a proper costratifying system of size t in* $\mathrm{mod}(\Lambda)$, *$X \in \mathcal{F}(\{\Psi(j)\colon j \leqslant i\})$ and $f \in \mathrm{Hom}_\Lambda(X, \Psi(i))$. If $f \neq 0$ then f is surjective and* $\mathrm{Ker}(f) \in \mathcal{F}(\{\Psi(j)\colon j \leqslant i\})$.

Proof. The proof in [AHLU] can be adapted directly, by using that the Ψ-filtration multiplicity is well defined (see Lemma 3.9), and Lemmas 3.8 and 3.10. □

Corollary 3.12. *Let $(\Psi, \mathbf{Q}, \leqslant)$ be a proper costratifying system of size t in $\mathrm{mod}(\Lambda)$. Then any non-zero map $f \in \mathrm{Hom}_\Lambda(Q(i), \Psi(i))$ is a right minimal $\mathrm{add}(Q)$-approximation of $\Psi(i)$.*

Proof. Let $0 \neq f \in \mathrm{Hom}_\Lambda(Q(i), \Psi(i))$. Then, by Lemma 3.11, we have that $0 \to \mathrm{Ker}(f) \to Q(i) \xrightarrow{f} \Psi(i) \to 0$ is an exact sequence in $\mathcal{F}(\{\Psi(j)\colon\ j \leqslant i\})$. Furthermore, since $\mathrm{Ext}^1_\Lambda(Q, \mathrm{Ker}(f)) = 0$ and $Q(i)$ is indecomposable, it follows that f is a right minimal $\mathrm{add}(Q)$-approximation of $\Psi(i)$. □

4. The standardly stratified algebra associated to a proper costratifying system

In this section we prove, for a proper costratifying system $(\Psi, \mathbf{Q}, \leqslant)$, that the pair $(\mathrm{End}_\Lambda(Q), \leqslant^{op})$ is a standardly stratified algebra. Moreover, the category of modules filtered by Ψ is dual to the category of modules filtered by the proper costandard modules over $\mathrm{End}_\Lambda(Q)$. Finally, we show that $\mathcal{F}(\Psi)$ is coresolving precisely when Ψ coincides with the costandard modules of a standardly stratified algebra.

The following proposition is important for our considerations, because it will allow us to apply the results in Section 2.

Proposition 4.1. *Let $(\Psi, \mathbf{Q}, \leqslant)$ be a proper costratifying system of size t in $\mathrm{mod}(\Lambda)$. Then, for each $M \in \mathcal{F}(\{\Psi(j)\colon\ j \leqslant i\})$, there exists an exact sequence $0 \to M' \to Q' \to M \to 0$ in $\mathcal{F}(\{\Psi(j)\colon\ j \leqslant i\})$ with $Q' \in \mathrm{add}(\bigoplus_{j \leqslant i} Q(j))$. In particular $\mathcal{F}(\Psi) \subseteq C^Q_m$ for all $m \geqslant 1$.*

Proof. Let $M \in \mathcal{F}(\{\Psi(j)\colon\ j \leqslant i\})$. We proceed by induction on $\ell_\Psi(M)$. For $\ell_\Psi(M) = 1$, we get the sequence from Definition 3.1(c).

Let $\ell_\Psi(M) > 1$. Then, there is an exact sequence in $\mathcal{F}(\{\Psi(j)\colon\ j \leqslant i\})$

$$0 \to \Psi(i_1) \xrightarrow{\alpha} M \xrightarrow{\gamma} M_1 \to 0,$$

with $\ell_\Psi(M_1) < \ell_\Psi(M)$. Hence, by induction, there exists an exact sequence in $\mathcal{F}(\{\Psi(j)\colon\ j \leqslant i\})$

$$0 \to M'_1 \to Q'_1 \xrightarrow{\beta} M_1 \to 0$$

with $Q'_1 \in \mathrm{add}(\bigoplus_{j \leqslant i} Q(j))$. Since $Q \in {}^{\perp_1}\mathcal{F}(\Psi)$, there is a morphism $\overline{\beta}: Q'_1 \to M$ such that $\beta = \gamma\overline{\beta}$. Hence we get an exact and commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z(i_1) & \longrightarrow & X_2 & \longrightarrow & M'_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q(i_1) & \xrightarrow{\binom{1}{0}} & Q(i_1) \oplus Q'_1 & \xrightarrow{(0,1)} & Q'_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \beta_{i_1}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \beta} & & \\
0 & \longrightarrow & \Psi(i_1) & \xrightarrow[\alpha]{} & M & \xrightarrow[\gamma]{} & M_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $f = (\alpha\beta_{i_1}, \overline{\beta})$ and $X_2 = \mathrm{Ker}(f)$. Then $X_2 \in \mathcal{F}(\{\Psi(j)\colon\ j \leqslant i\})$ and the middle vertical sequence is the desired one. $\square$

Corollary 4.2. *Let* $(\Psi, \mathbf{Q}, \leqslant)$ *be a proper costratifying system of size* t *in* $\mathrm{mod}(\Lambda)$, $\Gamma = \mathrm{End}_\Lambda(Q)^{op}$, $F = \mathrm{Hom}_\Lambda(Q, -) : \mathrm{mod}(\Lambda) \to \mathrm{mod}(\Gamma)$ *and* $G = Q \otimes_\Gamma - : \mathrm{mod}(\Gamma) \to \mathrm{mod}(\Lambda)$. *Then, the following statements hold.*

(a) *The restriction* $F|_{\mathcal{F}(\Psi)} : \mathcal{F}(\Psi) \to \mathcal{F}(F(\Psi))$ *is an exact equivalence of categories and* $G|_{\mathcal{F}(F(\Psi))} : \mathcal{F}(F(\Psi)) \to \mathcal{F}(\Psi)$ *is a quasi-inverse of* $F|_{\mathcal{F}(\Psi)}$.
(b) *If* $\mathcal{F}(F(\Psi))$ *is closed under kernels of epimorphisms, then*

$$\mathrm{add}(Q) = \mathcal{F}(\Psi) \cap {}^{\perp_1}\mathcal{F}(\Psi).$$

Proof. By Proposition 4.1, we know that $\mathcal{F}(\Psi) \subseteq C_2^Q$. On the other hand, since $Q(i)$ is indecomposable for each i and $\mathcal{F}(\Psi)$ is closed under extensions, it follows that $\mathrm{add}(Q) \subseteq \mathcal{F}(\Psi)$. Hence, the hypotheses of Theorem 2.10 are satisfied for $\mathcal{C} = \Psi$ and $M = Q$, and so the result follows. $\square$

We will prove that the family $\{F(\Psi(i))\}_{i=1}^t$ coincides with the family of proper standard modules over Γ. This fact and the previous corollary will lead us to the main result of this section, which we state in the following theorem.

Theorem 4.3. *Let* $(\Psi, \mathbf{Q}, \leqslant)$ *be a proper costratifying system of size* t *in* $\mathrm{mod}(\Lambda)$, $\Gamma = \mathrm{End}_\Lambda(Q)^{op}$, $F = \mathrm{Hom}_\Lambda(Q, -) : \mathrm{mod}(\Lambda) \to \mathrm{mod}(\Gamma)$ *and* $G = Q \otimes_\Gamma - : \mathrm{mod}(\Gamma) \to \mathrm{mod}(\Lambda)$. *Let* ${}_\Gamma\overline{\Delta} = \{{}_\Gamma\overline{\Delta}(i)\colon\ i \in [1, t]\}$ *be the proper standard modules corresponding to the pair* $({}_\Gamma P, \leqslant^{op})$, *where* $\leqslant^{op}$ *is the opposite order of* $\leqslant$. *Then, the following statements hold.*

(a) *The family* ${}_\Gamma P = \{F(Q(i))\colon\ i \in [1, t]\}$ *is a representative set of the indecomposable projective* Γ*-modules. In particular* Γ *is a basic algebra and* $\mathrm{rk}\, K_0(\Gamma) = t$.
(b) *The restriction* $F|_{\mathcal{F}(\Psi)} : \mathcal{F}(\Psi) \to \mathcal{F}({}_\Gamma\overline{\Delta})$ *is an exact equivalence of categories and* $G|_{\mathcal{F}({}_\Gamma\overline{\Delta})} : \mathcal{F}({}_\Gamma\overline{\Delta}) \to \mathcal{F}(\Psi)$ *is a quasi-inverse of* $F|_{\mathcal{F}(\Psi)}$.
(c) $F(\Psi(i)) \simeq {}_\Gamma\overline{\Delta}(i)$, *for all* $i \in [1, t]$.
(d) $(\Gamma^{op}, \leqslant^{op})$ *is a standardly stratified algebra.*
(e) $\mathrm{add}(Q) = \mathcal{F}(\Psi) \cap {}^{\perp_1}\mathcal{F}(\Psi)$.
(f) $\mathcal{F}({}_\Gamma\overline{\Delta})$ *is resolving and closed under direct summands in* $\mathrm{mod}(\Gamma)$.

Proof. It is well known that the functor $F : \mathrm{mod}(\Lambda) \to \mathrm{mod}(\Gamma)$ induces, by restriction, an equivalence from $\mathrm{add}(Q)$ to $\mathrm{proj}(\Gamma)$ (see [ARS, II Proposition 2.1]).

(a) Since $Q = \{Q(i)\}_{i=1}^t$ is a family of indecomposable and pairwise not isomorphic Λ-modules (see Lemma 3.9(b)), we get (a) from the above observation.

(b) and (c): From Corollary 4.2, we know that the restriction $F|_{\mathcal{F}(\Psi)} : \mathcal{F}(\Psi) \to \mathcal{F}(F(\Psi))$ is an exact equivalence of categories and $G|_{\mathcal{F}(F(\Psi))} : \mathcal{F}(F(\Psi)) \to \mathcal{F}(\Psi)$ is a quasi-inverse of $F|_{\mathcal{F}(\Psi)}$. So, to get (b) and (c), it is enough to prove that

$$F\big(\Psi(i)\big) \simeq {}_\Gamma\overline{\Delta}(i) = {}_\Gamma P(i)/\mathrm{Tr}_{\bigoplus_{j \geqslant^{op} i} {}_\Gamma P(j)}\big(\mathrm{rad}\, {}_\Gamma P(i)\big), \quad \text{for all } i \in [1, t].$$

Let $i \in [1, t]$. Then, from Definition 3.1(c), we have an exact sequence

$$0 \to Z(i) \xrightarrow{\alpha_i} Q(i) \xrightarrow{\beta_i} \Psi(i) \to 0,$$

where $Z(i) \in \mathcal{F}(\{\Psi(j)\colon\ j \leqslant i\})$. Hence, by Proposition 4.1, we get an exact sequence

$$0 \to \operatorname{Ker}(t) \to Q' \xrightarrow{t} Z(i) \to 0$$

in $\mathcal{F}(\{\Psi(j)\colon\ j \leqslant i\})$, where $Q' \in \operatorname{add}(\bigoplus_{j\leqslant i} Q(j))$. Therefore, since F is exact on $\mathcal{F}(\Psi)$, we have a presentation $Q' \xrightarrow{\alpha_i t} Q(i) \xrightarrow{\beta_i} \Psi(i) \to 0$ such that $F(Q') \xrightarrow{F(\alpha_i t)} F(Q(i)) \xrightarrow{F(\beta_i)} F(\Psi(i)) \to 0$ is exact in $\operatorname{mod}(\Gamma)$. So, applying Lemma 2.13(a), it follows that

$$\operatorname{Im}\bigl(F(\alpha_i)\bigr) = \operatorname{Im}\bigl(F(\alpha_i t)\bigr) \subseteq \operatorname{Tr}_{\bigoplus_{j\leqslant i} F(Q(j))}\bigl(\operatorname{rad} F\bigl(Q(i)\bigr)\bigr).$$

So, in order to prove that $F(\Psi(i)) \simeq {}_{\Gamma}\overline{\Delta}(i)$, it is enough to show the inclusion $\operatorname{Tr}_{\bigoplus_{j\leqslant i} F(Q(j))}(\operatorname{rad} F(Q(i))) \subseteq \operatorname{Im}(F(\alpha_i))$. To prove such inclusion, we assume that $j \leqslant i$ and consider a morphism $\delta : F(Q(j)) \to \operatorname{rad} F(Q(i))$. Let $\iota : \operatorname{rad} F(Q(i)) \to F(Q(i))$ be the inclusion map, which is not an isomorphism since $F(Q(i)) \in \operatorname{proj}(\Gamma)$ is indecomposable (see (a)). Furthermore, from the equivalence $F|_{\operatorname{add}(Q)} : \operatorname{add}(Q) \to \operatorname{proj}(\Gamma)$, there is a morphism $\eta : Q(j) \to Q(i)$ such that $\iota\delta = F(\eta)$. Hence $\operatorname{Im}(\delta) \subseteq \operatorname{Im}(F(\eta))$. We assert that $\operatorname{Im}(F(\eta)) \subseteq \operatorname{Im}(F(\alpha_i))$ and, from this, it follows that $\operatorname{Im}(\delta) \subseteq \operatorname{Im}(F(\alpha_i))$, proving that $\operatorname{Tr}_{\bigoplus_{j\leqslant i} F(Q(j))}(\operatorname{rad} F(Q(i))) \subseteq \operatorname{Im}(F(\alpha_i))$. So, to prove that $\operatorname{Im}(F(\eta)) \subseteq \operatorname{Im}(F(\alpha_i))$, we need to show that $F(\beta_i)F(\eta) = 0$ since we have the following exact sequence

$$0 \to \operatorname{Im}\bigl(F(\alpha_i)\bigr) \to F\bigl(Q(i)\bigr) \xrightarrow{F(\beta_i)} F\bigl(\Psi(i)\bigr) \to 0.$$

Thus, we only need to prove that the composition $Q(j) \xrightarrow{\eta} Q(i) \xrightarrow{\beta_i} \Psi(i)$ is zero. If $j < i$ this is true since, by Lemma 3.8, we know that $\operatorname{Hom}_{\Lambda}(Q(j), \Psi(i)) = 0$.

Let $i = j$ and suppose that $\beta_i\eta \neq 0$. By Corollary 3.12, we know that $\beta_i\eta : Q(i) \to \Psi(i)$ and $\beta_i : Q(i) \to \Psi(i)$ are both minimal right $\operatorname{add}(Q)$-approximations of $\Psi(i)$. Thus, from the commutative diagram

$$\begin{array}{ccc} Q(i) & \xrightarrow{\beta_i\eta} & \Psi(i) \\ \downarrow{\scriptstyle \eta} & & \| \\ Q(i) & \xrightarrow[\beta_i]{} & \Psi(i) \end{array}$$

we get that η is an isomorphism. Therefore $F(\eta) = \iota\delta$ is also an isomorphism, contradicting that the inclusion map $\iota : \operatorname{rad} F(Q(i)) \to F(Q(i))$ is not an isomorphism, and therefore $\beta_i\eta = 0$ as desired.

(d) The fact that $(\Gamma^{op}, \leqslant^{op})$ is a standardly stratified algebra is equivalent to the condition ${}_{\Gamma}\Gamma \in \mathcal{F}({}_{\Gamma}\overline{\Delta})$ (see [D1, 2.2], [ADL, 2.2] or [L]). It is easy to check the last claim. In fact, $Q \in \mathcal{F}(\Psi)$ and so $\Gamma_{\Gamma} \simeq F(Q) \in \mathcal{F}({}_{\Gamma}\overline{\Delta})$.

(e) and (f): Since $(\Gamma^{op}, \leqslant^{op})$ is a standardly stratified algebra (see (d)), it follows from [AHLU, Theorem 1.6(ii)] that $\mathcal{F}({}_{\Gamma^{op}}\overline{\nabla})$ is coresolving. We get by duality that $\mathcal{F}({}_{\Gamma}\overline{\Delta})$ is resolving. On the other hand, from (b), we know that $\mathcal{F}(F(\Psi)) = \mathcal{F}({}_{\Gamma}\overline{\Delta})$. Hence, (e) follows from Corollary 4.2. Finally, we prove that $\mathcal{F}({}_{\Gamma}\overline{\Delta})$ is closed under direct summands in $\operatorname{mod}(\Gamma)$. Indeed, we have that $D_{\Gamma}(\mathcal{F}({}_{\Gamma}\overline{\Delta})) = \mathcal{F}({}_{\Gamma^{op}}\overline{\nabla}) = \mathcal{F}({}_{\Gamma^{op}}\Delta)^{\perp_1}$ (the last equality follows from [AHLU, Theorem 1.6(iv)]), and so the result follows observing that $\mathcal{F}({}_{\Gamma^{op}}\Delta)^{\perp_1}$ is closed under direct summands in $\operatorname{mod}(\Gamma^{op})$. □

Remark 4.4. We recall that an algebra Λ is properly stratified if and only if ${}_{\Lambda}\Lambda \in \mathcal{F}({}_{\Lambda}\Delta) \cap \mathcal{F}({}_{\Lambda}\overline{\Delta})$ (see [D2]). In this case, the standard modules provide a stratifying system, and the proper standard modules a proper stratifying system.

Example 4.5. Let $({}_{\Lambda}\overline{\nabla}, \{T(i)\}_{i=1}^{n}, \leqslant)$ be the canonical proper costratifying system associated to the standardly stratified algebra $(\Lambda, \leqslant)$ (see Example 3.5). Then, by Theorem 4.3(d), $\Gamma^{op} = \operatorname{End}_{\Lambda}(T)$ is the 'Ringel dual' of Λ.

Example 4.6. Let $(\Psi, \mathbf{Q}, \leqslant)$ be the proper costratifying system considered in Example 3.6. In this case, the algebra $\Gamma^{op} = \mathrm{End}_\Lambda(Q)$ is given by the quiver

$$\underset{1}{\circ} \xrightarrow{\varepsilon} \underset{3}{\circ} \xrightarrow{\mu} \underset{2}{\circ}$$

with the relation $\mu\varepsilon = 0$. Then

$$_{\Gamma^{op}}\Gamma^{op} = \begin{smallmatrix}1\\3\end{smallmatrix} \oplus 2 \oplus \begin{smallmatrix}3\\2\end{smallmatrix}.$$

We consider $(\Gamma^{op}, \leqslant^{op})$, where $3 \leqslant^{op} 2 \leqslant^{op} 1$. Then the corresponding standard modules are $_{\Gamma^{op}}\Delta = \{_{\Gamma^{op}}\Delta(1) = \begin{smallmatrix}1\\3\end{smallmatrix},\ _{\Gamma^{op}}\Delta(2) = 2,\ _{\Gamma^{op}}\Delta(3) = 3\}$. In this case, it is easy to check directly that $_{\Gamma^{op}}\Gamma^{op} \in \mathcal{F}(_{\Gamma^{op}}\Delta)$. That is, $(\Gamma^{op}, \leqslant^{op})$ is a standardly stratified algebra.

Proposition 4.7. *Let $(\Psi, \mathbf{Q}, \leqslant)$ be a proper costratifying system of size t in $\mathrm{mod}(\Lambda)$, $\Gamma = \mathrm{End}_\Lambda(Q)^{op}$ and $F = \mathrm{Hom}_\Lambda(Q, -) : \mathrm{mod}(\Lambda) \to \mathrm{mod}(\Gamma)$. If $X, Y \in \mathcal{F}(\Psi)$ then the map $\rho_{X,Y} : \mathrm{Ext}^1_\Lambda(X, Y) \to \mathrm{Ext}^1_\Gamma(F(X), F(Y))$, induced by F, is an isomorphism.*

Proof. The result is a direct consequence of Proposition 2.4 applied to $\mathcal{X} = \mathcal{F}(\Psi)$ and $M = Q$, since Proposition 4.1 shows that $\mathcal{F}(\Psi) \subseteq C^Q_m$ for any $m \geqslant 1$. □

Let $\mathcal{C}$ be a class of Λ-modules such that $\mathrm{add}(Q) = \mathcal{F}(\mathcal{C}) \cap {}^{\perp_1}\mathcal{F}(\mathcal{C})$ for some Λ-module Q. Let $M \in \mathcal{F}(\mathcal{C})$. We recall that a *$\mathcal{C}$-projective cover* of M, is a surjective morphism $f : Q_M \to M$ of Λ-modules such that $Q_M \in \mathrm{add}(Q)$, $\mathrm{Ker}(f) \in \mathcal{F}(\mathcal{C})$ and f is a right minimal $\mathrm{add}(Q)$-approximation of M.

Proposition 4.8. *Let $(\Psi, \mathbf{Q}, \leqslant)$ be a proper costratifying system of size t in $\mathrm{mod}(\Lambda)$. Then $\mathcal{F}(\Psi)$ is closed under direct summands in $\mathrm{mod}(\Lambda)$, and any object in $\mathcal{F}(\Psi)$ admits a Ψ-projective cover.*

Proof. Recall that we have the exact equivalence $F = \mathrm{Hom}_\Lambda(Q, -) : \mathcal{F}(\Psi) \to \mathcal{F}(_\Gamma\overline{\Delta})$, and also that $\mathcal{F}(_\Gamma\overline{\Delta})$ is closed under direct summands in $\mathrm{mod}(\Gamma)$ (see Theorem 4.3). We will carry this property back to $\mathcal{F}(\Psi)$. In fact, let $G : \mathcal{F}(_\Gamma\overline{\Delta}) \to \mathcal{F}(\Psi)$ be a quasi-inverse of F and $M \in \mathcal{F}(\Psi)$, and let $M = \bigoplus_{i=1}^n M_i$ and $F(M) = \bigoplus_{j=1}^m X_j$ with M_i and X_j indecomposable modules for all i, j. Since $\mathcal{F}(_\Gamma\overline{\Delta})$ is closed under direct summands, then X_j belongs to it.

We have $M \simeq GF(M) \simeq \bigoplus_{j=1}^m G(X_j)$. Since G is faithful and full, G preserves indecomposables. Therefore, it follows from Krull–Schmidt Theorem that $M_i \simeq G(X_{i_j})$ for some i_j, proving that $M_i \in \mathcal{F}(\Psi)$, as desired.

We prove next that $\mathcal{F}(\Psi)$ admits Ψ-projective covers. Indeed, by Proposition 4.1 we know the existence of an exact sequence in $\mathcal{F}(\Psi)$

$$0 \to M' \to Q' \xrightarrow{f'} M \to 0,$$

where f' is a right $\mathrm{add}(Q)$-approximation of M. Therefore, since $\mathcal{F}(\Psi)$ is closed under direct summands, we get that the right minimal version $f : Q_M \to M$ of f' is the desired Ψ-projective cover. □

The following proposition gives sufficient conditions for $F(D(\Lambda_\Lambda))$ to be a cotilting Γ-module.

Proposition 4.9. *Let $(\Psi, \mathbf{Q}, \leqslant)$ be a proper costratifying system of size t in $\mathrm{mod}(\Lambda)$, $\Gamma = \mathrm{End}_\Lambda(Q)^{op}$, $F = \mathrm{Hom}_\Lambda(Q, -) : \mathrm{mod}(\Lambda) \to \mathrm{mod}(\Gamma)$ and $T = F(D(\Lambda_\Lambda))$. Let $_\Gamma\overline{\Delta}$ be the family of proper standard modules. If $D(\Lambda_\Lambda) \in \mathcal{F}(\Psi)$ and $t = \mathrm{rk}\, K_0(\Lambda)$, then the following statements hold.*

(a) *T is a cotilting Γ-module.*
(b) $\mathcal{F}({}_{\Gamma}\overline{\Delta}) = {}^{\perp}T$ *and* $\mathcal{F}({}_{\Gamma}\overline{\Delta}) \cap \mathcal{F}({}_{\Gamma}\overline{\Delta})^{\perp_1} = \operatorname{add}(T)$.

Proof. Let $D(\Lambda_{\Lambda}) \in \mathcal{F}(\Psi)$ and $t = \operatorname{rk} K_0(\Lambda)$. Since $\operatorname{Ext}^1_{\Lambda}(-, D(\Lambda_{\Lambda})) = 0$, by Proposition 4.7, it follows that $\operatorname{Ext}^1_{\Gamma}(F(X), F(D(\Lambda_{\Lambda}))) = 0$ for any $X \in \mathcal{F}(\Psi)$. Hence $T = F(D(\Lambda_{\Lambda})) \in \mathcal{F}({}_{\Gamma}\overline{\Delta})^{\perp_1}$ and so $\operatorname{add}(T) \subseteq \mathcal{F}({}_{\Gamma}\overline{\Delta}) \cap \mathcal{F}({}_{\Gamma}\overline{\Delta})^{\perp_1}$. In addition, from the fact that $(\Gamma^{op}, \leqslant^{op})$ is a standardly stratified algebra (see Theorem 4.3), the duals of [AHLU, Theorem 2.1, Proposition 2.2(i)] show that there is a basic cotilting Γ-module T' such that $\mathcal{F}({}_{\Gamma}\overline{\Delta}) = {}^{\perp}T'$ and $\mathcal{F}({}_{\Gamma}\overline{\Delta}) \cap \mathcal{F}({}_{\Gamma}\overline{\Delta})^{\perp_1} = \operatorname{add}(T')$. Finally, since T' and T have the same number of indecomposable direct summands and $\operatorname{add}(T) \subseteq \operatorname{add}(T')$, we have $T' \simeq T$ and the proof is complete. □

We know from Theorem 4.3(e) that the Ext-projective modules in $\mathcal{F}(\Psi)$ coincide with $\operatorname{add}(Q)$. The next proposition describes the Ext-injectives in $\mathcal{F}(\Psi)$.

Proposition 4.10. *Let $(\Psi, \mathbf{Q}, \leqslant)$ be a proper costratifying system of size t in $\operatorname{mod}(\Lambda)$, $\Gamma = \operatorname{End}_{\Lambda}(Q)^{op}$, $F = \operatorname{Hom}_{\Lambda}(Q, -) : \operatorname{mod}(\Lambda) \to \operatorname{mod}(\Gamma)$ and $G = Q \otimes_{\Gamma} - : \operatorname{mod}(\Gamma) \to \operatorname{mod}(\Lambda)$. If ${}_{\Gamma^{op}}T$ is the characteristic tilting module associated to the standardly stratified algebra $(\Gamma^{op}, \leqslant^{op})$, then*

$$\mathcal{F}(\Psi) \cap \mathcal{F}(\Psi)^{\perp_1} = \operatorname{add}\big(GD({}_{\Gamma^{op}}T)\big).$$

Proof. By Proposition 4.7, we know that

$$X \in \mathcal{F}(\Psi) \cap \mathcal{F}(\Psi)^{\perp_1} \quad \Leftrightarrow \quad F(X) \in \mathcal{F}({}_{\Gamma}\overline{\Delta}) \cap \mathcal{F}({}_{\Gamma}\overline{\Delta})^{\perp_1}.$$

On the other hand, by using [AHLU, Theorem 1.6(iii), Proposition 2.2(i)], it follows that

$$D\big(\mathcal{F}({}_{\Gamma}\overline{\Delta}) \cap \mathcal{F}({}_{\Gamma}\overline{\Delta})^{\perp_1}\big) = \mathcal{F}({}_{\Gamma^{op}}\overline{\nabla}) \cap {}^{\perp_1}\mathcal{F}({}_{\Gamma^{op}}\overline{\nabla}) = \operatorname{add}({}_{\Gamma^{op}}T).$$

Thus, we have that $X \in \mathcal{F}(\Psi) \cap \mathcal{F}(\Psi)^{\perp_1}$ if and only if $X \in \operatorname{add}(GD({}_{\Gamma^{op}}T))$. □

We recall that a class $\mathcal{X}$, of objects in $\operatorname{mod}(\Lambda)$, is *coresolving* if it is closed under extensions, cokernels of monomorphisms and contains the injective Λ-modules [AR]. In what follows, we characterize the situation when a proper costratifying system is the canonical one.

Theorem 4.11. *Let Λ be a basic artin algebra and $(\Psi, \mathbf{Q}, \leqslant)$ be a proper costratifying system of size t in $\operatorname{mod}(\Lambda)$. Let $\Gamma = \operatorname{End}_{\Lambda}(Q)^{op}$ and ${}_{\Gamma^{op}}T$ be the characteristic tilting module associated to the standardly stratified algebra $(\Gamma^{op}, \leqslant^{op})$. Then, the following statements are equivalent.*

(a) *$\mathcal{F}(\Psi)$ is coresolving.*
(b) $\mathcal{F}(\Psi) \cap \mathcal{F}(\Psi)^{\perp_1} = \operatorname{add}(D(\Lambda_{\Lambda}))$.
(c) *$D(\Lambda_{\Lambda}) \in \mathcal{F}(\Psi)$ and $t = \operatorname{rk} K_0(\Lambda)$.*
(d) *$\Lambda \simeq \operatorname{End}({}_{\Gamma^{op}}Q)$ and ${}_{\Gamma^{op}}Q \simeq {}_{\Gamma^{op}}T$.*
(e) *$t = \operatorname{rk} K_0(\Lambda)$ and there is a choice of the representative set ${}_{\Lambda}P = \{{}_{\Lambda}P(i)\colon i \in [1, t]\}$ of indecomposable projective Λ-modules such that ${}_{\Lambda}\overline{\nabla}(i) \simeq \Psi(i)$ for all $i \in [1, t]$ and $(\Lambda, \leqslant)$ is a standardly stratified algebra.*
(f) *$D(\Lambda_{\Lambda}) \in \mathcal{F}(\Psi)$ and Q is a generalized tilting Λ-module.*

Proof. Consider the quasi-inverse functors $F : \mathcal{F}(\Psi) \to \mathcal{F}({}_{\Gamma}\overline{\Delta})$ and $G : \mathcal{F}({}_{\Gamma}\overline{\Delta}) \to \mathcal{F}(\Psi)$, given in Theorem 4.3. Then, from [CE, p. 120], we have $G = Q \otimes_{\Gamma} - \simeq D\operatorname{Hom}_{\Gamma}(-, D(Q))$.

The implication (a) ⇒ (b) follows from the dual of Lemma 2.8.

(b) $\Rightarrow$ (d) Let $\mathcal{F}(\Psi)\cap\mathcal{F}(\Psi)^{\perp_1}=\operatorname{add}(D(\Lambda_\Lambda))$. Then $D(\Lambda_\Lambda)\simeq GF(D(\Lambda_\Lambda))=G(\operatorname{Hom}_\Lambda(Q, D(\Lambda_\Lambda)))\simeq G(D(Q))$. In addition, by hypothesis and Proposition 4.10, we have $\operatorname{add}(D(\Lambda_\Lambda))=\operatorname{add}(GD({}_{\Gamma^{op}}T))$. Then, since Λ is basic, we get that $GD({}_{\Gamma^{op}}T)\simeq G(D(Q))$ and therefore ${}_{\Gamma^{op}}Q\simeq{}_{\Gamma^{op}}T$. Now, we prove that $\Lambda\simeq\operatorname{End}({}_{\Gamma^{op}}Q)$. Indeed, the isomorphisms $D(\Lambda_\Lambda)\simeq G(D(Q))\simeq D\operatorname{Hom}_\Gamma(D(Q),D(Q))\simeq D\operatorname{Hom}_{\Gamma^{op}}(Q,Q)$ show that $\Lambda\simeq\operatorname{End}({}_{\Gamma^{op}}Q)$.

(d) $\Rightarrow$ (e) Let $\Lambda\simeq\operatorname{End}({}_{\Gamma^{op}}Q)$ and ${}_{\Gamma^{op}}Q\simeq{}_{\Gamma^{op}}T$. In particular, since ${}_{\Gamma^{op}}T$ is basic, it follows that ${}_{\Gamma^{op}}Q$ is so, and therefore $t=\operatorname{rk}K_0(\Lambda)$. On the other hand, by Example 3.5, we know that $({}_{\Gamma^{op}}\overline{\nabla},\{{}_{\Gamma^{op}}T(i)\}_{i=1}^t,\leqslant^{op})$ is a proper costratifying system of size t in $\operatorname{mod}(\Gamma^{op})$. Hence, applying Theorem 4.3 to this system, we get an exact equivalence $\widetilde{F}=\operatorname{Hom}_{\Gamma^{op}}(T,-):\mathcal{F}({}_{\Gamma^{op}}\overline{\nabla})\to\mathcal{F}({}_A\overline{\Delta})$ such that $\widetilde{F}({}_{\Gamma^{op}}\overline{\nabla}(i))\simeq{}_A\overline{\Delta}(i)$ for all $i\in[1,t]$, with $A=\operatorname{End}({}_{\Gamma^{op}}T)^{op}$. The same theorem implies that $(A^{op},\leqslant)$ is a standardly stratified algebra and the ${}_A\overline{\Delta}(i)$'s correspond to the pair $({}_AP,\leqslant)$, where ${}_AP=\{{}_AP(i)=\widetilde{F}(T(i))\}_{i=1}^t$.

Since we are assuming that ${}_{\Gamma^{op}}Q\simeq{}_{\Gamma^{op}}T$, we get that their endomorphism rings are isomorphic. We will identify Λ and A^{op} through this isomorphism. Then ${}_\Lambda\overline{\nabla}=D({}_A\overline{\Delta})$, where the projective A-modules are $({}_AP(i))^*=\operatorname{Hom}_A({}_AP(i),A)$.

Finally, it remains to show that ${}_\Lambda\overline{\nabla}(i)\simeq\Psi(i)$ for all $i\in[1,t]$. Let $i\in[1,t]$. Since $F(\Psi(i))\simeq{}_\Gamma\overline{\Delta}(i)$, we have

$$\begin{aligned}\Psi(i)&\simeq GD\big({}_{\Gamma^{op}}\overline{\nabla}(i)\big)\simeq D\operatorname{Hom}_\Gamma\big(D\big({}_{\Gamma^{op}}\overline{\nabla}(i)\big),D(Q)\big)\simeq D\operatorname{Hom}_{\Gamma^{op}}\big(Q,{}_{\Gamma^{op}}\overline{\nabla}(i)\big)\\&\simeq D\operatorname{Hom}_{\Gamma^{op}}\big(T,{}_{\Gamma^{op}}\overline{\nabla}(i)\big)\simeq D\big({}_A\overline{\Delta}(i)\big)\simeq{}_\Lambda\overline{\nabla}(i).\end{aligned}$$

(e) $\Rightarrow$ (f) Assume that (e) holds. In particular ${}_\Lambda\Lambda\in\mathcal{F}({}_\Lambda\Delta)$. Then, it follows from [D1, 2.2] (see also [L]) that $D(\Lambda_\Lambda)\in\mathcal{F}({}_\Lambda\overline{\nabla})=\mathcal{F}(\Psi)$. If ${}_\Lambda T=\bigoplus_{i=1}^t T(i)$ is the characteristic tilting module associated to the standardly stratified algebra $(\Lambda,\leqslant)$, we know that $({}_\Lambda\overline{\nabla},\{T(i)\}_{i=1}^t,\leqslant)$ is a proper costratifying system. From ${}_\Lambda\overline{\nabla}(i)\simeq\Psi(i)$, for all $i\in[1,t]$, and the uniqueness of proper costratifying systems proven in Remark 3.2, it follows that ${}_\Lambda Q\simeq{}_\Lambda T$. Hence ${}_\Lambda Q$ is a tilting module.

(e) $\Rightarrow$ (a) Since $(\Lambda,\leqslant)$ is a standardly stratified algebra, we know from [AHLU, Theorem 1.6(ii)] that $\mathcal{F}({}_\Lambda\overline{\nabla})$ is coresolving. Furthermore, $\mathcal{F}(\Psi)=\mathcal{F}({}_\Lambda\overline{\nabla})$ since ${}_\Lambda\overline{\nabla}(i)\simeq\Psi(i)$, for all $i\in[1,t]$, and so (e) follows.

(b) $\Rightarrow$ (c) Let $\mathcal{F}(\Psi)\cap\mathcal{F}(\Psi)^{\perp_1}=\operatorname{add}(D(\Lambda_\Lambda))$. Then, by Proposition 4.10, we get that $\operatorname{add}(D(\Lambda_\Lambda))=\operatorname{add}(GD({}_{\Gamma^{op}}T))$ and hence $t=\operatorname{rk}K_0(\Lambda)$.

(c) $\Rightarrow$ (b) Let $D(\Lambda_\Lambda)\in\mathcal{F}(\Psi)$ and $t=\operatorname{rk}K_0(\Lambda)$. Applying the functor G to the second equality in Proposition 4.9(b), we have the equalities $\mathcal{F}(\Psi)\cap\mathcal{F}(\Psi)^{\perp_1}=G(\mathcal{F}({}_\Gamma\overline{\Delta})\cap\mathcal{F}({}_\Gamma\overline{\Delta})^{\perp_1})=\operatorname{add}(GF(D(\Lambda_\Lambda)))=\operatorname{add}(D(\Lambda_\Lambda))$.

(f) $\Rightarrow$ (c) We have that $t=\operatorname{card}(\operatorname{ind}(\operatorname{add}(Q)))=\operatorname{rk}K_0(\Lambda)$, where the last equality holds since ${}_\Lambda Q$ is tilting, and this completes our proof. $\square$

Remark 4.12. Let $(\Psi,\mathbf{Q},\leqslant)$ be the proper costratifying system considered in Example 3.7. In this case, $\Gamma^{op}=\operatorname{End}({}_\Lambda Q)$ is given by the quiver

$$\underset{1}{\circ}\ \underset{\delta}{\overset{\mu}{\rightleftarrows}}\ \underset{3}{\circ}\ \xrightarrow{\ \varepsilon\ }\ \underset{2}{\circ}$$

with the relations $\varepsilon\mu=0$ and $\mu\delta\mu=0$. By Theorem 4.3 we know that $(\Gamma^{op},\leqslant^{op})$ is a standardly stratified algebra. The characteristic tilting module is

$${}_{\Gamma^{op}}T=\begin{matrix}3\\1\\3\\1\end{matrix}\oplus\begin{matrix}3\\2\end{matrix}\oplus 3,$$

and it is not isomorphic to

$$ {}_{\Gamma^{op}}Q = \begin{smallmatrix}2\\1\end{smallmatrix} \oplus \begin{smallmatrix} & 3 & \\ 1 & & 2 \\ 3 & & \\ 1 & & \end{smallmatrix} \oplus 2. $$

On the contrary, in Example 3.6 (see also Example 4.6), we have that ${}_{\Gamma^{op}}T = {}_{\Gamma^{op}}Q = \begin{smallmatrix}3\\2\end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix}1\\3\end{smallmatrix}$, but $\Lambda \not\simeq \mathrm{End}({}_{\Gamma^{op}}Q)$. Note that, in both cases, we have that ${}_{\Lambda}\overline{\nabla} \neq \Psi$.

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