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Measurable cardinals and the cardinality of Lindelöf spaces

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We obtain from the consistency of the existence of a measurable cardinal the consistency of "small" upper bounds on the cardinality of a large class of Lindelöf spaces whose singletons are G_δ sets.

Call a topological space in which each singleton is a G_δ set a *points* G_δ *space*. A.V. Arhangel'skii proved that any points G_δ Lindelöf space must have cardinality less than the least measurable cardinal and asked whether for T_2 spaces this cardinality upper bound could be improved. I. Juhasz constructed examples showing that for T_1 spaces this upper bound is sharp. F.D. Tall, investigating Arhangel'skii's problem, defined the class of *indestructibly* Lindelöf spaces. A Lindelöf space is *indestructible* if it remains Lindelöf after forcing with a countably closed forcing notion. He proved:

Theorem 1. *(F.D. Tall [15]) If it is consistent that there is a supercompact cardinal, then it is consistent that* $2^{\aleph_0} = \aleph_1$, *and every points* G_δ *indestructibly Lindelöf space has cardinality* $\leqslant \aleph_1$.

In this paper we show that the hypothesis that there is a supercompact cardinal can be weakened to the hypothesis that there exists a measurable cardinal. Our technique permits flexibility on the cardinality of the continuum.

In Section 1 we review relevant information about ideals and the weakly precipitous ideal game. The relevance of the weakly precipitous ideal game to points G_δ spaces is given in Lemma 2. In Section 2 we consider the indestructibly Lindelöf spaces. A variation of the weakly precipitous ideal game is introduced. This variation is featured in the main result, Theorem 4: a cardinality restriction is imposed on the indestructibly Lindelöf spaces with points G_δ. In Section 3 we give the consistency strength of the hypothesis used in Theorem 4 and point out that mere existence of a precipitous ideal is insufficient to derive a cardinality bound on the indestructibly Lindelöf points G_δ spaces. In Section 4 we describe models of set theory in which the Continuum Hypothesis fails while there is a "small" upper bound on the cardinality of points G_δ indestructibly Lindelöf spaces.

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Rothberger spaces are important examples of indestructibly Lindelöf spaces. A space X is a *Rothberger space* if for each ω-sequence of open covers of X there is a sequence of open sets, then n-th belonging to the n-th cover, such that the terms of the latter sequence form an open cover of X. Rothberger spaces are indestructibly Lindelöf (but not conversely). More details about Rothberger spaces in this context can be found in [14].

1. Weakly precipitous ideals and points G_δ spaces

For κ an infinite cardinal, $\mathcal{P}(\kappa)$ denotes the powerset of κ. A set $J \subseteq \mathcal{P}(\kappa)$ is said to be a *free ideal* on κ if: (i) each finite subset of κ is an element of J, (ii) κ is not an element of J, (iii) the union of any two elements of J is an element of J, and (iv) if $B \in J$ then $\mathcal{P}(B) \subset J$. For a free ideal J the symbol J^+ denotes $\{A \in \mathcal{P}(\kappa)\colon\ A \notin J\}$.

Let $\lambda \leqslant \kappa$ be a cardinal number. The free ideal J on κ is said to be λ-*complete* if: For each $\mathcal{A} \subset J$, if $|\mathcal{A}| < \lambda$ then $\bigcup \mathcal{A} \in J$. A free ideal which is ω_1-complete is said to be σ-complete.

Lemma 2. *Let κ be a cardinal let $J \subset \mathcal{P}(\kappa)$ be a σ-complete free ideal on κ. Let $X \supseteq \kappa$ be a topological space in which each point is a G_δ. Then for each $x \in X$ and each $B \in J^+$ and each sequence $(U_n(x)\colon\ n < \omega)$ of neighborhoods of x with $\{x\} = \bigcap_{n<\omega} U_n(x)$, there is a $C \subseteq B$ with $C \in J^+$ and an n such that $U_n(x) \cap C \in J$.*

Proof. For each x in X fix a sequence $(U_n(x)\colon\ n < \omega)$ of open neighborhoods such that for each n we have $U_{n+1}(x) \subseteq U_n(x)$, and $\{x\} = \bigcap_{n<\omega} U_n(x)$.

Suppose that contrary to the claim of the lemma, there is an $x \in X$ and a $B \in J^+$ such that for each $C \subset B$ with $C \in J^+$ and for each n, $U_n(x) \cap C \in J^+$. Fix x and B. Note that for each n we have by hypothesis that $U_n(x) \cap B \in J^+$.

Then for each n we have $B \cap (U_n(x) \setminus U_{n+1}(x))$ is in J. But then as the ideal J is σ-complete we find that $B \cap U_1(x) \setminus \{x\} = \bigcup_{n=1}^{\infty} B \cap (U_n(x) \setminus U_{n+1}(x)) \in J$, whence for each n, $B \cap U_n(x) \in J$, a contradiction. □

Lemma 2 is true for not only for points that are G_δ sets, but more generally for each element of J which is a G_δ subset of X.

For a free ideal J on κ Galvin et al. [3] investigated the game $G(J)$ of length ω, defined as follows: Two players, ONE and TWO, play an inning per finite ordinal n. In inning n ONE first chooses $O_n \in J^+$. TWO responds with $T_n \in J^+$. The players obey the rule that for each n, $O_{n+1} \subset T_n \subset O_n$. TWO wins a play

$$O_1, T_1, O_2, T_2, \ldots, O_n, T_n, \ldots$$

if $\bigcap_{n<\omega} T_n \neq \emptyset$; else, ONE wins.

It is easy to see that if J is not σ-complete, then ONE has a winning strategy in $G(J)$. It was shown in Theorem 2 of [3] that $J \subseteq \mathcal{P}(\kappa)$ is a *weakly precipitous* ideal if, and only if, ONE has no winning strategy in the game $G(J)$. We shall take this characterization of weak precipitousness as the definition.

Thus Lemma 2 applies to weakly precipitous ideals. In an earlier version of this paper I proved Lemma 2 under the hypothesis that the ideal J is weakly precipitous. M. Magidor observed that it is sufficient to assume that the ideal is σ-complete.

An ideal J on $\mathcal{P}(\kappa)$ is said to be *precipitous* if it is weakly precipitous and κ-complete. This distinction was not made in the earlier literature such as [3] and [7]. The κ-completeness requirement appears to have emerged in [8], and the "weakly precipitous" terminology for the σ-complete case seems to have been coined in [11].

2. The cardinality of points G_δ indestructibly Lindelöf spaces

For a space X define the game $G_1^{\omega_1}(\mathcal{O}, \mathcal{O})$ as follows: Players ONE and TWO play an inning for each $\gamma < \omega_1$. In inning γ ONE first chooses an open cover O_γ of X, and then TWO chooses $T_\gamma \in O_\gamma$. A play

$$O_0, T_0, \ldots, O_\gamma, T_\gamma, \ldots$$

is won by TWO if $\{T_\gamma\colon\ \gamma < \omega_1\}$ is a cover of X. Else, ONE wins.

In [14] we proved the following characterization of being indestructibly Lindelöf:

Theorem 3. *([14], Theorem 1) A Lindelöf space X is indestructibly Lindelöf if, and only if, ONE has no winning strategy in the game $G_1^{\omega_1}(\mathcal{O}, \mathcal{O})$.*

Of several natural variations on $G(J)$ we now need the following one: The game $G^+(J)$ proceeds like $G(J)$, but TWO wins a play only when $\bigcap_{n<\omega} T_n \in J^+$; else, ONE wins. Evidently, if TWO has a winning strategy in $G^+(J)$ then TWO has a winning strategy in $G(J)$. Similarly, if ONE has no winning strategy in $G^+(J)$, then ONE has no winning strategy in $G(J)$. A winning strategy in $G^+(J)$ for TWO which depends on only the most recent move of ONE is said to be a winning *tactic*.

Theorem 4. *Assume there is a free, σ-complete ideal J on κ such that TWO has a winning tactic in $G^+(J)$. Then each points G_δ indestructibly Lindelöf space has cardinality less than κ.*

Proof. Let X be a points G_δ Lindelöf space with $|X| \geqslant \kappa$. Let Y be a subset of X of cardinality κ and let $J \subset \mathcal{P}(Y)$ be a free ideal such that TWO has a winning tactic σ in $G^+(J)$. We define a winning strategy F for ONE of the game $\mathsf{G}_1^{\omega_1}(\mathcal{O}, \mathcal{O})$ and then cite Theorem 1 of [14]:

For each $x \in X$ fix a sequence of neighborhoods $(U_n(x)\colon n < \infty)$ such that for $m < n$ we have $U_m(x) \supset U_n(x)$, and $\{x\} = \bigcap_{n<\omega} U_n(x)$. First, ONE does the following: For each $x \in X$: Choose $D_x \subseteq Y$ and n so that $D_x \notin J$, $U_n(x) \cap D_x \in J$ and set $C(x) = \sigma(D_x)$. Choose $n(C(x), x) < \omega$ such that $C(x) \cap U_{n(C(x),x)} \in J$. ONE's first move in $\mathsf{G}_1^{\omega_1}(\mathcal{O}, \mathcal{O})$ is

$$F(\emptyset) = \{U_{n(C(x),x)}(x)\colon x \in X\}.$$

When TWO chooses $T_0 \in F(\emptyset)$, fix x_0 with $T_0 = U_{n(C(x_0),x_0)}(x_0)$. Define $C_0 = C(x_0)$, $D_0 = D_{x_0}$. Since $C_0 \in J^+$ we choose by Lemma 2 for each $x \in X$ a $D_{x_0,x}$ and an n with:

(1) $D_{x_0,x} \in J^+$ and $D_{x_0,x} \subset C(x_0)$ and
(2) $U_n(x) \cap D_{x_0,x} \in J$.

Put $C(x_0, x) = \sigma(D_{x_0,x})$, and choose $n(C(x_0, x), x)$ with $C(x_0, x) \cap U_{n(C(x_0,x),x)}(x) \in J$. ONE plays

$$F(T_0) = \{U_{n(C(x_0,x),x)}(x)\colon x \in X\}.$$

When TWO plays $T_1 \in F(T_0)$, fix x_1 so that $T_1 = U_{n(C(x_0,x_1),x_1)}(x_1)$. Define $C_1 = C(x_0, x_1)$ and $D_1 = D_{x_0,x_1}$. Since $C_1 \in J^+$ we choose by Lemma 2 for each $x \in X$ a $D_{x_0,x_1,x}$ and an n with:

(1) $D_{x_0,x_1,x} \in J^+$ and $D_{x_0,x_1,x} \subset C(x_0, x_1)$ and
(2) $U_n(x) \cap D_{x_0,x_1,x} \in J$.

Put $C(x_0, x_1, x) = \sigma(D_{x_0,x_1,x})$ and choose $n(C(x_0, x_1, x), x)$ with $C(x_0, x_1, x) \cap U_{n(C(x_0,x_1,x),x)}(x) \in J$. ONE plays

$$F(T_0, T_1) = \{U_{n(C(x_0,x_1,x),x)}(x)\colon x \in X\},$$

and so on.

At a limit stage $\alpha < \omega_1$ we have descending sequences $(C_\gamma\colon \gamma < \alpha)$ and $(D_\gamma\colon \gamma < \alpha)$ of elements of J^+ as well as a sequence $(x_\gamma : \gamma < \alpha)$ of elements of X such that:

(1) For each γ, $C_\gamma = C(x_\nu\colon \nu \leqslant \gamma)$ and $D_\gamma = D_{(x_\nu\colon \nu \leqslant \gamma)}$;
(2) For each γ, $D_{\gamma+1} \subset C_\gamma = \sigma(D_\gamma)$;
(3) $T_\gamma = U_{n(C_\gamma,x_\gamma)}(x_\gamma)$.

Since α is countable choose a cofinal subset $(\gamma_n\colon n < \omega)$ of ordinals increasing to α. Then as for each n we have $C_{\gamma_n} = \sigma(D_{\gamma_n})$ we see that $(C_{\gamma_n}\colon n < \omega)$ are moves by TWO, using the winning tactic σ, in $G^+(J)$. Thus we have $\bigcap_{\gamma<\alpha} C_\gamma = \bigcap_{n<\omega} C_{\gamma_n} \in J^+$.

Then by Lemma 2 choose for each $x \in X$ a $D_{(x_\nu\colon \nu \leqslant \gamma)\frown(x)}$ and n such that

(1) $D_{(x_\nu\colon \nu \leqslant \gamma)\frown(x)} \in J^+$ and $D_{(x_\nu\colon \nu \leqslant \gamma)\frown(x)} \subset \bigcap_{\gamma<\alpha} C_\gamma$ and
(2) $D_{(x_\nu\colon \nu \leqslant \gamma)\frown(x)} \cap U_n(x) \in J$.

Put

$$C\big((x_\nu\colon \nu < \alpha) \frown (x)\big) = \sigma(D_{((x_\nu\colon \nu<\alpha)\frown(x))})$$

and choose $n(C((x_\nu\colon \nu < \alpha) \frown x), x)$ such that: $C((x_\nu\colon \nu < \alpha) \frown x) \cap U_{n(C((x_\nu\colon \nu<\alpha)\frown x))}(x) \in J$.

Then ONE plays

$$F(T_\gamma\colon \gamma < \alpha) = \{U_{n(C((x_\nu\colon \nu<\alpha)\frown x),x)}(x)\colon x \in X\}.$$

This defines a strategy for ONE of the game $\mathsf{G}_1^{\omega_1}(\mathcal{O}, \mathcal{O})$. To see that it is winning, suppose that on the contrary there is an F-play won by TWO, say

$$O_0, T_0, \dots, O_\gamma, T_\gamma, \dots, \gamma < \omega_1,$$

where $O_0 = F(\emptyset)$ and for each $\gamma > 0$, $O_\gamma = F(T_\beta\colon \beta < \gamma)$. Since TWO wins $\mathsf{G}_1^{\omega_1}(\mathcal{O}, \mathcal{O})$, $X = \bigcup_{\gamma<\omega_1} T_\gamma$. But X is Lindelöf, and so we find a $\beta < \omega_1$ with $X = \bigcup_{\gamma<\beta} T_\gamma$. But then $C_\alpha = C(x_\nu\colon \nu < \alpha)$, $\alpha < \beta$ occurring in the definition of F are in J^+

and satisfy for $\alpha < \beta$ that $C_\beta \subset C_\alpha$. It follows that for each $\gamma < \beta$ we have $T_\gamma \cap C_\beta \in J$, and as J is σ-complete it follows that the T_γ do not cover $C_\beta \subset X$, a contradiction. □

At one point in the above proof we made use of the fact that TWO has a winning tactic in $G^+(J)$. Masaru Kada informed me that in fact, by known results of Foreman and of Velickovic the conclusion of the theorem can be deduced from simply assuming that TWO has a winning strategy in $G^+(J)$. I thank Kada for his kind permission to include the relevant remarks here. For notation and more information, see [6]: If TWO has a winning strategy in $\mathsf{G}^+(J)$, the partially ordered set $(J^+, \subseteq)$ is $\omega + 1$-strategically closed, and thus (see Corollary 3.2 in [6]) is strongly ω_1-strategically closed. In the proof of Theorem 4 use TWO's winning strategy in the game $\mathsf{G}^I_{\omega_1}(J^+)$ instead of a winning tactic in $\mathsf{G}^+(J)$.

Problem 1. Let J be a σ-complete free ideal on κ such that TWO has a winning strategy in $G^+(J)$. Does it follow that TWO has a winning tactic in $G^+(J)$?

Note that by Theorem 7 of [4], if TWO has a winning strategy in $G^+(J)$, then TWO has a winning strategy σ such that $T_1 = \sigma(\kappa, O_1)$, and for each n, $T_{n+1} = \sigma(T_n, O_{n+1})$. It is not known if the conclusion of Theorem 4 follows simply from assuming that TWO has a winning strategy in $G(J)$.

Problem 2. Let J be a σ-complete free ideal on κ such that TWO has a winning strategy in $G(J)$. Does it follow that TWO has a winning strategy in $G^+(J)$?

3. The hypothesis "TWO has a winning tactic in $G^+(J)$"

We now consider the strength of the hypothesis that TWO has a winning tactic in $G^+(J)$. First recall some concepts. Let $J \subset \mathcal{P}(\kappa)$ be a σ-complete ideal and let $\lambda \leqslant \kappa$ be an initial ordinal. For subsets X and Y of κ write $X \equiv Y \bmod J$ if the symmetric difference of X and Y is in J. Then $\mathcal{P}(\kappa)/J$ denotes the set of equivalence classes for this relation, and $[X]_J$ denotes the equivalence class of X.

A subset D of the Boolean algebra $\mathcal{P}(\kappa)/J$ is said to be *dense* if there is for each $b \in \mathcal{P}(\kappa)/J$ a $d \in D$ with $d < b$. A dense set $D \subset \mathcal{P}(\kappa)/J$ is said to be λ-*dense* if for each $\beta < \lambda$, for each β-sequence $b_0 > b_1 > \cdots > b_\gamma > \cdots$, $\gamma < \beta < \lambda$ of elements of D there is a $d \in D$ such that for all $\gamma < \beta$, $d < b_\gamma$.

The *Dense Ideal Hypothesis for* $\kappa \geqslant \lambda$, denoted $\mathbf{DIH}(\kappa, \lambda)$, is the statement:

There is a σ-complete free ideal $J \subset \mathcal{P}(\kappa)$ such that the Boolean algebra $\mathcal{P}(\kappa)/J$ has a λ-dense subset.

Note that if $\mu < \lambda$ then $\mathbf{DIH}(\kappa, \lambda) \Rightarrow \mathbf{DIH}(\kappa, \mu)$.

Consider the following five statements:

I There exists a measurable cardinal.
II There is an ω_1 dense free ideal J on an infinite set S.
III There is a free ideal J on a set S such that TWO has a winning tactic in $\mathsf{G}^+(J)$.
IV There is a free ideal J on a set S such that TWO has a winning strategy in $\mathsf{G}^+(J)$.
V There is a precipitous[1] ideal J on an infinite set S.

Then I ⇒ II (let J be the dual ideal of the ultrafilter witnessing measurability), II ⇒ III (see remarks (1), (3) and (4) on page 292 of [3]), and evidently III ⇒ IV and IV ⇒ V.

In ZFC, for a statement P, let CON(P) denote "P is consistent". Then we have CON(I) if, and only if, CON(V):

Proposition 5. *The existence of a free ideal J on an uncountable cardinal such that TWO has a winning tactic* (*or strategy*) *in $G^+(J)$ is equiconsistent with the existence of a measurable cardinal.*

Proof. When TWO has a winning tactic in $G^+(J)$, then TWO has a winning strategy in $G(J)$, and thus ONE has no winning strategy in $G(J)$. It follows that J is a weakly precipitous ideal. Jech et al. [7] show that the existence of a weakly precipitous ideal is equiconsistent with the existence of a measurable cardinal. This shows that consistency of the existence of a free ideal J on an uncountable set, such that TWO has a winning tactic in $G^+(J)$ implies the consistency of the existence of a measurable cardinal.

[1] Indeed, weakly precipitous works here.

For the other direction: The argument in Section 4 of [3] can be adapted to show that if it is consistent that there is a measurable cardinal κ, then for any infinite cardinal $\lambda < \kappa$ it is consistent that $\mathbf{DIH}(\lambda^{++}, \lambda^{+})$ holds.[2] A free ideal J on λ^{++} witnessing $\mathbf{DIH}(\lambda^{++}, \lambda^{+})$ is a free σ-complete ideal such that TWO has a winning tactic in $G^{+}(J)$. □

Theorem 4 cannot be proved from the equiconsistent but formally weaker hypothesis that the ideal in question is (weakly) precipitous. This can be seen as follows: Property V is preserved when adding $\aleph_1$ or more Cohen reals. This follows from the following result, Theorem 2.1 of [13]:

Lemma 6. *(M. Magidor [13]) Let $(\mathbb{P}, <)$ be a countable chain condition partially ordered set. If J is a precipitous[3] ideal on a set S, then*

$$\Vdash_{\mathbb{P}} \text{“}J \text{ is a precipitous ideal on } S\text{”}.$$

Now I. Juhász has proven that for each infinite cardinal κ less than the first measurable cardinal there is a points G_δ Lindelöf space X with $\kappa < |X|$ (see [9] and [15] for details). But adding $\aleph_1$ Cohen reals converts each such groundmodel space to a Rothberger space (and thus indestructibly Lindelöf space) in the generic extension (see [14]). Thus if it is consistent that there is a weakly precipitous ideal on a cardinal μ less than the first measurable cardinal then it is consistent that there is a (weakly) precipitous ideal on μ, and yet there is an indestructibly Lindelöf points G_δ space of cardinality larger than μ.

In Problem 1 it is asked if the existence of a winning strategy for TWO in $\mathsf{G}^{+}(J)$ is equivalent to the existence of a winning tactic for TWO. Some partial positive information is available. If the Boolean algebra $\mathcal{P}(\kappa)/J$ has a dense subset of cardinality at most $2^{\aleph_0}$ and if TWO has a winning strategy in $\mathsf{G}^{+}(J)$, then $\mathcal{P}(\kappa)/J$ has a countably closed dense subset: See [16] Corollary 1.3 and [2]. But then TWO has a winning tactic in $\mathsf{G}^{+}(J)$: TWO plays from the countably closed dense set, and only consults ONE's most recent move to decide which element to play. It is not clear if the restriction on the cardinality of a dense subset of $\mathcal{P}(\kappa)/J$ is necessary.

4. The continuum and the cardinality of points G_δ indestructibly Lindelöf spaces

The first consequence of the work above is that the hypothesis of the consistency of the existence of a supercompact cardinal in Theorem 1 can be reduced to the hypothesis of the consistency of the existence of a measurable cardinal:

Corollary 7. *If it is consistent that there is a measurable cardinal, then it is consistent that $2^{\aleph_0} = \aleph_1$ and all points G_δ indestructibly Lindelöf spaces are of cardinality $\leqslant \aleph_1$.*

In what follows we demonstrate that a bound on the cardinality of points G_δ indestructibly Lindelöf spaces does not have a strong influence on the cardinality of the real line. Since separable metric spaces are points G_δ indestructibly Lindelöf spaces, there are always points G_δ indestructibly Lindelöf spaces of each cardinality less than or equal to $2^{\aleph_0}$.

Corollary 8. *If it is consistent that there is a measurable cardinal κ, then for each infinite cardinal $\aleph_\alpha$ with $\kappa > \aleph_\alpha > \aleph_0$ it is consistent that $2^{\aleph_0} = \aleph_{\alpha+1}$ and there are no points G_δ indestructibly Lindelöf spaces of cardinality $> 2^{\aleph_0}$.*

Proof. First raise the continuum to $\aleph_{\alpha+1}$ by adding reals. Next Lévy collapse the measurable cardinal to $\aleph_{\alpha+2}$. In the resulting model $2^{\aleph_0} = \aleph_{\alpha+1}$ and $\mathsf{DIH}(\aleph_{\alpha+2}, \aleph_{\alpha+1})$ holds. By Theorem 1 each indestructibly Lindelöf space with points G_δ has cardinality $\leqslant \aleph_{\alpha+1}$ in this generic extension. □

Corollary 9. *If it is consistent that there is a measurable cardinal κ, then for each pair of regular cardinals $\aleph_\alpha < \aleph_\beta < \kappa$ with $\aleph_\beta^{\aleph_1} = \aleph_\beta$ it is consistent that $2^{\aleph_0} = \aleph_\alpha$ and $2^{\aleph_1} = \aleph_\beta$ and there is a points G_δ indestructibly Lindelöf space of cardinality $\aleph_\beta$, but there are no points G_δ indestructible Lindelöf spaces of cardinality $> 2^{\aleph_1}$.*

Proof. We may assume the ground model is $\mathbf{L}[\mathcal{U}]$ where $\mathcal{U}$ is a normal ultrafilter witnessing measurability. GCH holds in $\mathbf{L}[\mathcal{U}]$. First use Gorelic's cardinal- and cofinality-preserving forcing to raise $2^{\aleph_1}$ to $\aleph_\beta$ while maintaining CH. This gives a points G_δ Lindelöf T_3-space X of cardinality $2^{\aleph_1}$. Tall shows in [15] that this space X is indestructibly Lindelöf. Then add $\aleph_\alpha$ Cohen reals to get $2^{\aleph_0} = \aleph_\alpha$. In this extension the space X from the first step still is a points G_δ indestructibly Lindelöf T_3-space since all these properties are preserved by Cohen reals [14]. The cardinal κ is, in this generic extension, still measurable [12]. Finally, Levy collapse the measurable cardinal to $\aleph_{\beta+1}$. This forcing is countably closed (and more) and thus preserves indestructibly Lindelöf spaces from the ground model. The resulting model is the one for the corollary. □

[2] The model in [3] is obtained as follows: For κ a measurable in the ground model, collapse all cardinals below κ to $\aleph_1$ using the Levy collapse. One can show that with $\mu < \kappa$ an uncountable regular cardinal, collapsing all cardinals below κ to μ produces a model of $\mathbf{DIH}(\mu^{+}, \mu)$, by verifying that Lemmas 1, 2 and 3 and the subsequent claims in [3] apply *mutatis mutandis*.

[3] An examination of Magidor's proof reveals that "weakly precipitous" suffices.

5. Regarding a problem of Hajnal and Juhász

Hajnal and Juhász asked if an uncountable T_2-Lindelöf space must contain a Lindelöf subspace of cardinality $\aleph_1$. Baumgartner and Tall showed in [1] that there are ZFC examples of uncountable T_1 Lindelöf spaces with points G_δ which have no Lindelöf subspaces of cardinality $\aleph_1$. In [10], Section 3, Koszmider and Tall showed that the answer to Hajnal and Juhász's question is "no". They also show that the existence of their example is independent of ZFC. Recall that a topological space is said to be a P-space if each G_δ-subset is open. It is known that Lindelöf P-spaces are Rothberger spaces [14]. They show in [10], Theorem 4, that:

Theorem 10 *(Koszmider–Tall). The following is consistent relative to the consistency of ZFC: CH holds, $2^{\aleph_1} > \aleph_2$ and every T_2 Lindelöf P-space of cardinality $\aleph_2$ contains a convergent ω_1-sequence (thus a Rothberger subspace of cardinality $\aleph_1$).*

And then in Section 3 of [10] they obtain their (consistent) example:

Theorem 11 *(Koszmider–Tall). It is consistent, relative to the consistency of ZFC, that CH holds and there is an uncountable T_3-Lindelöf P-space which has no Lindelöf subspace of cardinality $\aleph_1$.*

One may ask if the problem of Hajnal and Juhász has a solution in certain subclasses of the class of Lindelöf spaces. Koszmider and Tall's results show that even in the class of Rothberger spaces the Hajnal–Juhász problem has answer "no". In the class of Rothberger spaces with small character the following is known [14]:

Theorem 12. *If it is consistent that there is a supercompact cardinal, then it is consistent that $2^{\aleph_0} = \aleph_1$, and every uncountable Rothberger space of character $\leqslant \aleph_1$ has a Rothberger subspace of cardinality $\aleph_1$.*

F.D. Tall communicated to me that the techniques of this paper can also be used to reduce the strength of the hypothesis in Theorem 12 from supercompact to measurable. A small additional observation converts Tall's remark to the following.

Theorem 13. *Assume there is a free ideal J on ω_2 such that TWO has a winning tactic in $G^+(J)$. Then every indestructibly Lindelöf space of cardinality larger than $\aleph_1$ and of character $\leqslant \aleph_1$ has a Rothberger subspace of cardinality $\aleph_1$.*

Proof. If an indestructibly Lindelöf space has cardinality larger than $\aleph_1$ then Theorem 4 implies it has a point that is not G_δ. Hajnal and Juhász proved that if a Lindelöf space has character $\leqslant \aleph_1$ and if some element is not a G_δ-point, then the space has a convergent ω_1-sequence (see Theorem 7 in [1]). Such a sequence together with its limit is a Rothberger subspace. $\square$

6. Remarks

If in a ground model V we have an ideal J on an ordinal α, then in generic extensions of V let J^* denote the ideal on α generated by J. It is of interest to know which forcings increase $2^{\aleph_1}$ but preserve for example the statement: "There is a σ-complete free ideal J on ω_2 such that TWO has a winning strategy in $G^+(J^*)$". Not all ω_1-complete ω_2-chain condition partial orders preserve this statement: In [5] Gorelic shows that for each cardinal number $\kappa > \aleph_1$ it is consistent that CH holds, that $2^{\aleph_1} > \kappa$, and there is a T_3 points G_δ Lindelöf space X of cardinality $2^{\aleph_1}$. Tall showed in [15] that Gorelic's space is indestructibly Lindelöf. Since the model in Section 4 of [3] is a suitable ground model for Gorelic's construction, Theorem 4 implies that in the model obtained by applying Gorelic's extension to the model from [3], there is no free ideal J on ω_2 such that TWO has a winning strategy in $G^+(J)$.

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