ELSEVIER
Available online at www.sciencedirect.com
ScienceDirect
Advances in Mathematics 223 (2010) 1731–1781
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

Homology of perfect complexes ☆

Luchezar L. Avramov [a], Ragnar-Olaf Buchweitz [b], Srikanth B. Iyengar [a,*], Claudia Miller [c]

[a] *Department of Mathematics, University of Nebraska, Lincoln, NE 68588, USA*
[b] *Department of Computer and Mathematical Sciences, University of Toronto Scarborough, Toronto, ON M1A 1C4, Canada*
[c] *Mathematics Department, Syracuse University, Syracuse, NY 13244, USA*

Received 18 March 2009; accepted 9 October 2009

Available online 21 October 2009

Communicated by Karen Smith

Abstract

It is proved that the sum of the Loewy lengths of the homology modules of a finite free complex F over a local ring R is bounded below by a number depending only on R. This result uncovers, in the structure of modules of finite projective dimension, obstructions to realizing R as a closed fiber of some flat local homomorphism. Other applications include, as special cases, uniform proofs of known results on free actions of elementary abelian groups and of tori on finite CW complexes. The arguments use numerical invariants of objects in general triangulated categories, introduced here and called levels. They allow one to track, through changes of triangulated categories, homological invariants like projective dimension, as well as structural invariants like Loewy length. An intermediate result sharpens, with a new proof, the New Intersection Theorem for commutative algebras over fields. Under additional hypotheses on the ring R stronger estimates are proved for Loewy lengths of modules of finite projective dimension.

MSC: 13D05; 13H10; 13D40; 13B10; 13D07; 13D25; 18E30

☆ Research partly supported by NSF grants DMS 0201904 and DMS 0803082 (L.L.A.); NSERC grant 3-642-114-80 (R.-O.B.); NSF grants DMS 0602498 and DMS 0903493 (S.B.I.); NSF grant DMS 0434528 and NSA grant H98230-06-1-0035 (C.M.).

* Corresponding author.
E-mail addresses: avramov@math.unl.edu (L.L. Avramov), ragnar@utsc.utoronto.ca (R.-O. Buchweitz), iyengar@math.unl.edu (S.B. Iyengar), clamille@syr.edu (C. Miller).

0001-8708/$ – see front matter
doi:10.1016/j.aim.2009.10.009

Keywords: Bernstein–Gelfand–Gelfand equivalence; Conormal module; Koszul complex; Loewy length; New Intersection Theorem; Perfect complex; Triangulated category

Contents

0. Introduction

We study homological properties of finite free complexes over noetherian local rings. The results uncover novel links between the structure of the homology modules of such complexes and conormal modules of the ring. Their statements incorporate intuition coming from research in algebraic topology. The arguments use techniques from commutative algebra, differential graded homological algebra, and triangulated categories, and develop new tools for these fields.

Let $(R, \mathfrak{m}, k)$ be a local ring with maximal ideal $\mathfrak{m}$ and residue field k. One theme that runs through the paper is that when R has 'small' modules of finite projective dimension it is 'not far' from being regular. Here we measure the 'size' of an R-module M in terms of its *Loewy length*, defined to be the number

$$\ell\ell_R M = \inf\{i \geqslant 0 \mid \mathfrak{m}^i M = 0\}.$$

When M is non-zero and of finite projective dimension, Loewy length one or two occurs if and only if R is regular or a quadratic hypersurface, respectively.

We provide uniform lower bounds on Loewy lengths of modules of finite projective dimension in terms of invariants depending only on the ring R. The first one involves the Castelnuovo–Mumford regularity of R; see Section 1 for a definition.

Theorem 1. *If R is Gorenstein and its associated graded ring $\mathrm{gr}_{\mathfrak{m}}(R)$ is Cohen–Macaulay, then each non-zero R-module M of finite projective dimension satisfies*

$$\ell\ell_R M \geqslant \operatorname{reg} R + 1.$$

This result applies, for example, when $\mathrm{gr}_{\mathfrak{m}}(R)$ is a graded complete intersection, that is, $\mathrm{gr}_{\mathfrak{m}}(R) \cong k[x_1, \dots, x_e]/I$ where I is generated by a homogeneous regular sequence $g_1, \dots, g_c$; in this case $\operatorname{reg} R = \sum_{j=1}^{c}(\deg g_j - 1)$. Even this special case of the theorem above was known only for $c = 1$, where it was proved by Ding [17].

Applications of Theorem 1 are restricted by the fact that its hypothesis is on the associated graded ring, rather than on the ring of interest. For instance, $\mathrm{gr}_{\mathfrak{m}}(R)$ need not be Cohen–Macaulay even when R is a local complete intersection.

In the remaining results of this work the hypotheses bear on the structure of R itself. Recall that its *embedding dimension* is the number $\operatorname{edim} R = \operatorname{rank}_k(\mathfrak{m}/\mathfrak{m}^2)$.

Theorem 2. *Let $(P, \mathfrak{p}) \to (Q, \mathfrak{q})$ be a flat local homomorphism and set $R = Q/\mathfrak{p}Q$.*
Every non-zero R-module M of finite projective dimension then satisfies

$$\ell\ell_R M \geqslant \operatorname{edim} P - \operatorname{edim} Q + \operatorname{edim} R + 1.$$

In particular, one has $\mathfrak{q}^l \nsubseteq \mathfrak{p}Q$ for $l = \operatorname{edim} P - \operatorname{edim} Q + \operatorname{edim} R$.

The special case $\operatorname{edim} R = \operatorname{edim} Q$ captures an important aspect of the theorem: If $\mathfrak{m}^h M = 0$ for some R-module M of finite projective dimension, then R admits no embedded (that is to say, with $\mathfrak{p}Q \subseteq \mathfrak{q}^2$) flat deformation over a base of embedding dimension greater than h.

Theorem 1 is proved in Section 1, using invariants of Gorenstein local rings defined by M. Auslander and studied by S. Ding. The proof of Theorem 2 is an altogether different affair. It is derived from a more general result, which gives information on the structure of the homology of *finite free complexes*; that is, complexes of finitely generated free R-modules of the form

$$F = 0 \longrightarrow F_s \longrightarrow F_{s-1} \longrightarrow \cdots \longrightarrow F_{t+1} \longrightarrow F_t \longrightarrow 0.$$

In our main result we estimate the sum of the Loewy lengths of the homology modules of F in terms of another invariant of R, which we call the *conormal free rank* and denote $\operatorname{cf-rank} R$. When R is complete equicharacteristic or is essentially of finite type over a field, $\operatorname{cf-rank} R$ equals the maximal rank of a free summand of its conormal module; see Section 8. The inequality in the next theorem is sharp, in the sense that equality holds in some cases.

Theorem 3. *Every finite free complex F with $\mathrm{H}(F) \neq 0$ satisfies an inequality:*

$$\sum_{n \in \mathbb{Z}} \ell\ell_R \mathrm{H}_n(F) \geqslant \operatorname{cf-rank} R + 1.$$

Evidently, to apply this result one needs lower bounds for the conormal free rank.

In the situation of Theorem 2 the free R-module $\mathfrak{p}Q/(\mathfrak{p}Q)^2$ is the conormal module of the surjection $Q \to R$, which can be used to show that $\operatorname{cf-rank} R$ is not less than l. This allows one to deduce Theorem 2 from Theorem 3.

One case when the value of $\operatorname{cf-rank} R$ is known is when R is complete intersection: it is equal to the codimension of R. Remarkably, over such rings one can bound Loewy length of homology for every homologically finite complex, see Theorem 7.

The proof of Theorem 3 is presented in Section 10. It draws on the results of Sections 2 through 9 and involves a number of concepts and techniques that are not traditional for commutative algebra.

The first move is to replace complexes over rings with DG (that is, differential graded) modules over DG algebras. Such a procedure was developed by Avramov to study homological invariants of rings and modules; see [5] for a survey. It utilizes the possibility of adjusting the algebraic properties of a DG algebra by replacing it with a quasi-isomorphic one, while replacing its derived category with an equivalent category. However, the situation here is different because structural properties, such as Loewy length, depend on the underlying graded algebra of a DG algebra and need not be preserved by equivalences of derived categories.

A crucial new idea is to bound Loewy length by numerical invariants of objects of the derived category of the ring that behave predictably under applications of exact functors. Their introduction is motivated in part by work of Dwyer, Greenlees, and Iyengar [19]. These authors transported from homotopy theory into commutative algebra the concept of *building* an object X in a triangulated category T from some fixed object C of T. Here we define a number, $\operatorname{level}_{\mathsf{T}}^{C}(X)$, that we call the *level* of X with respect to C in T. It measures the number of *extensions* needed in the 'building process'. Further suggestions that such a notion might be useful came from the papers of D. Christensen [16], Bondal and Van den Bergh [11], and Rouquier [32,33] dealing with dimensions of triangulated categories.

To show how these ideas fit together we sketch an outline of the proof of Theorem 3. It also serves as an overview of the content of the paper.

In Section 2 we define levels and record their elementary properties.

In Section 3 we specialize to the case of the derived category $\mathsf{D}(A)$ of DG modules over a DG algebra A; for simplicity, we write $\operatorname{level}_{A}^{C}(X)$ in place of $\operatorname{level}_{\mathsf{D}(A)}^{C}(X)$. Two levels over A play a special role in this work.

Levels with respect to the DG A-module A extend the concept of projective dimension from modules over a ring to DG modules over a DG algebra. For instance, for every complex F of finite free R-modules the definitions give an inequality

$$\operatorname{level}_{R}^{R}(F) \leqslant \operatorname{card}\{n \in \mathbb{N} \mid F_n \neq 0\}. \tag{$*$}$$

A structure theorem for DG A-modules of finite A-level is proved in Section 4. In Section 5 it is used, along with the main theorem of [6], to prove the result below. In view of $(*)$, the inequality on the right generalizes and sharpens the classical New Intersection Theorem for commutative noetherian algebras over fields.

Theorem 4. *Let A be a DG algebra with zero differential, let M be a DG module over A, and let I denote the annihilator of $\mathrm{H}(M) = \bigoplus_{n\in\mathbb{Z}} \mathrm{H}_n(M)$ in the ring $A^\flat = \bigoplus_{n\in\mathbb{Z}} A_n$.*

When $A^\flat$ is a commutative noetherian algebra over a field one has

$$\operatorname{proj\,dim}_A \mathrm{H}(M) + 1 \geqslant \operatorname{level}_{A}^{A}(M) \geqslant \operatorname{height} I + 1.$$

Of major importance here is also the level with respect to a semi-simple DG A-module k; its behavior is akin to that of Loewy length of modules over rings. The basic properties of $\operatorname{level}_{A}^{k}(-)$ are derived in Section 6. In particular, we prove

Proposition 5. *If M is a complex of R-modules and $\mathrm{H}(M)$ has finite length, then*

$$\sum_{n\in\mathbb{Z}} \ell\ell_R \mathrm{H}_n(M) \geqslant \mathrm{level}_R^k(M) \geqslant \max_{n\in\mathbb{Z}}\bigl\{\ell\ell_R \mathrm{H}_n(M)\bigr\}.$$

The next stage in the proof of Theorem 3 is to construct a chain of exact functors

$$\mathsf{D}(R) \xrightarrow{\ \mathsf{t}\ } \mathsf{D}(K) \underset{\equiv}{\xrightarrow{\ \mathsf{j}\ }} \mathsf{D}(\Lambda \otimes_k B) \xrightarrow{\ \mathsf{i}\ } \mathsf{D}(\Lambda)$$

of derived categories of DG modules, where $\equiv$ flags an equivalence. The DG algebras are: the ring R viewed as a DG algebra concentrated in degree 0; the Koszul complex K on a minimal generating set for $\mathfrak{m}$; a DG k-algebra B with $\mathrm{rank}_k B < \infty$; and an exterior algebra $\Lambda = k\langle \xi_1, \ldots, \xi_c\rangle$ with $\deg \xi_i = 1$, zero differential, $c = \text{cf-rank}\, R$.

Two functors are easy to describe: $\mathsf{t}(-) = K \otimes_R -$ and i is the forgetful functor defined by the canonical morphism $\Lambda \to \Lambda \otimes_k B$. On the other hand, the construction of the equivalence of categories j takes up all of Section 9, where we also prove that the DG module $\mathsf{jt}(k) \in \mathsf{D}(\Lambda \otimes_k B)$ is a direct sum of suspensions of copies of k; as a consequence, levels with respect to $\mathsf{jt}(k)$ are equal to k-levels. This fact, together with the formal property that functors do not raise levels and equivalences preserve them, justify all but the initial step in the following sequence:

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \ell\ell_R \mathrm{H}_n(F) &\geqslant \mathrm{level}_R^k(F) \geqslant \mathrm{level}_K^{\mathsf{t}(k)}\bigl(\mathsf{t}(F)\bigr)\\ &= \mathrm{level}_{\Lambda\otimes_k B}^{\mathsf{jt}(k)}\bigl(\mathsf{jt}(F)\bigr) = \mathrm{level}_{\Lambda\otimes_k B}^{k}\bigl(\mathsf{jt}(F)\bigr) \geqslant \mathrm{level}_\Lambda^k(N),\end{aligned}$$

where we have set $N = \mathsf{ijt}(F)$. The first inequality is provided by Proposition 5.

To finish the proof of Theorem 3 one needs an estimate for $\mathrm{level}_\Lambda^k(N)$. The key to obtaining one is to show that N is isomorphic in $\mathsf{D}(\Lambda)$ to a DG module with finite free underlying graded Λ-module. To do this we first remark that formula $(*)$ implies that the finite free complex F has finite R-level, then use results on persistence of levels to show that N has finite Λ-level, and finally prove that the last condition is equivalent to N being finite free as a graded Λ-module.

It remains to use the equality in the following result of independent interest:

Theorem 6. *If N is a DG Λ-module with finite free underlying graded Λ-module and with $\mathrm{H}(N) \neq 0$, then one has*

$$\mathrm{card}\bigl\{n \bigm| \mathrm{H}_n(N) \neq 0\bigr\} \geqslant \mathrm{level}_\Lambda^k(N) = c + 1.$$

The proof of Theorem 6 itself consists of two independent steps. The first one is to reduce the computation of $\mathrm{level}_\Lambda^k(N)$ to that of $\mathrm{level}_S^S(M)$, where S is a polynomial ring in c indeterminates over k, and M is a DG S-module with $\mathrm{rank}_k \mathrm{H}(M)$ finite. For this we use a DG version of the Bernstein–Gelfand–Gelfand equivalence, presented in Section 7. Theorem 4 then gives $\mathrm{level}_S^S(M) = c + 1$.

All the threads of the delicate proof of Theorem 3 come together in Section 10. In the final count, the argument hinges on the ability to simultaneously track numerical invariants, both structural and homological, under the actions of various functors. Its success attests to the remarkable versatility of the concept of level.

Theorem 3 is restricted to finite free complexes. Using results from [4], over certain rings we extend it to a statement about all complexes with finite homology:

Theorem 7. *If R is a complete intersection local ring and M is a complex of R-modules with $\mathrm{H}(M)$ finite and non-zero, then one has an inequality*

$$\sum_{n\in\mathbb{Z}} \ell\ell_R \mathrm{H}_n(M) \geqslant \operatorname{codim} R - \operatorname{cx}_R M + 1.$$

This is proved in Section 11. The number $\operatorname{cx}_R M$, known as the *complexity* of M, is the least non-negative integer d such that the ranks of the free modules in a minimal free resolution of M are bounded by a polynomial of degree $d-1$. Theorem 7 can also be deduced from a strengthening of Theorem 6 that covers all DG Λ-modules with finite underlying graded module; see [8].

The last two theorems above have antecedents in the study of the homology of a finite CW complex X with an action of an elementary abelian group or of a torus.

When G is an elementary abelian p-group and X has a G-equivariant cellular decomposition, its cellular chain complex with coefficients in $\mathbb{F}_p$ is finite free over the ring $T \cong \mathbb{F}_p[x_1,\dots,x_c]/(x_1^p,\dots,x_c^p)$. Carlsson [13] established that

$$\sum_{n} \ell\ell_T \mathrm{H}_n(X;\mathbb{F}_p) \geqslant c+1$$

holds by proving Theorem 7 for M a finite free complex over T and $p=2$. Allday, Baumgartner, and Puppe, see [2], proved Theorem 7 for odd p and for all complexes of T-modules with finite homology; the arguments depend on the parity of p.

When the induced action of G on $\mathrm{H}_*(X;\mathbb{F}_p)$ is trivial the inequality above simply states that X has at least $c+1$ non-trivial homology groups with coefficients in $\mathbb{F}_p$. Allday and Puppe [2] proved a similar estimate for almost free torus actions:

$$\operatorname{card}\{n \mid \mathrm{H}_n(X;\mathbb{Q}) \neq 0\} \geqslant c+1.$$

The algebraic core of their proof is a property of DG modules over a polynomial ring S in c indeterminates over $\mathbb{Q}$, which is implied by Theorem 4.

The original proofs of the theorems on group actions heavily depend on the structure of the rings S and T. Our results demonstrate that these theorems are manifestations of general properties of complexes over commutative noetherian rings.

1. Loewy length of modules of finite projective dimension

Let R be a commutative local noetherian ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. The *Loewy length* of an R-module M is the number

$$\ell\ell_R M = \inf\{n \in \mathbb{N} \mid \mathfrak{m}^n M = 0\}.$$

When M is finitely generated, $\ell\ell_R M$ is finite if and only if its (Jordan–Hölder) *length*, denoted $\ell_R M$, is finite. Often the Loewy length of M carries more structural information than does its length.

Let G denote the associated graded ring $\mathrm{gr}_{\mathfrak{m}}(R)$; thus, $G_0 = k$ and G is generated over G_0 by G_1. Choose a presentation $G \cong P/J$ with $P = k[x_1, \dots, x_e]$, a polynomial ring in indeterminates x_i of degree 1, and J a homogeneous ideal. Set

$$\operatorname{reg} R = \sup\bigl\{ j \bigm| \operatorname{Tor}_i^P(G, P/P_{\geqslant 1})_{i+j} \neq 0 \bigr\}.$$

In other words, reg R is the *Castelnuovo–Mumford regularity*, $\operatorname{reg}_P G$, of the graded P-module G; it is independent of the choice of P, see e.g. Şega [36, 1.4].

The *order* of the ring R is given for singular R by the formula

$$\operatorname{ord} R = \inf\left\{ n \in \mathbb{N} \;\middle|\; \ell_R\bigl(R/\mathfrak{m}^{n+1}\bigr) < \binom{n+e}{e} \right\}$$

where $e = \operatorname{edim} R$; when the ring R is regular we set $\operatorname{ord} R = 1$.

The next result contains Theorem 1 from the introduction. It applies, in particular, when R is the localization or the completion of a standard graded Gorenstein ring G at the maximal ideal $\bigoplus_{n \geqslant 1} G_n$.

Theorem 1.1. *If R is Gorenstein and the associated graded ring $\mathrm{gr}_{\mathfrak{m}}(R)$ is Cohen–Macaulay, then for each non-zero R-module M of finite projective dimension*

$$\ell\ell_R M \geqslant \operatorname{reg} R + 1 \geqslant \operatorname{ord} R.$$

When k is infinite the first inequality becomes an equality for some such module M.

The proof uses numerical invariants introduced by M. Auslander.

Let R be a Gorenstein local ring and M a finite R-module. Let M^{cm} denote the sum of all submodules $\lambda(L) \subseteq M$, when L ranges over all maximal Cohen–Macaulay R-modules with no non-zero free direct summand, and λ ranges over all R-linear homomorphisms $L \to M$. The minimal number of generators of the R-module M/M^{cm} is called the *delta invariant* of M and is denoted $\delta_R(M)$.

The basic properties of $\delta_R(-)$, due to Auslander, are collected in the next result. Their proofs in the literature are scattered and some use alternative characterizations, so we provide complete details.

Lemma 1.2. *Let M, N be finite modules over a Gorenstein local ring $(R, \mathfrak{m}, k)$.*

The following (in)equalities then hold:

$$\delta_R(M) = \delta_R(M') + \delta_R(M'') \quad \text{when } M = M' \oplus M'', \tag{1.2.1}$$

$$\delta_R(M) \geqslant \delta_R(N) \quad \text{when } M \to N \text{ is a surjective homomorphism}, \tag{1.2.2}$$

$$\delta_R(M) = \operatorname{rank}_k(M/\mathfrak{m}M) \quad \text{when } \operatorname{proj\,dim}_R M < \infty, \tag{1.2.3}$$

$$\delta_R\bigl(R/\mathfrak{m}^n\bigr) = 1 \quad \text{for all } n \gg 0, \tag{1.2.4}$$

$$\delta_R(k) = 1 \quad \text{if and only if } R \text{ is regular}. \tag{1.2.5}$$

Proof. The first two assertions are evident.

For the third one, it suffices to prove that if L is a maximal Cohen–Macaulay module and $\lambda : L \to M$ is a homomorphism of R-modules with $\lambda(L) \nsubseteq \mathfrak{m}M$, then L has a non-zero free summand. Consider an exact sequence

$$0 \longrightarrow E \longrightarrow F \xrightarrow{\pi} M \longrightarrow 0$$

with F free and $E \subseteq \mathfrak{m}F$. By [12, 3.3.10(d)] and induction on $\operatorname{proj\,dim}_R E$, one gets $\operatorname{Ext}^1_R(L, E) = 0$, so the following map is surjective:

$$\operatorname{Hom}_R(L, \pi) : \operatorname{Hom}_R(L, F) \longrightarrow \operatorname{Hom}_R(L, M).$$

Thus, there is a homomorphism $\rho : L \to F$ with $\pi\rho = \lambda$. Choosing $x \in L$ with $\lambda(x)$ not in $\mathfrak{m}M$ we get $y = \rho(x) \notin \mathfrak{m}F$, so Ry is a non-zero free direct summand of F. The composition $L \to F \to Ry$ is then surjective, as desired.

To prove the fourth formula, choose a maximal R-regular sequence $\boldsymbol{x}$. Since $\ell_R(R/R\boldsymbol{x})$ is finite, for all $n \gg 0$ one has $\mathfrak{m}^n \subseteq R\boldsymbol{x}$. Since $\operatorname{proj\,dim}_R(R/R\boldsymbol{x})$ is finite the surjection $R/\mathfrak{m}^n \to R/R\boldsymbol{x}$ implies $\delta_R(R/\mathfrak{m}^n) \geqslant 1$, see (1.2.3). It remains to note that any finite R-module M satisfies $\delta_R(M) \leqslant \operatorname{rank}_k(M/\mathfrak{m}M)$.

As to the last one, R regular implies $\operatorname{proj\,dim}_R k$ finite, hence $\delta_R(k) = 1$ by (1.2.3). If R is not regular, then it has a maximal Cohen–Macaulay R-module L with no non-zero free summands. Picking any surjection $L \to k$ one gets $\delta_R(k) = 0$. □

The *index* of R is defined by Auslander to be the number

$$\operatorname{index} R = \inf\{n \in \mathbb{N} \mid \delta_R(R/\mathfrak{m}^n) \geqslant 1\}.$$

It is a positive integer, by (1.2.4), and equals 1 if and only if R is regular, by (1.2.5).

The next result is the first step in the proof of Theorem 1.1.

Lemma 1.3. *Let R be a Gorenstein local ring and M a finite, non-zero R-module. If the projective dimension of M is finite, then one has*

$$\ell\ell_R M \geqslant \operatorname{index} R.$$

Proof. We may assume $\ell\ell_R M = l < \infty$. For $r = \operatorname{rank}_k(M/\mathfrak{m}M)$ there is then a surjection $(R/\mathfrak{m}^l)^r \to M$. The formulas in Lemma 1.2 yield (in)equalities

$$r \cdot \delta_R(R/\mathfrak{m}^l) = \delta_R((R/\mathfrak{m}^l)^r) \geqslant \delta_R(M) = r.$$

They imply $\delta_R(R/\mathfrak{m}^l) \geqslant 1$, which means $l \geqslant \operatorname{index} R$. □

To get lower bounds for $\operatorname{index} R$ we use a result of Ding. In order to state it we recall that a sequence $x_1, \ldots, x_d$ of elements of $\mathfrak{m}$ is said to be *super-regular* if their initial forms in the associated graded ring $\operatorname{gr}_{\mathfrak{m}}(R)$ form a regular sequence.

1.4. When R is Gorenstein, $\operatorname{gr}_{\mathfrak{m}}(R)$ is Cohen–Macaulay, and $\boldsymbol{x}$ is a super-regular sequence of length $\dim R$ in $\mathfrak{m} \smallsetminus \mathfrak{m}^2$, the *proof* of [17, 2.1] yields an equality:

$$\operatorname{index} R = \ell\ell_R(R/R\boldsymbol{x}).$$

Lemma 1.5. *If the ring R is Gorenstein and $\operatorname{gr}_{\mathfrak{m}}(R)$ is Cohen–Macaulay, then*

$$\operatorname{index} R = \operatorname{reg} R + 1.$$

Proof. Set $G = \operatorname{gr}_{\mathfrak{m}}(R)$, and choose a presentation $G \cong P/J$, where P is a polynomial ring over k, generated by finitely many indeterminates of degree 1.

After a flat base change one may assume that k is infinite. One can then find in P_1 a sequence $\boldsymbol{y} = y_1, \ldots, y_d$, with $d = \dim G$, which is both G-regular and P-regular. For $i = 1, \ldots, d$ choose $x_i \in \mathfrak{m} \smallsetminus \mathfrak{m}^2$ with initial form $y_i \in G_1$. The sequence $\boldsymbol{x} = x_1, \ldots, x_d$ is then super-regular, and one has a chain of equalities

$$\begin{aligned} \operatorname{index} R &= \ell\ell_R(R/R\boldsymbol{x}) \\ &= \inf\bigl\{n \in \mathbb{N} \bigm| (G_1)^n\bigl(\operatorname{gr}_{\mathfrak{m}}(R/R\boldsymbol{x})\bigr) = 0\bigr\} \\ &= \inf\bigl\{n \in \mathbb{N} \bigm| (G_1)^n(G/G\boldsymbol{y}) = 0\bigr\} \\ &= \operatorname{reg}_P(G/G\boldsymbol{y}) + 1 \\ &= \operatorname{reg}_P G + 1 \end{aligned}$$

that come from 1.4, the definition of Loewy length, the isomorphism $\operatorname{gr}_{\mathfrak{m}}(R/R\boldsymbol{x}) \cong G/G\boldsymbol{y}$, see [34, 0.1], and standard properties of regularity. □

One last observation is needed before proving the theorem.

1.6. For every local ring R one has $\operatorname{reg} R \geqslant \operatorname{ord} R - 1$.

Indeed, let $\widehat{R} \cong Q/I$ be a minimal regular presentation; see 8.1. By passing to associated graded rings one gets from it an isomorphism $G \cong P/J$, where P is a polynomial ring in $e = \operatorname{edim} R$ indeterminates. One has $\operatorname{rank}_k P/(P_1)^{n+1} = \binom{n+e}{e}$, so $\operatorname{ord} R$ is equal to the least degree of a non-zero element in J. The isomorphism $\operatorname{Tor}_1^P(G,k) \cong J \otimes_P k$ of graded vector spaces then yields the desired inequality.

Proof of Theorem 1.1. Lemmas 1.3, 1.5, and 1.6, imply

$$\ell\ell_R M \geqslant \operatorname{reg} R + 1 \geqslant \operatorname{ord} R.$$

When k is infinite the R-module $M' = R/R\boldsymbol{x}$ from the proof of Lemma 1.5 has $\operatorname{proj\,dim}_R M' < \infty$. The computation there yields $\ell\ell_R M' = \operatorname{reg}_P G + 1$. □

In many situations good estimates of the regularity of graded rings are known. The next result provides one that seems to be new. It involves conormal modules, which also appear in our main result, see Theorem 10.1.

Proposition 1.7. *Suppose $G \cong P/J$ with P a standard graded polynomial ring over k and J a homogeneous ideal.*

If J/J^2 is isomorphic to $(\bigoplus_{j=1}^{c} G(-n_j)) \oplus C$ as graded G-modules, then

$$\operatorname{reg}_P G \geqslant \begin{cases} \sum_{j=1}^{c}(n_j - 1) & \text{when } C = 0, \\ \sum_{j=1}^{c}(n_j - 1) + (m-1) & \text{otherwise} \end{cases}$$

where m is the least degree of a non-zero element in J.

Proof. Let K be the Koszul complex on a k-basis of G_1, considered as a complex of graded G-modules. For each i one then has $\operatorname{Tor}_i^P(G,k) \cong \operatorname{H}_i(K)$ as graded k-vector spaces. The arguments in the proofs of [23, (2.3) and (2.1)], see Theorem 9.2, show that K is quasi-isomorphic to a complex of graded k-vector spaces $V \otimes_k W$, where

$$V = \cdots \xrightarrow{0} \bigwedge^{i+1}\left(\bigoplus_{j=1}^{c} k(-n_j)\right) \xrightarrow{0} \bigwedge^{i}\left(\bigoplus_{j=1}^{c} k(-n_j)\right) \xrightarrow{0} \cdots$$

and W satisfies $W_0 = k$ and $\partial(W_1) = 0 = \partial(W_2)$. The isomorphisms

$$V_c \cong k(-n) \quad \text{and} \quad \operatorname{H}(V \otimes_k W) \cong V \otimes_k \operatorname{H}(W),$$

where $n = \sum_{j=1}^{c} n_j$, define injective linear maps of graded vector spaces

$$k(-n) \otimes_k W_0 \longrightarrow \operatorname{Tor}_c^P(G,k), \tag{1.7.1}$$

$$k(-n) \otimes_k W_1 \longrightarrow \operatorname{Tor}_{c+1}^P(G,k). \tag{1.7.2}$$

From (1.7.1) one gets $\operatorname{reg}_P G \geqslant n - c$, which is the first inequality of the proposition. Set $q = \inf\{j \in \mathbb{Z} \mid (W_1)_j \neq 0\}$. The isomorphisms

$$\left(\bigoplus_{j=1}^{c} k(-n_j)\right) \oplus (C \otimes_P k) \cong J \otimes_P k \cong \operatorname{Tor}_1^P(G,k) \cong V_1 \oplus W_1$$

of graded k-vector spaces yield $W_1 \cong C \otimes_P k \subseteq J \otimes_P k$. Thus, $C \neq 0$ implies $q \geqslant m$, so (1.7.2) gives $\operatorname{reg} G \geqslant (m+n) - (c+1)$. This is the second desired inequality. □

The theorem is readily applicable to a class of complete intersection rings that includes all hypersurface rings. The notation is that of Theorem 1.1.

Example 1.8. Suppose the local ring $(R, \mathfrak{m}, k)$ is a *strict complete intersection* of type $(n_1, \ldots, n_c)$, meaning that for some isomorphism $\operatorname{gr}_{\mathfrak{m}}(R) \cong P/J$ the ideal J can be generated by a homogeneous P-regular sequence $\boldsymbol{g} = g_1, \ldots, g_c$ with $\deg g_j = n_j$.

For every R-module M of finite projective dimension one has

$$\ell\ell_R M \geqslant \sum_{j=1}^{c}(n_j - 1) + 1.$$

Indeed, the hypothesis implies an isomorphism $J/J^2 \cong \bigoplus_{j=1}^{c} \operatorname{gr}_{\mathfrak{m}}(R)(-n_j)$ of graded modules over $\operatorname{gr}_{\mathfrak{m}}(R)$, so Theorem 1.1 and Proposition 1.7 apply.

2. Levels in triangulated categories

To any object, say X, in a triangulated category we associate a numerical invariant that measures the minimal number of steps necessary to 'build' X out of objects belonging to some fixed reference class. Its definition uses techniques developed by Beilinson, Bernstein, and Deligne [9, §1.3], Bondal and Van den Bergh [11, §1], D. Christensen [16], and Rouquier [33], and is partly motivated by the work of these authors. Where our approach differs from theirs is in the focus on properties of individual objects, rather than on global invariants of triangulated categories.

In this section we review the constructions allowed in the building process alluded to above, define levels, and record some general features. Only basic properties of triangulated categories are needed, and they can be found in Krause's succinct exposition [25, §§1–3]; for more details we refer the reader to Neeman's book [28].

We use Σ to denote the suspension functor in a triangulated category. Let S be a subcategory—always assumed non-empty—of a triangulated category T. We say that S is *strict* if it is closed under isomorphisms in T; it is *full* if every morphism in T between objects in S is contained in S. A strict full subcategory S is *thick* if it is additive, closed under direct summands, and in any exact triangle $L \to M \to N \to \Sigma L$ in T when two of the objects L, M, N are in S so is the third. Note that every thick subcategory is triangulated.

2.1. *Operations on subcategories.* Let T be a triangulated category and A a subcategory of T. We define several closure operations on A in T.

2.1.1. The intersection of all strict and full subcategories of T that contain A and are closed under finite direct sums and all suspensions is denoted $\mathsf{add}^{\Sigma}(\mathsf{A})$; in [11] this subcategory is denoted $\operatorname{add}\mathsf{A}$.

2.1.2. The intersection of all full subcategories of T that contain A and are closed under retracts (equivalently, isomorphisms and direct summands) is denoted $\mathsf{smd}(\mathsf{A})$.

2.1.3. Given strict and full triangulated subcategories A and B of T, let $\mathsf{A} \star \mathsf{B}$ be the full subcategory whose objects are described as follows:

$$\mathsf{A} \star \mathsf{B} = \left\{ M \in \mathsf{T} \;\middle|\; \begin{array}{l} \text{there is an exact triangle} \\ L \to M \to N \to \Sigma L \\ \text{with } L \in \mathsf{A} \text{ and } N \in \mathsf{B} \end{array} \right\}.$$

This subcategory is strict. For every strict and full subcategory C of T one has

$$\mathsf{A} \star (\mathsf{B} \star \mathsf{C}) = (\mathsf{A} \star \mathsf{B}) \star \mathsf{C}$$

see [9, (1.3.10)]. Thus, the following notation is unambiguous:

$$\mathsf{A}^{\star n} = \begin{cases} \{0\} & \text{for } n = 0; \\ \mathsf{A} & \text{for } n = 1; \\ \overbrace{\mathsf{A} \star \cdots \star \mathsf{A}}^{n \text{ copies}} & \text{for } n \geqslant 2. \end{cases}$$

We refer to the objects of $\mathsf{A}^{\star n}$ as $(n-1)$-*fold extensions* of objects from A.

2.1.4. Let C be a subcategory of T. The intersection of all thick subcategories of T containing C is itself a thick subcategory, called the *thick closure* of C in T; in this paper it is denoted by $\mathsf{thick}_{\mathsf{T}}(\mathsf{C})$.

Properties of objects in a subcategory often propagate to its thick closure:

2.1.5. Let $\mathcal{P}$ denote a property of objects in T, and assume that the full subcategory consisting of the objects with property $\mathcal{P}$ is thick.

If each $C \in \mathsf{C}$ has property $\mathcal{P}$, then so does every object in $\mathsf{thick}_{\mathsf{T}}(\mathsf{C})$.

One can approximate $\mathsf{thick}_{\mathsf{T}}(\mathsf{C})$ 'from below' by a process used in [16], [11], [33].

2.2. *Thickenings.* Let C be a subcategory of T.

For each $n \in \mathbb{N}$ the nth *thickening* of C is the full subcategory with objects

$$\mathsf{thick}^n_{\mathsf{T}}(\mathsf{C}) = \begin{cases} \{0\} & \text{if } n = 0, \\ \mathsf{smd}(\mathsf{add}^{\Sigma}(\mathsf{C})) & \text{if } n = 1, \\ \mathsf{smd}(\mathsf{thick}^{n-1}_{\mathsf{T}}(\mathsf{C}) \star \mathsf{thick}^1_{\mathsf{T}}(\mathsf{C})) & \text{if } n \geqslant 2. \end{cases}$$

This subcategory appears implicitly in [16] and explicitly in [11], where it is denoted $\langle \mathsf{C} \rangle_n$ but is not named. It is closed under suspensions, finite direct sums, and retracts, but it is not necessarily closed under formation of exact triangles.

2.2.1. The equality below, see [11, §2.1, p. 5], provides an alternative description:

$$\mathsf{thick}^n_{\mathsf{T}}(\mathsf{C}) = \mathsf{smd}\big(\mathsf{add}^{\Sigma}(\mathsf{C})^{\star n}\big).$$

In words: The objects in the nth thickening of C are retracts of $(n-1)$-fold extensions of objects in $\mathsf{add}^{\Sigma}(\mathsf{C})$. Thus, the nth thickening can also be built out of the $(n-1)$st one by gluing objects to the left.

2.2.2. If C is contained in some thick subcategory S of T, then it follows from the definitions that for each integer $n \geqslant 0$ one has $\mathsf{thick}^n_{\mathsf{S}}(\mathsf{C}) = \mathsf{thick}^n_{\mathsf{T}}(\mathsf{C})$.

2.2.3. Letting $(-)^{\mathrm{op}}$ denote passage to the opposite category, from 2.2.1 one gets

$$\mathsf{thick}^n_{\mathsf{T}}(\mathsf{C})^{\mathrm{op}} = \mathsf{thick}^n_{\mathsf{T}^{\mathrm{op}}}\big(\mathsf{C}^{\mathrm{op}}\big).$$

2.2.4. The thickenings of C provide a natural filtration of its thick closure in T:

$$\{0\} = \mathsf{thick}^0_{\mathsf{T}}(\mathsf{C}) \subseteq \mathsf{thick}^1_{\mathsf{T}}(\mathsf{C}) \subseteq \cdots \subseteq \bigcup_{n\in\mathbb{Z}} \mathsf{thick}^n_{\mathsf{T}}(\mathsf{C}) = \mathsf{thick}_{\mathsf{T}}(\mathsf{C}).$$

In [11,33] the filtration above is used to define the dimension of T as the infimum of the integers $d \geqslant 0$ with the property that $\mathsf{thick}^{d+1}_{\mathsf{T}}(C) = \mathsf{T}$ for some object $C \in \mathsf{T}$. Here we derive from the filtration numerical invariants of the objects of T.

2.3. *Levels.* Let C be a subcategory of T.

To each object M in T we associate the number

$$\operatorname{level}^{\mathsf{C}}_{\mathsf{T}}(M) = \inf\bigl\{n \in \mathbb{N} \bigm| M \in \mathsf{thick}^n_{\mathsf{T}}(\mathsf{C})\bigr\}$$

and call it the C-*level of* M. It measures the number of steps required to build M out of $\mathsf{add}^{\Sigma}(\mathsf{C})$. Evidently, $\operatorname{level}^{\mathsf{C}}_{\mathsf{T}}(M) < \infty$ is equivalent to $M \in \mathsf{thick}_{\mathsf{T}}(\mathsf{C})$.

When C consists of a single object, C, we write $\operatorname{level}^{C}_{\mathsf{T}}(M)$ in place of $\operatorname{level}^{\{C\}}_{\mathsf{T}}(M)$.

Lemma 2.4. *For each object $M \in \mathsf{T}$ the following statements hold.*

(1) $\operatorname{level}^{\mathsf{C}}_{\mathsf{T}}(\Sigma^i M) = \operatorname{level}^{\mathsf{C}}_{\mathsf{T}}(M)$ *for all* $i \in \mathbb{Z}$.
(2) *If* $N \to M \to P \to \Sigma N$ *is an exact triangle in* T, *then*

$$\operatorname{level}^{\mathsf{C}}_{\mathsf{T}}(M) \leqslant \operatorname{level}^{\mathsf{C}}_{\mathsf{T}}(N) + \operatorname{level}^{\mathsf{C}}_{\mathsf{T}}(P).$$

(3) $\operatorname{level}^{\mathsf{C}}_{\mathsf{T}}(M' \oplus M'') = \max\{\operatorname{level}^{\mathsf{C}}_{\mathsf{T}}(M'), \operatorname{level}^{\mathsf{C}}_{\mathsf{T}}(M'')\}$.
(4) *If* C *is contained in a thick subcategory* S *of* T, *then*

$$\operatorname{level}^{\mathsf{C}}_{\mathsf{S}}(M) = \operatorname{level}^{\mathsf{C}}_{\mathsf{T}}(M).$$

(5) $\operatorname{level}^{\mathsf{C}}_{\mathsf{T}}(M) = \operatorname{level}^{\mathsf{C}^{\mathrm{op}}}_{\mathsf{T}^{\mathrm{op}}}(M)$.
(6) *If* $\mathsf{f}:\mathsf{T} \to \mathsf{U}$, *respectively,* $\mathsf{f}:\mathsf{T}^{\mathrm{op}} \to \mathsf{U}$, *is an exact functor of triangulated categories, then the following inequality holds:*

$$\operatorname{level}^{\mathsf{f}(\mathsf{C})}_{\mathsf{U}}\bigl(\mathsf{f}(M)\bigr) \leqslant \operatorname{level}^{\mathsf{C}}_{\mathsf{T}}(M).$$

Equality holds if there is a functor $\mathsf{g}:\mathsf{U} \to \mathsf{T}$ *with* $\mathsf{gf} \simeq \mathrm{id}^{\mathsf{T}}$, *respectively,* $\mathsf{g}:\mathsf{U} \to \mathsf{T}^{\mathrm{op}}$ *with* $\mathsf{gf} \simeq \mathrm{id}^{\mathsf{T}^{\mathrm{op}}}$; *in particular, when* f *is an equivalence.*

Proof. The (in)equalities in (1), (2), and (3) come from the definition of levels. Parts (4) and (5) come from 2.2.2 and 2.2.3, respectively. Part (6) clearly holds when f is defined on T; given (5), the case when it is defined on T^{op} follows. □

We close the general discussion of levels with some special features of these invariants in the case of the derived category of an associative ring.

2.5. *Levels of complexes of modules.* Let R be an associative ring.

In this paper R always acts on its modules from the left. Our gradings are usually 'homological'. Thus, complexes of R-modules have the form

$$M = \cdots \longrightarrow M_{n+1} \xrightarrow{\partial_{n+1}} M_n \xrightarrow{\partial_n} M_{n-1} \longrightarrow \cdots.$$

For each integer d the dth *suspension* of M is the complex defined by

$$\left(\Sigma^d M\right)_n = M_{n-d} \quad \text{and} \quad \partial^{\Sigma^d M} = (-1)^d \partial^M.$$

We identify R-modules with complexes concentrated in degree 0, and graded R-modules with complexes with zero differential. Quasi-isomorphisms are morphisms of complexes that induce isomorphisms in homology; they are flagged with the symbol $\simeq$, while $\cong$ is reserved for isomorphisms.

A complex of R-modules M is *bounded above*, respectively, *bounded below*, when $M_i = 0$ holds for all $i \gg 0$, respectively, for all $i \ll 0$; it is *bounded* when both conditions hold; it is *perfect* if it is quasi-isomorphic to a bounded complex of finite projective R-modules.

Let $\mathsf{D}(R)$ denote the unbounded derived category of the category of R-modules; see [25, 3.2] for a description of $\mathsf{D}(R)$ as a triangulated category. We extend the use of $\simeq$ to mark also isomorphisms in $\mathsf{D}(R)$.

2.5.1. Let S be a thick subcategory of $\mathsf{D}(R)$, such as the derived category of homologically bounded above (respectively, bounded below, bounded, or perfect) complexes. For each subcategory C in S, each $n \geqslant 0$, and each $M \in \mathsf{S}$ one has

$$\mathsf{thick}^n_{\mathsf{S}}(\mathsf{C}) = \mathsf{thick}^n_{\mathsf{D}(R)}(\mathsf{C}) \quad \text{and} \quad \operatorname{level}^{\mathsf{C}}_{\mathsf{S}}(M) = \operatorname{level}^{\mathsf{C}}_{\mathsf{D}(R)}(M)$$

by 2.2.2 and Lemma 2.4(4). We use the abbreviations

$$\mathsf{thick}^n_R(\mathsf{C}) = \mathsf{thick}^n_{\mathsf{D}(R)}(\mathsf{C}) \quad \text{and} \quad \operatorname{level}^{\mathsf{C}}_R(M) = \operatorname{level}^{\mathsf{C}}_{\mathsf{D}(R)}(M).$$

Levels of a complex have useful relations to the corresponding levels of its components. A similar relation to homology is contained in Proposition 3.10(2).

Lemma 2.5.2. *For every complex of R-modules M there is an inequality*

$$\operatorname{level}^{\mathsf{C}}_R(M) \leqslant \inf\left\{ \sum_{n\in\mathbb{Z}} \operatorname{level}^{\mathsf{C}}_R(L_n) \;\middle|\; L \simeq M \text{ in } \mathsf{D}(R) \right\}.$$

Proof. As $\operatorname{level}^{\mathsf{C}}_R(M) = \operatorname{level}^{\mathsf{C}}_R(L)$, it suffices to assume that only finitely many components L_n are non-zero and to show that then one has

$$\operatorname{level}^{\mathsf{C}}_R(L) \leqslant \sum_{n\in\mathbb{Z}} \operatorname{level}^{\mathsf{C}}_R(L_n).$$

The complex L admits a filtration $\cdots \subseteq L_{\leqslant n-1} \subseteq L_{\leqslant n} \subseteq \cdots$, and there are isomorphisms $L_{\leqslant n}/L_{\leqslant n-1} \cong \Sigma^n L_n$. Lemma 2.4(1) yields $\operatorname{level}^{\mathsf{C}}_R(\Sigma^n L_n) = \operatorname{level}^{\mathsf{C}}_R(L_n)$, so one gets the desired inequality by a repeated application of Lemma 2.4(2). $\square$

As an example, we deduce an inequality that appears in the introduction.

Example 2.5.3. If P is a complex of finite projective R-modules, then

$$\operatorname{level}_R^R(P) \leqslant \operatorname{card}\{n \in \mathbb{Z} \mid P_n \neq 0\}.$$

Just note that each P_n is in $\mathsf{thick}_R^1(R)$ and apply Lemma 2.5.2.

3. Levels of DG modules

In this section we move from modules over rings to DG (= differential graded) modules over DG algebras. We start by recalling some concepts and collecting notation and basic facts concerning DG algebras and DG modules.

Throughout this paper, we consider DG objects as collections of abelian groups indexed by the integers, rather than direct sums of such groups. This point view, prevalent among topologists, is systematically developed in MacLane's books; see especially [27, §VI.2]. A consequence is that every element m of a graded object M has a degree, denoted $|m|$; namely, one has $|m| = i$ if and only if m belongs to M_i. Differentials have degree -1. DG algebras act on their DG modules from the left.

Rings are identified with DG algebras concentrated in degree 0. DG modules over a ring are just complexes of modules, and modules over it are identified with complexes concentrated in degree 0.

A *graded algebra* is a DG algebra with zero differential. It should be noted that in a DG algebra elements of different degrees cannot be added. Furthermore, DG modules over a graded algebra A should not be confused with complexes over it: The first ones have a unique grading and their differentials satisfy the identity $\partial(am) = (-1)^{|a|}a\partial(m)$, while the second ones are equipped with two gradings (homological and internal), and their differentials are A-linear.

Let A be a DG algebra and M, N be left DG modules.

A *homomorphism* $\beta : M \to N$ of degree d is a family $(\beta_i : M_i \to N_{i+d})_{i\in\mathbb{Z}}$ of additive maps satisfying $\beta(am) = (-1)^{d|a|}a\beta(m)$ for $a \in A$ and $m \in M$. All such homomorphisms form the dth component of a complex, $\operatorname{Hom}_A(M, N)$, whose differential is given by $\partial(\beta) = \partial^N\beta - (-1)^{|\beta|}\beta\partial^M$. A *morphism* of DG modules $M \to N$ is a homomorphism β of degree 0, satisfying $\partial^N\beta = \beta\partial^M$. A *quasi-isomorphism* is a morphism that induces an isomorphism in homology.

We let $A^\natural$ and $M^\natural$ denote the underlying graded algebra and graded module over it. For each integer s let $\Sigma^s M^\natural$ denote the graded $\mathbb{Z}$-module with $(\Sigma^s M^\natural)_i = M_{i-s}$. The map $m \mapsto m$ is a homomorphism $\sigma^s : M^\natural \to \Sigma^s M^\natural$ of degree s. Let $\Sigma^s M$ denote the DG A-module with underlying graded $\mathbb{Z}$-module $\Sigma^s M^\natural$, action of A given by $a\sigma^s(m) = (-1)^{|a|s}\sigma^s(am)$, and differential by $\partial(\sigma^s(m)) = (-1)^s\sigma^s(\partial(m))$.

Proofs of statements in 3.1–3.4 below can be found in [7].

3.1. *Semi-free DG modules.* A DG module F over a DG algebra A is *semi-free* if it admits a family $(F^n)_{n\in\mathbb{Z}}$ of DG A-submodules satisfying the conditions:

$$F^n \subseteq F^{n+1}, \qquad F^{-1} = 0, \qquad \bigcup_{n\in\mathbb{Z}} F^n = F, \quad \text{and}$$

$$F^{n+1}/F^n \quad \text{is isomorphic to a direct sum of suspensions of } A.$$

In this case, the functors $\operatorname{Hom}_A(F,-)$ and $-\otimes_A F$, defined on the category of DG A-modules and right DG A-modules, respectively, preserve quasi-isomorphisms.

Thus, if A is an algebra over a field k the functor $\operatorname{Hom}_A(-,\operatorname{Hom}_k(F,k))$ preserves quasi-isomorphisms of right DG A-modules, as it is isomorphic to $\operatorname{Hom}_k(-\otimes_A F,k)$.

3.2. *Semi-free resolutions.* Each DG A-module M admits a quasi-isomorphism $F\to M$ with F a semi-free DG A-module. Such a *semi-free resolution* of M is unique up to homotopy of DG A-modules.

3.3. *Derived categories.* Let A be a DG algebra. We let $\mathsf{D}(A)$ denote the derived category of DG A-modules. Its objects are DG A-modules, and it can be realized as the homotopy category of semi-free DG A-modules. The derived category is triangulated, see [24, §4] or [25, 3.2] for constructions.

We consider every ring R as a DG algebra concentrated in degree zero. Its DG modules are then simply the complexes of R-modules, and the derived category $\mathsf{D}(R)$ coincides with the derived category of R-modules.

3.4. *Derived functors.* For each DG A-module one sets

$$\mathbf{R}\operatorname{Hom}_A(M,-)=\operatorname{Hom}_A(F,-)\quad\text{and}\quad\left(-\otimes^{\mathbf{L}}_A M\right)=(-\otimes_A F),$$

where $F\to M$ is some semi-free resolution. This yields well defined exact functors on the derived category of DG A-modules and right DG A-modules, respectively.

Given a subcategory C of $\mathsf{D}(A)$ and a DG A-module M we write $\mathsf{thick}^n_A(\mathsf{C})$ and $\operatorname{level}^{\mathsf{C}}_A(M)$ in place of $\mathsf{thick}^n_{\mathsf{D}(A)}(\mathsf{C})$ and $\operatorname{level}^{\mathsf{C}}_{\mathsf{D}(A)}(M)$, respectively.

We record some easy consequences of general properties of levels.

3.5. Let A be a DG algebra, M a DG A-module, and $\mathsf{C}\subseteq\mathsf{D}(A)$ a subcategory.

3.5.1. If M has a filtration by DG submodules

$$0=M^0\subseteq M^1\subseteq\cdots\subseteq M^m=M$$

then the following inequality holds:

$$\operatorname{level}^{\mathsf{C}}_A(M)\leqslant\sum_{i=1}^{m}\operatorname{level}^{\mathsf{C}}_A\left(M^i/M^{i-1}\right).$$

Indeed, this is seen through iterated applications of Lemma 2.4(2) to the triangles in $\mathsf{D}(A)$ defined by the exact sequences of DG A-modules

$$0\to M^{i-1}\to M^i\to M^i/M^{i-1}\to 0.$$

3.5.2. Let B be a DG algebra and L a left–right B–A-bimodule with $\mathrm{H}(L\otimes^{\mathbf{L}}_A C)$ a noetherian, respectively, artinian, $\mathrm{H}(B)$-module for each $C\in\mathsf{C}$. If $\operatorname{level}^{\mathsf{C}}_A(M)$ is finite, then the $\mathrm{H}(B)$-module $\mathrm{H}(L\otimes^{\mathbf{L}}_A M)$ is noetherian, respectively, artinian.

Indeed, as $(L \otimes^{\mathbf{L}}_A -)$ is an exact functor from $\mathsf{D}(A)$ to $\mathsf{D}(B)$, Lemma 2.4(6) shows that $L \otimes^{\mathbf{L}}_A M$ has finite $(L \otimes^{\mathbf{L}}_A \mathsf{C})$-level, so the desired assertion follows from 2.1.5.

Next we consider how levels behave under changes of DG algebras.

3.6. *Morphisms of DG algebras.* Let $\varphi : A \to B$ be a morphism of DG algebras.

The following property is used implicitly in many arguments.

3.6.1. The map φ induces an adjoint pair of functors of triangulated categories

$$\mathsf{D}(A) \underset{\varphi_*}{\overset{(B \otimes^{\mathbf{L}}_A -)}{\rightleftarrows}} \mathsf{D}(B)$$

where φ_* is the functor restricting the action of B to A. In particular, for every $M \in \mathsf{D}(A)$ and every $N \in \mathsf{D}(B)$ there are canonical morphisms

$$M \longrightarrow \varphi_*\big(B \otimes^{\mathbf{L}}_A M\big) \quad \text{and} \quad B \otimes^{\mathbf{L}}_A \varphi_*(N) \longrightarrow N.$$

3.6.2. If φ is a quasi-isomorphism, then $(B \otimes^{\mathbf{L}}_A -)$ and φ_* are inverse equivalences.

Indeed, with U a semi-free resolution of the DG module M over A, see 3.2, the morphism $M \to \varphi_*(B \otimes^{\mathbf{L}}_A M)$ is represented by $\varphi \otimes_A U$; it is a quasi-isomorphism since φ is one. Let $\nu : V \to N$ be a semi-free resolution of $\varphi_*(N)$ over A. The morphism $B \otimes^{\mathbf{L}}_A \varphi_*(N) \to N$ is represented by the morphism $\mu : B \otimes_A V \to N$, where $b \otimes v \mapsto b\nu(v)$. The quasi-isomorphism ν factors as

$$V \xrightarrow{\varphi \otimes_A V} B \otimes_A V \xrightarrow{\mu} N.$$

Since V is semi-free, $\varphi \otimes_A V$ is a quasi-isomorphism, and hence so is μ.

3.6.3. Let M and N be DG modules over A and B, respectively, and let $\mu : M \to N$ be a morphism of complexes of abelian groups satisfying $\mu(am) = \varphi(a)\mu(m)$ for all $a \in A$ and $m \in M$. If φ and μ are quasi-isomorphisms, then in $\mathsf{D}(A)$ and $\mathsf{D}(B)$, respectively, one has canonical isomorphisms

$$M \simeq \varphi_*(N) \quad \text{and} \quad B \otimes^{\mathbf{L}}_A M \simeq N.$$

Indeed, $M \simeq \varphi_*(N)$ holds by assumption, so 3.6.2 yields isomorphisms

$$B \otimes^{\mathbf{L}}_A M \simeq B \otimes^{\mathbf{L}}_A \varphi_*(N) \simeq N.$$

Now we can track the behavior of levels under change of DG algebras.

Proposition 3.7. *Let $\varphi : A \to B$ be a morphism of DG algebras.*

For all DG A-modules C, M and all DG B-modules D, N the following hold.

(1) *There are inequalities*

$$\operatorname{level}_A^C(M) \geqslant \operatorname{level}_B^{B\otimes_A^{\mathbf{L}}C}\big(B \otimes_A^{\mathbf{L}} AM\big),$$
$$\operatorname{level}_A^{\varphi_*(D)}\big(\varphi_*(N)\big) \leqslant \operatorname{level}_B^D(N).$$

Equalities hold when φ *is a quasi-isomorphism.*

(2) *If both* $\operatorname{level}_A^A(\varphi_*(B))$ *and* $\operatorname{level}_B^B(N)$ *are finite, then so is* $\operatorname{level}_A^A(\varphi_*(N))$.

Proof. (1) Lemma 2.4(6) yields the inequalities and shows that they become equalities if $B \otimes_A^{\mathbf{L}} -$ and φ_* are equivalences; now refer to 3.6.2.

(2) Since $\operatorname{level}_B^B(N)$ is finite, (1) shows that so is $\operatorname{level}_A^{\varphi_*(B)}(\varphi_*(N))$. It follows that $\varphi_*(N)$ is in $\mathsf{thick}_A(\varphi_*(B))$. As $\operatorname{level}_A^A(\varphi_*(B))$ is finite, one has an inclusion

$$\mathsf{thick}_A\big(\varphi_*(B)\big) \subseteq \mathsf{thick}_A(A).$$

Thus, $\varphi_*(N)$ is in $\mathsf{thick}_A(A)$, which means $\operatorname{level}_A^A(\varphi_*(N))$ is finite, as desired. □

The balance of this section deals with special properties of DG algebras concentrated either in non-negative degrees or in non-positive degrees.

3.8. A DG algebra A is *non-negative* if it has $A_n = 0$ for all $n < 0$.

When A is a non-negative DG algebra the subcomplex

$$J = \cdots \longrightarrow A_2 \longrightarrow A_1 \longrightarrow \operatorname{Im}(\partial_1) \longrightarrow 0 \longrightarrow \cdots$$

is a DG ideal of A, and A/J is naturally isomorphic to $\mathrm{H}_0(A)$. The morphism of DG algebras $\varepsilon : A \to \mathrm{H}_0(A)$ is called the *canonical augmentation* of A. Via the functor $\varepsilon_* : \mathsf{D}(\mathrm{H}_0(A)) \to \mathsf{D}(A)$ we identify complexes of $\mathrm{H}_0(A)$-modules with DG A-modules; the same letter denotes a complex in $\mathsf{D}(\mathrm{H}_0(A))$ and its image in $\mathsf{D}(A)$.

Proposition 3.9. *Let A and B be non-negative DG algebras.*

If $\mathsf{f} : \mathsf{D}(A) \to \mathsf{D}(B)$ *is an equivalence of triangulated categories induced by a chain of quasi-isomorphisms of DG algebras, then in* $\mathsf{D}(B)$ *one has an isomorphism*

$$\mathsf{f}\big(\mathrm{H}_0(A)\big) \simeq \mathrm{H}_0(B).$$

Proof. By hypothesis, there exists a sequence of quasi-isomorphisms

$$A \xrightarrow{\simeq} A^0 \xleftarrow{\simeq} A^1 \xrightarrow{\simeq} \cdots \xleftarrow{\simeq} A^{i-1} \xrightarrow{\simeq} A^i \xleftarrow{\simeq} B$$

of DG algebras. For $j = 0, \dots, i$ the subcomplex

$$A_+^j = \cdots \longrightarrow A_2^j \longrightarrow A_1^j \longrightarrow \operatorname{Ker}\big(\partial_0^{A^j}\big) \longrightarrow 0 \longrightarrow \cdots$$

is a functorially defined DG subalgebra of A^j, and the inclusion $A^j_+ \subseteq A^j$ is a quasi-isomorphism. Thus, the original sequence induces a commutative diagram

$$\begin{array}{ccccccccccccc}
A & \xrightarrow{\simeq} & A^0_+ & \xleftarrow{\simeq} & A^1_+ & \xrightarrow{\simeq} & \cdots & \xleftarrow{\simeq} & A^{i-1}_+ & \xrightarrow{\simeq} & A^i_+ & \xleftarrow{\simeq} & B \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{H}_0(A) & \xrightarrow{\cong} & \mathrm{H}_0(A^0_+) & \xleftarrow{\cong} & \mathrm{H}_0(A^1_+) & \xrightarrow{\cong} & \cdots & \xleftarrow{\cong} & \mathrm{H}_0(A^{i-1}_+) & \xrightarrow{\cong} & \mathrm{H}_0(A^i_+) & \xleftarrow{\cong} & \mathrm{H}_0(B)
\end{array}$$

of isomorphisms of DG algebras, where all DG algebras in the top row are non-negative and all vertical arrows are canonical augmentations. Iterated applications of 3.6.3 produce the desired isomorphism in $\mathsf{D}(B)$. □

Proposition 3.10. *Let A be a non-negative DG algebra.*
For every DG A-module the following hold.

(1) *If* $\inf\{n \in \mathbb{Z} \mid \mathrm{H}_n(M) \neq 0\} = i > -\infty$, *then there is an exact triangle*

$$M' \longrightarrow M \longrightarrow \Sigma^i\big(\mathrm{H}_i(M)\big) \longrightarrow \Sigma M'$$

in $\mathsf{D}(A)$, *with* $\mathrm{H}_i(M') = 0$ *and* $\mathrm{H}_n(M') \cong \mathrm{H}_n(M)$ *for* $n \neq i$.

(2) *For every class* C *of* $\mathrm{H}_0(A)$*-modules one has*

$$\operatorname{level}^{\mathsf{C}}_A(M) \leqslant \sum_{n\in\mathbb{Z}} \operatorname{level}^{\mathsf{C}}_{\mathrm{H}_0(A)}\big(\mathrm{H}_n(M)\big).$$

Proof. (1) Since A is non-negative, the following subcomplexes

$$M' = \cdots \longrightarrow M_{i+1} \longrightarrow \operatorname{Im}(\partial_{i+1}) \longrightarrow 0 \longrightarrow \cdots,$$

$$M'' = \cdots \longrightarrow M_{i+1} \longrightarrow \operatorname{Ker}(\partial_i) \longrightarrow 0 \longrightarrow \cdots$$

of M are closed under multiplication by elements on A; in other words, they are DG submodules of M. It remains to observe that the inclusion $M'' \subseteq M$ is a quasi-isomorphism, and that one has an exact sequence of DG A-modules

$$0 \longrightarrow M' \longrightarrow M'' \longrightarrow \Sigma^i \mathrm{H}_i(M) \longrightarrow 0.$$

It gives rise to an exact triangle with the desired properties.

(2) We may assume that the number $w(M) = \operatorname{card}\{n \in \mathbb{N} \mid \mathrm{H}_n(M) \neq 0\}$ is finite. As $w(M) = 0$ means $M \simeq 0$, the desired inequality is evident in this case, so we may assume that it holds for DG modules N with $w(N) < r$ for some $r \geqslant 1$.

Set $i = \inf\{n \mid \mathrm{H}_n(M) \neq 0\}$. Lemma 2.4(2) applied to the exact triangle in (1) and Lemma 2.4(1) yield the first relation below:

$$\begin{aligned}
\operatorname{level}^{\mathsf{C}}_{A}(M) &\leqslant \operatorname{level}^{\mathsf{C}}_{A}\big(\mathrm{H}_i(M)\big) + \operatorname{level}^{\mathsf{C}}_{A}(M')\\
&\leqslant \operatorname{level}^{\mathsf{C}}_{\mathrm{H}_0 A}\big(\mathrm{H}_i(M)\big) + \operatorname{level}^{\mathsf{C}}_{A}(M')\\
&\leqslant \operatorname{level}^{\mathsf{C}}_{\mathrm{H}_0 A}\big(\mathrm{H}_i(M)\big) + \sum_{n\geqslant i+1} \operatorname{level}^{\mathsf{C}}_{\mathrm{H}_0 A}\big(\mathrm{H}_n(M')\big)\\
&= \sum_{n\geqslant i} \operatorname{level}^{\mathsf{C}}_{\mathrm{H}_0 A}\big(\mathrm{H}_n(M)\big).
\end{aligned}$$

Proposition 3.7(1), applied to the morphism $\varepsilon : A \to \mathrm{H}_0(A)$, yields the second one. As $w(M') = r-1$ by part (1), the third inequality is the induction hypothesis. The equality uses the expressions for the modules $\mathrm{H}_n(M')$, again from (1). □

3.11. A DG algebra A with $A_n = 0$ for $n > 0$ is said to be *non-positive*. Such a DG algebra has no augmentation to $\mathrm{H}_0(A)$ in general. However, the subcomplex

$$J = \cdots \longrightarrow 0 \longrightarrow A_{-1} \longrightarrow A_{-2} \longrightarrow \cdots$$

is a DG ideal of A. Thus, when $\partial(A_0) = 0$ one has natural isomorphisms $A/J \cong A_0 \cong \mathrm{H}_0(A)$, and hence a *canonical augmentation* $\varepsilon : A \to \mathrm{H}_0(A)$. As in 3.8, for every complex $M \in \mathsf{D}(\mathrm{H}_0(A))$ we let M denote also the DG A-module $\varepsilon_*(M)$.

Proposition 3.12. *If A is a non-positive DG algebra with $\partial(A_0) = 0$ and A_0 semi-simple, and M is a DG A-module, then the following hold.*

(1) *If* $\sup\{n \in \mathbb{Z} \mid \mathrm{H}_n(M) \neq 0\} = s < \infty$, *then there is an exact triangle*

$$M' \longrightarrow M \longrightarrow \Sigma^s \mathrm{H}_s(M) \longrightarrow \Sigma M'$$

in $\mathsf{D}(A)$, with $\mathrm{H}_s(M') = 0$ and $\mathrm{H}_n(M') \cong \mathrm{H}_n(M)$ for $n \neq s$.

(2) *For every class C of $\mathrm{H}_0(A)$-modules one has*

$$\operatorname{level}^{\mathsf{C}}_{A}(M) \leqslant \sum_{n\in\mathbb{Z}} \operatorname{level}^{\mathsf{C}}_{\mathrm{H}_0 A}\big(\mathrm{H}_n(M)\big).$$

Proof. (1) The hypothesis $\partial(A_0) = 0$ implies ∂^M is A_0-linear. As A_0 is semi-simple,

$$0 \longrightarrow \operatorname{Ker}(\partial_s) \longrightarrow M_s \longrightarrow \operatorname{Im}(\partial_s) \longrightarrow 0,$$
$$0 \longrightarrow \operatorname{Im}(\partial_{s+1}) \longrightarrow \operatorname{Ker}(\partial_s) \longrightarrow \mathrm{H}_s(M) \longrightarrow 0$$

are split-exact sequences of A_0-modules. Thus, there are isomorphisms

$$M_s \cong \operatorname{Im}(\partial_{s+1}) \oplus \mathrm{H}_s(M) \oplus C \quad \text{and} \quad C \cong \operatorname{Im}(\partial_s)$$

of A_0-modules, the latter induced by ∂_s. As A is non-positive,

$$M' = \cdots \longrightarrow 0 \longrightarrow C \xrightarrow{\partial_s} M_{s-1} \longrightarrow \cdots,$$

$$M'' = \cdots \longrightarrow 0 \longrightarrow \mathrm{H}_s(M) \oplus C \xrightarrow{\partial_s} M_{s-1} \longrightarrow \cdots$$

are DG A-submodules of M and there is an exact sequence of DG A-modules

$$0 \longrightarrow M' \longrightarrow M'' \longrightarrow \Sigma^s \mathrm{H}_s(M) \longrightarrow 0.$$

The inclusion $M'' \subseteq M$ is a quasi-isomorphism, so it yields the desired exact triangle; the homology of M' is computed from the homology exact sequence.

(2) follows from (1) by an argument parallel to that for Proposition 3.10(2). □

4. Perfect DG modules

We say that a DG module M over a DG algebra A is *perfect* if $\mathrm{level}_A^A(M)$ is finite. The first result of this section describes the structure of perfect DG modules. Specialized to the case of rings it shows that our terminology is consistent with the traditional notion for complexes, see 2.5. We extend this characterization to DG modules over non-negative DG algebras.

Over certain DG algebras which arise in many applications we establish a homological and hence verifiable criterion for perfection. We finish the section with examples showing that the hypotheses of this last result cannot be relaxed easily.

4.1. *Semi-freeness.* A *semi-free filtration* of a DG A-module F is a family $(F^n)_{n\in\mathbb{Z}}$ of DG A-submodules satisfying the conditions in 3.1. Such a filtration $(F^n)_{n\in\mathbb{Z}}$ has *class at most l* if $F^l = F$ holds for some integer l; it is *finite* if, in addition, its subquotients are finitely generated.

A DG A-module admitting a (finite) semi-free filtration (of class at most l) is said to be (*finite*) *semi-free* (*of class at most l*). Note that 0 is the only DG module that is semi-free of class at most -1.

The next theorem suggests that the A-level of a DG A-module may be thought of as a kind of 'projective dimension'.

Theorem 4.2. *Let A be a DG algebra and l a non-negative integer.*

A DG A-module M has $\mathrm{level}_A^A(M) \leqslant l$ *if and only if it is a retract of some finite semi-free DG module of class at most $l-1$.*

Proof. Let F denote the full subcategory $\mathsf{add}^{\Sigma}(A)$ of $\mathsf{D}(A)$; its objects are the DG modules isomorphic to finite direct sums of suspended copies of A, see 2.1.1. In view of 2.2.1, it suffices to prove that $\mathsf{F}^{\star l}$ is the smallest strict subcategory of $\mathsf{D}(A)$ that contains all finite semi-free DG modules of class at most $l-1$.

Indeed, a semi-free filtration $(F^n)_{n\in\mathbb{Z}}$ of class at most $l-1$ yields exact triangles

$$F^{n-1} \longrightarrow F^n \longrightarrow F^n/F^{n-1} \longrightarrow \Sigma F^{n-1} \quad \text{for } 1 \leqslant n \leqslant l-1.$$

Since F^0 and F^n/F^{n-1} are in F, induction shows that F is in $\mathsf{F}^{\star l}$.

Conversely, assume that M is in $\mathsf{F}^{\star l}$ for some $l \geqslant 0$. By induction on l we prove that M is isomorphic to a finite semi-free DG module F of class at most $l-1$. For $l=0$ the assertion is evident. For $l \geqslant 1$ one has in $\mathsf{D}(A)$ an exact triangle

$$G \xrightarrow{\gamma} L \longrightarrow M \longrightarrow \Sigma G$$

with L in $\mathsf{F}^{(l-1)\star}$ and G in F. By the induction hypothesis, L is isomorphic to a DG A-module with a finite semi-free filtration of class $l-2$. Thus, there is an exact triangle as above, where L has a finite semi-free filtration (L^n) with $L^{l-2}=L$.

Since G is in F, the triangle above is isomorphic to a triangle

$$\widetilde{G} \xrightarrow{\widetilde{\gamma}} L \longrightarrow M \longrightarrow \Sigma\widetilde{G}$$

where $\widetilde{G}$ is a finite direct sum of suspended copies of A. Set $F = \mathsf{cone}(\widetilde{\gamma})$. One then has $M \simeq F$ and an exact sequence of DG A-modules

$$0 \longrightarrow L \xrightarrow{\lambda} F \longrightarrow \Sigma\widetilde{G} \longrightarrow 0.$$

One gets a finite semi-free filtration of F with $F^{l-1}=F$ by setting

$$F^n = \begin{cases} \lambda(L^n) & \text{for } n \leqslant l-2; \\ F & \text{for } n \geqslant l-1. \end{cases}$$

It shows that the class of F is at most $l-1$, as desired. □

By Theorem 4.2 a DG A-module M of finite A-level is isomorphic in $\mathsf{D}(A)$ to a DG module P with $P^\natural$ finite projective over $A^\natural$. Next we prove that the converse holds when A is non-negative, in particular, when M is a complex over a ring.

In the argument, we use the following classical fact on the structure of graded projective modules over graded rings; see [38, 6.6].

4.3. Let B be a non-negatively (respectively, non-positively) graded algebra.

For each bounded below (respectively, bounded above) projective graded B-module N there is an isomorphism of graded B-modules $N \cong V \otimes_{B_0} B$, with V a bounded below (respectively, bounded above) projective graded B_0-module.

The B_0-module V is defined uniquely up to isomorphism: $V \cong (B/J) \otimes_B N$, where J denotes the ideal of elements of positive (respectively, negative) degree.

Proposition 4.4. *Let A be a non-negative DG algebra.*

A bounded below DG A-module M is a direct summand of some (finite) semi-free DG A-module if and only if the underlying graded $A^\natural$-module is (finite) projective.

Proof. The 'only if' part is evident. For the converse, set $R = A_0$. By 4.3, one has $M^\natural \cong V \otimes_R A^\natural$ for some bounded below graded R-module V with each V_n projective. Pick for each $n \in \mathbb{Z}$ an R-module W_n so that $V_n \oplus W_n$ is free. If $M^\natural$ is finite over $A^\natural$, each V_n is finite over R, so choose W_n finite, as one may. Let W be the complex of R-modules having W_n as nth component and $\partial^W = 0$.

Form the DG A-module $F = M \oplus (W \otimes_R A)$. As A is non-negative,

$$F^n = (V \oplus W)_{\leqslant n} \otimes_R A$$

is a DG submodule of F, and $\cdots \subseteq F^n \subseteq F^{n+1} \subseteq \cdots$ is a semi-free filtration of F; it is finite when the R-module W is finite. □

Theorem 4.2 and Proposition 4.4 yield:

Corollary 4.5. *A DG module over a non-negative DG algebra A is perfect if it is quasi-isomorphic to a DG module P with $P^\natural$ finite projective over $A^\natural$.*

From the preceding corollary and Proposition 3.7(2) one gets:

Corollary 4.6. *Let $\varphi : A \to B$ be a morphism of DG algebras such that A is non-negative and the graded $A^\natural$-module $\varphi_*(B)^\natural$ is finite projective.*

If N is a perfect DG B-module, then the DG A-module $\varphi_*(N)$ is perfect.

Our next goal is to obtain a homological criterion for perfection. We start by noting that 3.5.2 applied to the class $\mathsf{C} = \{A\}$ gives a homological obstruction:

Remark 4.7. Let A and B be DG algebras, L a left–right B–A-bimodule, and M a perfect DG A-module.

If the graded $\mathrm{H}(B)$-module $\mathrm{H}(L)$ is noetherian (respectively, artinian) then the graded $\mathrm{H}(B)$-module $\mathrm{H}(L \otimes^{\mathbf{L}}_A M)$ is noetherian (respectively, artinian).

Under additional hypotheses, we prove that finiteness is the only obstruction; for the definition of the canonical augmentation, see Remarks 3.8 and 3.11.

Theorem 4.8. *Let A be a DG algebra and M a DG A-module, such that*

(a) *A is non-negative, $\partial(A_1) = 0$, and $\mathrm{H}(M)$ is bounded below; or*
(b) *A is non-positive, $A_{-1} = 0$, and $\mathrm{H}(M)$ is bounded above.*

Set $k = A_0$ and let $\varepsilon : A \to \mathrm{H}_0(A) = k$ denote the canonical augmentation.

If k is a field, then the following conditions are equivalent.

(i) *M is perfect over A.*
(ii) *$\mathrm{H}(k \otimes^{\mathbf{L}}_A M)$ is finite over k.*
(iii) *$M \simeq F$ for some finite semi-free $F \in \mathsf{D}(A)$.*
(iv) *$M \simeq F$ for some $F \in \mathsf{D}(A)$ with $F^\natural$ a finite projective graded $A^\natural$-module.*

When they are satisfied, the inequality below holds:

$$\mathrm{level}^A_A(M) \leqslant \mathrm{card}\bigl\{n \in \mathbb{Z} \bigm| \mathrm{H}_n\bigl(k \otimes^{\mathbf{L}}_A M\bigr) \neq 0\bigr\}.$$

In our argument, we use the existence of minimal semi-free resolutions, see [7]:

4.9. *Minimal resolutions.* Let A be a DG algebra, and M a DG A-module, as in Theorem 4.8. Assume $\mathrm{H}_0(A)$ is a field. Set $J = \{a \in A\colon |a| \neq 0\}$; this is a DG ideal of A, and there exists a semi-free resolution $F \to M$ with the property $\partial(F) \subseteq JF$.

Proof of Theorem 4.8. Set $J = \mathrm{Ker}(\varepsilon)$.

(i) $\Longrightarrow$ (iv). This follows from Theorem 4.2.

(iv) $\Longrightarrow$ (iii). By 4.3 the $A^\natural$-module $F^\natural$ is free. We induce on $r = \mathrm{rank}_{A^\natural} F^\natural$.

For $r = 0$ the assertion is obvious. Fix $r \geqslant 1$ and assume that the assertion holds for all DG modules with underlying graded module of smaller rank. Choose a non-zero element e of F, with $|e|$ minimal in case (a) and maximal in case (b). One can then find a basis $\{e_1, \dots, e_r\}$ of $F^\natural$ over $A^\natural$, with $e_1 = e$.

If $\partial(e) = 0$, then $\overline{F} = F/Ae$ is a DG A-module with $(\overline{F})^\natural$ free of rank $r - 1$, so by the induction hypothesis it has a finite semi-free filtration $(\overline{F}^n)$. Setting $F^0 = Ae$ and letting F^n denote the inverse image in F of $\overline{F}^{n-1}$ for $n \geqslant 1$, we obtain a finite semi-free filtration of F. This covers case (a), as well as case (b) for $r = 1$.

Finally, assume (b) holds, $r \geqslant 2$, and $\partial(e) = f \neq 0$. Set $|e| = j$ and $J = A_{\leqslant -2}$. As $|f| = j - 1$, and $(JF)_i = 0$ for $i \geqslant j - 1$, we get $f \notin JF$. By Nakayama's Lemma $F^\natural$ has a basis $\{e_1, \dots, e_r\}$ with $e_1 = e$ and $e_2 = f$. The graded submodule $E^\natural$ of F generated by e and f is a DG submodule, and has $\mathrm{H}(E) = 0$. The exact sequence of DG modules $0 \to E \to F \to F/E \to 0$ yields a quasi-isomorphism $F \simeq F/E$ and shows that $(F/E)^\natural$ is free of rank $r - 2$ over $A^\natural$. Thus we get $M \simeq F/E$ in $\mathsf{D}(A)$, and the induction hypothesis implies that F/E is semi-free.

(iii) $\Longrightarrow$ (ii). This is due to the isomorphism $\mathrm{H}(k \otimes^{\mathbf{L}}_A M) \cong \mathrm{H}(k \otimes_A F)$.

(ii) $\Longrightarrow$ (i). It suffices to show that (ii) implies the inequality in the statement of the theorem. Setting $V = \mathrm{H}(k \otimes^{\mathbf{L}}_A M)$ we argue by induction on $v = \mathrm{rank}_k V$.

Choose a quasi-isomorphism $F \to M$ with F a semi-free DG A-module and $\partial(F) \subseteq JF$; see 4.9. One then has isomorphisms of graded k-vector spaces

$$k \otimes_A F = \mathrm{H}(k \otimes_A F) \cong \mathrm{H}\big(k \otimes^{\mathbf{L}}_A M\big) = V.$$

From 4.3 one gets $F^\natural \cong V \otimes_k A^\natural$. In particular, $v = 0$ implies $F = 0$, and hence $\mathrm{level}^A_A(F) = 0$. This is the basis for our induction.

Assume $v \geqslant 1$. For $i = \inf\{n \in \mathbb{Z} \mid V_n \neq 0\}$ and $s = \sup\{n \in \mathbb{Z} \mid V_n \neq 0\}$ we have

$$\partial(V_i \otimes_k 1) \subseteq (V \otimes_k A_{\geqslant 1})_{i-1} = 0 \quad \text{in case (a);}$$
$$\partial(V_s \otimes_k 1) \subseteq (V \otimes_k A_{\leqslant -1})_{s-1} = V_s \otimes_k A_{-1} = 0 \quad \text{in case (b).}$$

Setting $j = i$ in case (a) and $j = s$ in case (b), we get an exact sequence

$$0 \longrightarrow \big(\Sigma^j V_j\big) \otimes_k A \longrightarrow F \longrightarrow G \longrightarrow 0$$

of DG A-modules. The homology exact sequence of the functor $(k \otimes^{\mathbf{L}}_A -)$ splits:

$$0 \longrightarrow \Sigma^j V_j \longrightarrow V \longrightarrow \mathrm{H}\big(k \otimes^{\mathbf{L}}_A G\big) \longrightarrow 0.$$

In particular, we get $\mathrm{rank}_k \mathrm{H}(k \otimes^{\mathbf{L}}_A G) < v$. The DG module G is bounded below in case (a) and above in case (b), so the induction hypothesis applies to it. Since $\mathrm{level}^A_A(V \otimes_k A) = 1$ holds by definition, 3.5.1 yields the inequality below:

$$\begin{aligned}\operatorname{level}_A^A(M) &= \operatorname{level}_A^A(F)\\ &\leqslant 1 + \operatorname{level}_A^A(G)\\ &= 1 + \operatorname{card}\{n \in \mathbb{Z} \mid \mathrm{H}_n(k \otimes_A^{\mathrm{L}} G) \neq 0\}\\ &= \operatorname{card}\{n \in \mathbb{Z} \mid \mathrm{H}_n(k \otimes_A^{\mathrm{L}} M) \neq 0\}.\end{aligned}$$

The last equality comes from the homology exact sequence above. □

Remark 4.10. The proof above depends on the existence of minimal semi-free resolutions, which are more widely available. For example, when A is non-negative and M is bounded below they exist if the ring A_0 is artinian, and also if A_0 is local, $\mathrm{H}(A)$ is noetherian, and $\mathrm{H}(M)$ is finite over $\mathrm{H}(A)$. Minor modifications in our arguments extend to these cases the validity of the theorem.

In the sequel we use only the following special case.

Proposition 4.11. *Let A be a DG algebra with zero differential and let M be a DG A-module, such that there is an isomorphism $M \simeq \mathrm{H}(M)$ in $\mathsf{D}(A)$.*

If A is non-negative or non-positive, and the ring A_0 is artinian and local, then M is perfect if and only if the graded $A^\natural$-module $\mathrm{H}(M)$ has a finite free resolution.

Proof. Let k denote the residue field of A_0 and J the kernel of the canonical augmentation $A \to k$. The graded $A^\natural$-module $N = \mathrm{H}(M)$ has a *minimal* free resolution

$$F^\bullet = \cdots \longrightarrow F^i \xrightarrow{\delta^i} F^{i-1} \longrightarrow \cdots \longrightarrow F^0 \longrightarrow 0 \longrightarrow \cdots$$

where each δ^i is a homomorphism of degree 0 and satisfies $\delta^i(F^i) \subseteq JF^{i-1}$, see [20, 15 and §4]. Totaling the complex $F^\bullet$ one gets a DG A-module F with $F^\natural = \coprod_{i=0}^{\infty} \Sigma^i F^i$ and $\partial(F) \subseteq JF$. It comes with a quasi-isomorphism $F \to N$ and a semi-free filtration $(F^n)_{n\in\mathbb{Z}}$ with $F^{n\natural} = \coprod_{i=0}^{n} \Sigma^i F^i$. If N has a finite free resolution $G^\bullet$, then $F^\bullet$ is isomorphic to as a direct summand of $G^\bullet$ by [20, 8], and hence N is perfect by Theorem 4.2. Conversely, if $\operatorname{level}_A^A(M)$ is finite, then 3.5.2 implies that the graded k-vector space $\mathrm{H}(k \otimes_A^{\mathbf{L}} M)$ has finite rank. The isomorphisms

$$\mathrm{H}(k \otimes_A^{\mathbf{L}} M) \cong \mathrm{H}(k \otimes_A^{\mathbf{L}} N) \cong \mathrm{H}(k \otimes_A F) = \coprod_{i=0}^{\infty} \Sigma^i (k \otimes_A F^i)$$

yield $\operatorname{rank}_k \mathrm{H}(k \otimes_A^{\mathbf{L}} M) = \sum_{i=0}^{\infty} \operatorname{rank}_A F^i$, so $F^\bullet$ is a finite free resolution. □

We conclude with examples that show that for non-positive algebras condition (b) in Theorem 4.8 cannot be relaxed significantly; compare with Remark 4.10.

Examples 4.12. Let k be a field and A a DG algebra with

$$A^\natural = k[y, z]/(y^2, yz, z^2) \quad \text{and} \quad \partial^A = 0,$$

where y and z are indeterminates over k. The exact sequence

$$0 \longrightarrow \Sigma^{|y|} ky \oplus \Sigma^{|z|} kz \longrightarrow A \longrightarrow k \longrightarrow 0$$

implies that in every minimal free resolution $F^\bullet$ of the graded $A^\natural$-module k one has $\operatorname{rank}_A F^i = 2^i$ for each $i \geqslant 0$; by Proposition 4.11, this yields

$$\operatorname{level}_A^A(k) = \infty.$$

4.12.1. For $|y| = -1 = |z|$ one has $A_0 = k$ and $A_{-1} \neq 0$.
Let M be the DG module with $M^\natural = A$ and $\partial(a) = ya$.
The graded $A^\natural$-module $M^\natural$ is free of rank 1, and one has $\operatorname{level}_A^A(M) = \infty$.

Indeed, the map $\Sigma^{-1}k \to M$ sending $\sigma^{-1}(1)$ to z is a quasi-isomorphism of DG A-modules, so one has $\operatorname{level}_A^A(M) = \operatorname{level}_A^A(k) = \infty$.

4.12.2. For $|y| = 0$ and $|z| = -2$ one has $A_0 \neq k$ and $A_{-1} = 0$.
Let M be the DG module with $M^\natural = A \oplus \Sigma^{-1}A$ and $\partial(a, \sigma^{-1}(b)) = (zb, \sigma^{-1}(ya))$.
The graded $A^\natural$-module $M^\natural$ is free of rank 2, and one has $\operatorname{level}_A^A(M) = \infty$.

Indeed, the map $(a, \sigma^{-3}(b)) \mapsto (ya, \sigma^{-1}(zb))$ is a quasi-isomorphism of DG A-modules $k \oplus \Sigma^{-3}k \to M$, which yields $\operatorname{level}_A^A(M) = \operatorname{level}_A^A(k \oplus \Sigma^{-3}k) = \infty$.

5. A New Intersection Theorem for DG algebras

The New Intersection Theorem is a central result in the homological theory of commutative noetherian rings. Here we generalize it to certain DG modules, using Hochster's notion of super height of an ideal I in a commutative ring R:

$$\operatorname{super height} I = \sup\left\{\operatorname{height}(IS) \;\middle|\; \begin{array}{l} R \to S \text{ is a homomorphism} \\ \text{of rings and } S \text{ is noetherian} \end{array}\right\}.$$

Evidently, when R is noetherian one has that $\operatorname{super height} I \geqslant \operatorname{height} I$.

Theorem 5.1. *Let A be a DG algebra with zero differential, let M be a DG module over A, and let I denote the annihilator of $\bigoplus_{n\in\mathbb{Z}} \mathrm{H}_n(M)$ in the ring $A^\flat = \bigoplus_{n\in\mathbb{Z}} A_n$.*
If $A^\flat$ is commutative and noetherian and is an algebra over a field, then one has

$$\operatorname{level}_A^A(M) \geqslant \operatorname{super height} I + 1.$$

When the ring $A^\flat$ is Cohen–Macaulay, or when its dimension is at most 3, one has

$$\operatorname{level}_A^A(M) \geqslant \operatorname{height} I + 1.$$

Addendum. Whenever $A^\flat$ is commutative and noetherian, one has

$$\operatorname{level}_A^A(M) \geqslant \operatorname{super height} I.$$

For the next remark, recall that a ring is just a DG algebra concentrated in degree 0, and a DG module over a ring is nothing but a complex.

Remark 5.2. We recall the statement of the New Intersection Theorem:

In a bounded complex P of finite free modules with non-zero homology of finite length over a local ring R one has $P_n \neq 0$ for at least $(\dim R + 1)$ values of n.

For algebras over a field it is due to Peskine and Szpiro [29], P. Roberts [31], and Hochster [22]; this case follows from Theorem 5.1, because Example 2.5.3 yields

$$\operatorname{card}\{n \in \mathbb{Z} \mid P_n \neq 0\} \geqslant \operatorname{level}_R^R(P).$$

However, we do not recover the New Intersection Theorem over arbitrary local rings, proved by Roberts; see [31]. The reason is that in the proof of Theorem 5.1 we need a result from [6], which uses Hochster's [22] big Cohen–Macaulay modules.

Over the rings it covers, Theorem 5.1 may provide a significantly sharper bound on heights than the one given by the New Intersection Theorem, as the difference $\operatorname{card}\{n \mid F_n \neq 0\} - \operatorname{level}_A^A(F)$ can be arbitrarily large, even when R is very nice:

Example 5.3. Let $(R, \mathfrak{m}, k)$ be a regular local ring of dimension $d \geqslant 2$ and l an integer, $l \geqslant d$. Let I be an $\mathfrak{m}$-primary ideal minimally generated by l elements; one always exists. The Koszul complex F on a minimal generating set for I has

$$\operatorname{level}_R^R(F) = d + 1 \quad \text{and} \quad \operatorname{card}\{n \mid F_n \neq 0\} = l + 1.$$

Indeed, the equality on the right reflects the construction of F. Theorem 5.1 yields $\operatorname{level}_R^R(F) \geqslant d + 1$, because $\operatorname{Ann}_R \mathrm{H}(K) = I$ and $\operatorname{height} I = d$. On the other hand, F is quasi-isomorphic to the complex D obtained by totaling its Cartan–Eilenberg resolution C. Since $\operatorname{gl\,dim} R = d$, by [15, XVII.1.4] one can choose C to have $d + 1$ columns; this yields a semi-free filtration of D of class at most d.

A consequence of the New Intersection Theorem is that a local ring with a non-zero module of finite length and finite projective dimension is Cohen–Macaulay. Using this and the argument above, one can prove a stronger statement:

Remark 5.4. If M is a complex over a local ring R and the module $\bigoplus_n \mathrm{H}_n(M)$ has non-zero finite length and finite projective dimension, then $\operatorname{level}_R^R(M) = \dim R + 1$.

The next result provides an upper bound for A-levels that, together with Theorem 5.1, completes the proof of Theorem 4 from the introduction.

Recall that a graded algebra is said to be *coherent* if finite graded submodules of finitely presented graded modules are also finitely presented.

Theorem 5.5. *Let A be a DG algebra with zero differential, which is left coherent as a graded ring, and let M be a DG A-module.*

If $\mathrm{H}(M)$ is finitely presented over A, then the following inequalities hold:

$$\operatorname{level}_A^A(M) \leqslant \operatorname{proj\,dim}_A \mathrm{H}(M) + 1 \leqslant \operatorname{gl\,dim} A + 1.$$

Remark 5.6. The second inequality in Theorem 5.5 holds by definition.

When A is a ring and M is an A-module, Krause and Kussin [26, 2.4] prove that the first inequality in Theorem 5.5 becomes an equality.

In view of the Syzygy Theorem, Theorems 5.1 and 5.5 yield:

Corollary 5.7. *Let S be a graded polynomial algebra in c indeterminates over a field k with $\partial^S = 0$. Each DG S-module M with* $\operatorname{rank}_k \mathrm{H}(M) \neq 0, \infty$ *satisfies*

$$\operatorname{level}_S^S(M) = c + 1.$$

Proof of Theorem 5.5. It suffices to deal with the case $\operatorname{proj\,dim}_A \mathrm{H}(M) = p < \infty$.

When $p = 0$ the graded A-module $\mathrm{H}(M)$ is projective. As A has zero differential, the cycles of M form a graded A-submodule $\mathrm{Z}(M)$, and the canonical surjection $\mathrm{Z}(M) \to \mathrm{H}(M)$ is an A-linear map. Choosing an A-linear splitting $\sigma : \mathrm{H}(M) \to \mathrm{Z}(M)$ and composing it with the inclusion $\mathrm{Z}(M) \subseteq M$ one gets a quasi-isomorphism $\mathrm{H}(M) \to M$. The DG module $\mathrm{H}(M)$ is a direct summand of some finite free graded A-module, so in $\mathsf{D}(A)$ one has $\mathrm{H}(M) \in \mathsf{add}^{\Sigma}(A)$. Thus, one has

$$\operatorname{level}_A^A(M) = \operatorname{level}_A^A\big(\mathrm{H}(M)\big) \leqslant 1.$$

Let now p be a positive integer, and assume that the desired inequality holds for all DG A-modules whose homology has projective dimension strictly smaller than p. In M, pick cycles $z_1, \dots, z_s$ whose homology classes generate $\mathrm{H}(M)$, set $L = \bigoplus_{i=1}^{s} \Sigma^{|z_i|} A$. The map of graded A-modules $\lambda : L \to M$ that sends $1 \in \Sigma^{|z_i|} A$ to z_i is a morphism of DG A-modules. In $\mathsf{D}(A)$ it fits into an exact triangle $L \xrightarrow{\lambda} M \xrightarrow{\mu} N \to \Sigma L$. In the induced exact sequence of graded A-modules

$$\Sigma^{-1}\mathrm{H}(M) \xrightarrow{\Sigma^{-1}\mathrm{H}(\mu)} \Sigma^{-1}\mathrm{H}(N) \longrightarrow \mathrm{H}(L) \xrightarrow{\mathrm{H}(\lambda)} \mathrm{H}(M) \xrightarrow{\mathrm{H}(\mu)} \mathrm{H}(N)$$

the map $\mathrm{H}(\lambda)$ is surjective by construction. This implies $\mathrm{H}(\mu) = 0 = \Sigma^{-1}\mathrm{H}(\mu)$, so the exact sequence shows that $\mathrm{H}(N)$ is finitely presented over A, and one has $\operatorname{proj\,dim}_A \mathrm{H}(N) = p - 1$. From the induction hypothesis we now obtain $\operatorname{level}_A^A(N) \leqslant p$, hence we get $\operatorname{level}_A^A(M) \leqslant p + 1$ from Lemma 2.4(2). □

In order to obtain lower bounds in Theorem 5.1 on A-levels of DG A-modules we use our recent results on invariants of a related structure, which we define next.

5.8. *Differential modules.* Let R be an associative ring.

A *differential R-module* is a pair (D, δ), where D is an R-module and $\delta : D \to D$ an R-linear map with $\delta^2 = 0$; the module $\mathrm{H}(D) = \operatorname{Ker}(\delta)/\operatorname{Im}(\delta)$ is the *homology* of D. Every module M supports a differential module $(M, 0)$, also called M.

A *projective flag* in a differential R-module D is a family $(D^n)_{n \in \mathbb{Z}}$ of differential R-submodules of D, such that for each $n \in \mathbb{Z}$ the following hold:

$$D^n \subseteq D^{n+1}, \qquad D^{-1} = 0, \qquad \bigcup_{n \in \mathbb{Z}} D^n = D, \quad \text{and}$$

$$D^n / D^{n-1} \cong (P_n, 0) \quad \text{for some projective } R\text{-module } P_n.$$

In [6, 2.8] the *projective class* of D is defined to be the number

$$\operatorname{proj\,class}_R D = \inf\left\{ l \in \mathbb{Z} \;\middle|\; \begin{array}{l} D \text{ admits a projective flag} \\ (D^n)_{n\in\mathbb{Z}} \text{ with } D^l = D \end{array} \right\}.$$

The statement below is one of the main results of [6]:

5.9. *Class Inequality.* If R is a commutative noetherian ring, F a finitely generated differential R-module, D a retract of F, and $I = \operatorname{Ann}_R \mathrm{H}(D)$, then one has

$$\operatorname{proj\,class}_R F \geqslant \operatorname{super\,height} I - 1,$$

with strict inequality when R is an algebra over a field; see [6, 4.2]. Moreover, when $\dim R \leqslant 3$ or R is Cohen–Macaulay [6, 4.1] yields an inequality

$$\operatorname{proj\,class}_R F \geqslant \operatorname{height} I.$$

There is an obvious parallel between the notion of semi-free filtration for DG modules and that of projective flag for differential modules. Under additional conditions we turn it into a direct comparison, by using the following construction.

5.10. *DG algebras with zero differential.* Let A be a DG algebra with $\partial^A = 0$.

Using the product of the graded algebra A one defines an associative ring

$$A^\flat = \bigoplus_{n\in\mathbb{Z}} A_n,$$

and then to each DG A-module M one associates a differential $A^\flat$-module

$$M^\flat = \left(\bigoplus_{n\in\mathbb{Z}} M_n, \bigoplus_{n\in\mathbb{Z}} (-1)^n \partial_n^M \right).$$

The simplicity of the construction notwithstanding, a couple of caveats may be in order: The signs appearing in the formula for the differential of $M^\flat$ are necessary to ensure that it is $A^\flat$-linear. The action of $A^\flat$ on $M^\flat$ respects the obvious internal gradings of these objects, but the differential of $M^\flat$ need not.

Lemma 5.11. *Let A be a DG algebra with trivial differential.*

The assignment $M \mapsto M^\flat$ defines an exact functor from the abelian category of DG A-modules to that of differential $A^\flat$-modules. It preserves finite generation and transforms semi-free filtrations into projective flags.

If F is a semi-free DG A-module of class at most c, then one has

$$c \geqslant \operatorname{proj\,class}_{A^\flat}\left(F^\flat\right).$$

Proof. The first two assertions are evident from the definitions and constructions preceding the lemma. If (F^n) is a semi-free filtration of F with $F^c = F$, then $((F^\flat)^n)$ is a projective flag with $(F^\flat)^c = F^\flat$: this gives the inequality above. □

Proof of Theorem 5.1. Set $\operatorname{level}_A^A(M) = l$. We may assume that l is finite.

Theorem 4.2 shows that M is isomorphic, in $\mathsf{D}(A)$, to a retract of some finite semi-free DG A-module F of class $l-1$. Levels do not change under isomorphisms, so we may assume that M itself is a direct summand of F. In that case $M^\flat$ is a retract of $F^\flat$, and the $A^\flat$-module $F^\flat$ is finite. Thus, Lemma 5.11 yields

$$l - 1 \geqslant \operatorname{proj\,class}_{A^\flat}\left(F^\flat\right).$$

To finish the proof, invoke the Class Inequality from 5.9. □

6. Levels and semi-simplicity

In this section we analyze levels of DG modules related to two classical notions of length for modules over rings.

6.1. *Restricted lengths of modules.* Let S be a ring and C a finite non-empty set of simple S-modules. A C-*filtration* of an S-module N is a sequence of submodules

$$0 = N^0 \subsetneq N^1 \subsetneq \cdots \subsetneq N^l = N$$

where every N^i/N^{i-1} is isomorphic to a direct sum of modules from C.

The C-*length of* N, denoted $\ell_S^{\mathsf{C}}(N)$, is the largest integer l for which N has a C-filtration with $N^l = N$; when no such filtration exists we set $\ell_S^{\mathsf{C}}(N) = \infty$.

The C-*Loewy length of* N, denoted $\ell\ell_S^{\mathsf{C}}(N)$, is the least integer l for which N has a C-filtration with $N^l = N$; when none exists we set $\ell\ell_S^{\mathsf{C}}(N) = \infty$.

6.1.1. For each exact sequence $0 \to N' \to N \to N'' \to 0$ of S-modules one has

$$\ell_S^{\mathsf{C}}(N) = \ell_S^{\mathsf{C}}(N') + \ell_S^{\mathsf{C}}(N''), \tag{6.1.1.1}$$

$$\ell\ell_S^{\mathsf{C}}(N) \leqslant \ell\ell_S^{\mathsf{C}}(N') + \ell\ell_S^{\mathsf{C}}(N''), \tag{6.1.1.2}$$

$$\ell\ell_S^{\mathsf{C}}(N) \geqslant \max\left\{\ell\ell_S^{\mathsf{C}}(N'), \ell\ell_S^{\mathsf{C}}(N'')\right\} \tag{6.1.1.3}$$

with equality in (6.1.1.3) when the sequence splits. From here one gets

$$\ell\ell_S^{\mathsf{C}}(N) \leqslant \ell\ell_S^{\mathsf{C}}(S). \tag{6.1.1.4}$$

Lemma 6.1.2. *Let N be an S-module.*

If $\ell_S^{\mathsf{C}}(N)$ is finite, then so is $\ell\ell_S^{\mathsf{C}}(N)$.

When N is noetherian the converse holds as well.

Proof. By definition, one has $\ell\ell_S^{\mathsf{C}}(N) \leqslant \ell_S^{\mathsf{C}}(N)$, whence the first assertion. For the converse, assume $\ell\ell_S^{\mathsf{C}}(N) = l < \infty$ and fix a filtration

$$0 = N^0 \subsetneq N^1 \subsetneq \cdots \subsetneq N^l = N$$

where each N^i/N^{i-1} a direct sum of simple modules from C. When N is noetherian each direct sum is finite, so N has finite C-length. $\square$

Recall that S is *semi-local* if it has finitely many isomorphism classes of simple modules; equivalently, if $S/\mathfrak{n}$, where $\mathfrak{n}$ is the Jacobson radical, is semi-simple.

6.1.3. Let S be a semi-local ring, let $\mathfrak{n}$ denote its Jacobson radical, and let C contain representatives of every isomorphism class of simple S-modules. The C-length and C-Loewy length of N are then equal to their classical counterparts, denoted $\ell_S N$ and $\ell\ell_S N$, respectively. Furthermore, one has

$$\ell\ell_S N = \inf\{n \in \mathbb{N} \mid \mathfrak{n}^n N = 0\} = \inf\{n \in \mathbb{N} \mid (0 : \mathfrak{n}^n)_N = N\}.$$

Now we return to DG algebras. As in 3.8 and 3.11, we use canonical augmentations $\varepsilon : A \to \mathrm{H}_0(A)$ to identify $\mathrm{H}_0(A)$-modules with DG A-modules.

Parts (3) and (4) of the next result contain Proposition 5 from the introduction.

Theorem 6.2. *Let A be a non-negative DG algebra. Set $S = \mathrm{H}_0(A)$, let C be a finite set of simple S-modules, and set $k = \bigoplus_{N \in \mathsf{C}} N$.*

For each DG A-module M the following statements hold.

(1) $\mathrm{level}_A^k(M) = \mathrm{level}_A^{\mathsf{C}}(M)$.
(2) $\mathrm{level}_A^k(M) \geqslant \max_{n \in \mathbb{Z}}\{\ell\ell_S^{\mathsf{C}}(\mathrm{H}_n(M))\}$.
(3) *The numbers* $\mathrm{level}_A^k(M)$ *and* $\ell_S^{\mathsf{C}}(\mathrm{H}(M))$ *are finite simultaneously; when they are, they are linked by the following inequality:*

$$\mathrm{level}_A^k(M) \leqslant \sum_{n \in \mathbb{Z}} \ell\ell_S^{\mathsf{C}}(\mathrm{H}_n(M)).$$

(4) *If N is an S-module and $\ell_S^{\mathsf{C}}(N)$ is finite, then one has*

$$\mathrm{level}_A^k(N) = \ell\ell_S^{\mathsf{C}}(N).$$

Proof. (1) This follows from the equality $\mathsf{smd}(\mathsf{add}^\Sigma(k)) = \mathsf{smd}(\mathsf{add}^\Sigma(\mathsf{C}))$ in $\mathsf{D}(S)$.

(2) We may assume $\mathrm{level}_A^k(M) = l < \infty$. By (1) and 2.2.1, this means that M is a direct summand of a complex $L \in \mathsf{add}^\Sigma(\mathsf{C})^{\star l}$. When $l = 1$ one may assume that each L_i is a finite direct sum of modules from C; then so is each $\mathrm{H}_i(L)$, and thus $\ell\ell_S^{\mathsf{C}}(\mathrm{H}_i(L)) \leqslant 1$. For $l \geqslant 2$ there is an exact triangle $L' \to L \to L'' \to \Sigma L'$ in $\mathsf{D}(S)$ with $\mathrm{level}_A^k(L') \leqslant l-1$ and $\mathrm{level}_A^k(L'') \leqslant 1$. It induces an exact sequence $\mathrm{H}_i(L') \to \mathrm{H}_i(L) \to \mathrm{H}_i(L'')$ of S-modules. Induction and sub-additivity, see (6.1.1.2), yield $\ell\ell_S^{\mathsf{C}}(\mathrm{H}_i(L)) \leqslant l$. Thus, one gets $\ell\ell_S^{\mathsf{C}}(\mathrm{H}_i(M)) \leqslant l$.

(3) It is easy to verify that the subcategory of DG A-modules M, such that $\ell_S^k(\mathrm{H}(M))$ is finite, is thick. It contains k, hence also $\mathsf{thick}_A(k)$. Thus, when $\mathrm{level}_A^k(M)$ is finite, so is $\ell_S^{\mathsf{C}}(\mathrm{H}(M))$, see 2.1.5. One now has

$$\mathrm{level}_A^k(M) \leqslant \sum_{n \in \mathbb{Z}} \mathrm{level}_S^k(\mathrm{H}_n(M))$$

from Proposition 3.10. To finish, for each S-module N of finite C-length we prove:

$$\operatorname{level}_S^k(N) \leqslant \ell\ell_S^C(N).$$

Indeed, N has a filtration by submodules with $N^0 = 0$, $N^l = N$ for $l = \ell\ell_S^C(N)$, and each N^i/N^{i-1} a direct sum of modules from C. The sums are finite because $\ell_S^{\mathsf{C}}(N)$ is, so each N^i/N^{i-1} has C-level 1. Thus 3.5.1 yields $\operatorname{level}_S^k(N) \leqslant l$.

(4) This is a formal consequence of (2) and (3). □

We pause to remark that the finiteness hypothesis in Theorem 6.2(4), as well as the noetherian hypothesis in Lemma 6.1.2 are essential:

Example 6.3. When k is a simple S-module the module $N = k^{(\mathbb{N})}$ satisfies

$$\ell\ell_S^k(N) = 1 < \infty = \operatorname{level}_S^k(N) = \ell_S^k(N).$$

More precise conclusions can be made when $\mathrm{H}_0(A)$ is semi-simple.

Theorem 6.4. *Let A be a DG algebra satisfying one of the conditions below:*

(a) *A is non-negative.*
(b) *A is non-positive and $\partial(A_0) = 0$.*

Set $S = \mathrm{H}_0(A)$ and assume that the ring S is semi-simple.

For a DG A-module M the number $\operatorname{level}_A^S(M)$ is finite if and only if the S-module $\mathrm{H}(M)$ is finitely generated; when it is, the following inequality holds:

$$\operatorname{level}_A^S(M) \leqslant \operatorname{card}\{n \in \mathbb{Z} \mid \mathrm{H}_n(M) \neq 0\}.$$

Proof. The semi-simple ring S is noetherian. Let C be a set of representatives of the isomorphism classes of simple S-modules, and note that C is finite.

If $\operatorname{level}_A^S(M)$ is finite, then the S-module $\mathrm{H}(M)$ is noetherian; see 3.5.2.

Conversely, assume that the graded S-module $\mathrm{H}(M)$ is finitely generated. It is isomorphic to a direct sum of suspensions of modules from C, so for each $n \in \mathbb{Z}$ one has $\mathrm{H}_n(M) \neq 0$ if and only if $\operatorname{level}_S^S(\mathrm{H}_n(M)) = 1$. The desired inequality follows from Proposition 3.10(2) in case (a) and from Proposition 3.12(2) in case (b). □

The next two results link the structure of a semi-local ring to k-levels.

Proposition 6.5. *Let S be a left artinian ring and $\mathfrak{n}$ its Jacobson radical.*

If M is a complex of S-modules with $\mathrm{H}(M)$ finitely generated, then one has

$$\operatorname{level}_S^{S/\mathfrak{n}}(M) \leqslant \ell\ell_S S.$$

Proof. Replacing M, if necessary, by a quasi-isomorphic complex, we may assume that M is a bounded complex of finite S-modules. Since $\mathfrak{n}^l = 0$ for $l = \ell\ell_S S$, setting $M^i = \mathfrak{n}^{l-i}M$ we get a filtration of M by subcomplexes with $M^0 = 0$, $M^l = M$ and M^i/M^{i-1} a bounded complex of finite semi-simple S-modules for $i = 0, \dots, l$. In $\mathsf{D}(S)$ such a complex is quasi-isomorphic to its homology. The latter, being a bounded complex of finite semi-simple modules with trivial differential, has k-level 1 by definition. Thus 3.5.1 now yields $\operatorname{level}_S^{S/\mathfrak{n}}(M) \leqslant l$, as desired. □

Proposition 6.6. *For a local ring $(R, \mathfrak{m}, k)$ the following conditions are equivalent.*

- (i) *The ring R is regular.*
- (ii) *There exists a finite free complex of k-level* 1.
- (iii) *The Koszul complex K on a minimal set of generators of $\mathfrak{m}$ has k-level* 1.

Proof. When R is regular in $\mathsf{D}(R)$ one has $K \simeq k$, hence $\operatorname{level}_R^k(K) = 1$.

Let F be a finite free complex with $\operatorname{level}_R^k(F) = 1$. By definition, in $\mathsf{D}(R)$ one then has $F \simeq V$, where V is a complex of k-vector spaces with $\operatorname{rank}_k \mathrm{H}(V)$ finite non-zero. As k is a retract of $\Sigma^s V$ for some $s \in \mathbb{Z}$, in $\mathsf{D}(R)$ it is also a retract of $\Sigma^s F$, and so is quasi-isomorphic to a finite free complex. This means that the R-module k has finite projective dimension, hence R is regular; see [12, 2.2.7]. □

We finish the section with examples of strict inequalities in the preceding results.

Example 6.7. Let $(R, \mathfrak{m}, k)$ be a local ring and K the Koszul complex on a minimal set of generators of $\mathfrak{m}$. Each module $\mathrm{H}_n(K)$ is a vector space over k, and so has k-level 1; see Theorem 6.2(4). When R is singular, one gets the inequality below

$$\operatorname{level}_R^k(K) > 1 = \max_{n \in \mathbb{Z}} \bigl\{\ell\ell_R^k\bigl(\mathrm{H}_n(K)\bigr)\bigr\},$$

so the inequality in Theorem 6.2(2) can be strict. On the other hand, $\mathrm{H}_n(K) \neq 0$ holds precisely when n satisfies $0 \leqslant n \leqslant \operatorname{edim} R - \operatorname{depth} R$, so one has

$$\sum_{n \in \mathbb{Z}} \ell\ell_R^k\bigl(\mathrm{H}_n(K)\bigr) = \operatorname{card}\bigl\{n \in \mathbb{Z} \bigm| \mathrm{H}_n(K) \neq 0\bigr\} = \operatorname{edim} R - \operatorname{depth} R + 1.$$

The number on the right-hand side is independent of $\ell\ell_R R$, so by fixing one and varying the other it is easy to conjure artinian local rings for which a strict inequality holds in Theorem 6.2(3), Theorem 6.4, or Proposition 6.5.

7. Perfect DG modules over exterior algebras

Our goal is to prove a slightly enhanced version of Theorem 6 in the introduction. It is an algebraic analogue of, and can be used to deduce, [2, 4.4.5], which is a statement about the toral rank of spaces.

Theorem 7.1. *Let k be a field and Λ a DG k-algebra with $\partial^\Lambda = 0$ and $\Lambda^\natural$ an exterior algebra on c alternating indeterminates of positive odd degrees.*

For every perfect DG Λ-module N with $\mathrm{H}(N) \neq 0$ *one then has*

$$\operatorname{level}_\Lambda^k(N) = c + 1 \leqslant \operatorname{card}\bigl\{n \in \mathbb{Z} \bigm| \mathrm{H}_n(N) \neq 0\bigr\}.$$

It will be deduced from Corollary 5.7 by using equivalences between certain thick subcategories of $\mathsf{D}(\Lambda)$ and $\mathsf{D}(S)$, where S is a graded polynomial ring in c indeterminates over k. The prototype of such results is a classical theorem of J. Bernstein, I.M. Gelfand, and S.I. Gelfand [10], dealing with subcategories of the derived category of graded modules over these graded algebras. The situation here is different. We provide a self-contained treatment, as none of the results that we have located in the literature covers it with the detail and in the generality that we need; see Remarks 7.7 and 7.8.

7.2. Let k be a field and let c be a non-negative integer.

Let Λ be the DG algebra with $\partial^{\Lambda}=0$ and $\Lambda^{\natural}$ an exterior algebra on alternating indeterminates $\xi_1,\dots,\xi_c$ of positive odd degrees.

Let S denote the DG algebra with $\partial^{S}=0$ and $S^{\natural}$ a polynomial ring on commuting indeterminates $x_1,\dots,x_c$, with $|x_i|=-|\xi_i|-1$ for $i=1,\dots,c$.

Set $(-)^*=\operatorname{Hom}_k(-,k)$, viewed as a functor on the category of complexes of k-vector spaces; see Section 3. The DG algebras S and Λ are graded commutative, so for every DG module L over either one of them the complex L^* of k-vector spaces carries a canonical structure of DG module over the same DG algebra.

7.3. As S and Λ are graded commutative with elements of odd degree squaring to zero, the DG algebra $\Lambda\otimes_k S$ has the same property. In particular,

$$\delta=\sum_{h=1}^{c}\xi_h\otimes x_h\in(\Lambda\otimes_k S)_{-1}\quad\text{satisfies}\quad\delta^2=0.$$

Let E be a DG $(\Lambda\otimes_k S)$-module. An elementary calculation shows that the map

$$\partial:E\to E\quad\text{defined by}\quad\partial(e)=\partial^{E}(e)+\delta\cdot e$$

for all $e\in E$ is k-linear of degree -1, and satisfies $\partial^2=0$ and $\partial(ae)=(-1)^{|a|}a\partial(e)$ for all $a\in\Lambda\otimes_k S$. Thus, $E^{\delta}=(E^{\natural},\partial)$ is a DG $(\Lambda\otimes_k S)$-module.

For the DG Λ-module Λ^*, see 7.2, and the DG S-module S, the tensor product $\Lambda^*\otimes_k S$ is naturally a DG module with zero differential over $\Lambda\otimes_k S$. Set

$$X=\left(\Lambda^*\otimes_k S\right)^{\delta}.$$

The next result provides the last ingredient needed in the proof of Theorem 7.1.

Theorem 7.4. *Let X be the DG module described above. The functors*

$$\mathsf{t}=X\otimes_S-\quad\text{and}\quad\mathsf{h}=\operatorname{Hom}_{\Lambda}\left(X^*,-\right)$$

induce an adjoint pair (t,h) *of exact functors of triangulated categories.*

They restrict to inverse equivalences of triangulated categories:

$$\begin{array}{ccc} \mathsf{D}(S) & \underset{\mathsf{h}}{\overset{\mathsf{t}}{\rightleftarrows}} & \mathsf{D}(\Lambda) \\ \cup\! | & & \cup\! | \\ \mathsf{thick}_S(S) & \overset{\equiv}{\longleftrightarrow} & \mathsf{thick}_\Lambda(k) \\ \cup\! | & & \cup\! | \\ \mathsf{thick}_S(k) & \overset{\equiv}{\longleftrightarrow} & \mathsf{thick}_\Lambda(\Lambda) \end{array} \tag{7.4.1}$$

For $d = |\xi_1| + \cdots + |\xi_c|$ there are isomorphisms in $\mathsf{D}(S)$ and $\mathsf{D}(\Lambda)$, respectively:

$$\begin{array}{lcl} \mathsf{h}(\Lambda) \simeq \Sigma^d k, & & \mathsf{t}(k) \simeq \Sigma^{-d}\Lambda, \\ & \textit{and} & \\ \mathsf{h}(k) \simeq S & & \mathsf{t}(S) \simeq k. \end{array} \tag{7.4.2}$$

The theorem is proved at the end of this section. The thick subcategories that appear in its statement admit very explicit descriptions.

Remark 7.5. For each DG S-module M the following hold:

$$M \in \mathsf{thick}_S(S) \iff M \simeq M' \text{ with } M'^{\natural} \text{ finite free over } S \tag{7.5.1}$$

$$\iff \mathrm{H}(M) \text{ is finite over } S, \tag{7.5.1$'$}$$

$$M \in \mathsf{thick}_S(k) \iff \mathrm{H}(M) \text{ is finite over } k. \tag{7.5.2}$$

For each DG Λ-module N the following hold:

$$N \in \mathsf{thick}_\Lambda(\Lambda) \iff N \simeq N' \text{ with } N'^{\natural} \text{ finite free over } \Lambda, \tag{7.5.3}$$

$$N \in \mathsf{thick}_\Lambda(k) \iff \mathrm{H}(N) \text{ is finite over } k \tag{7.5.4}$$

$$\iff \mathrm{H}(N) \text{ is finite over } \Lambda. \tag{7.5.4$'$}$$

Indeed, (7.5.1) and (7.5.3) are special cases of Theorem 4.8, while (7.5.2) and (7.5.4) are special cases of Theorem 6.4. When $\mathrm{level}_S^S(M)$ is finite the graded S-module $\mathrm{H}(M)$ is noetherian by 3.5.2: this establishes one direction of (7.5.1$'$). Conversely, when $\mathrm{H}(M)$ is finite from Theorem 5.5 and the Syzygy Theorem one gets $\mathrm{level}_S^S(M) \leqslant \mathrm{gl\,dim}\, S = c + 1$. Finally, (7.5.4$'$) is evident as $\mathrm{rank}_k \Lambda$ is finite.

7.6. *The Koszul DG module.* For use in the proof of Theorem 7.4, we collect homological properties of the DG module X introduced in 7.3.

7.6.1. Let $\eta^\Lambda : k \to \Lambda$ denote the structure map and $\varepsilon^S : S \to k$ the canonical augmentation. The following map of DG S-modules is a quasi-isomorphism:

$$\pi : X = \left(\Lambda^* \otimes_k S\right)^\delta \xrightarrow{(\eta^\Lambda)^* \otimes \varepsilon^S} k \otimes_k k = k.$$

Indeed, bigrading the complex of k-vector spaces underlying X by assigning to the indeterminates ξ_i and x_i homological degrees 1 and 0, respectively, one obtains the graded Koszul complex on $x_1, \dots, x_c \in S$; its homology is equal to k.

7.6.2. Let $\eta^S : k \to S$ denote the structure map and $\varepsilon^\Lambda : \Lambda \to k$ the canonical augmentation. The following map is a quasi-isomorphism of DG Λ-modules:

$$\iota : k = k \otimes_k k \xrightarrow{(\varepsilon^\Lambda)^* \otimes \eta^S} \left(\Lambda^* \otimes S\right)^\delta = X.$$

As $\pi\iota = \mathrm{id}^k$ and π is a quasi-isomorphism, ι is one as well. The induced map

$$\rho = \mathrm{Hom}_k(\iota, k) : X^* \longrightarrow k$$

is a quasi-isomorphism of DG Λ-modules.

7.6.3. For every DG S-module M the complex $X \otimes_S M$ of k-vector spaces has a canonical structure of DG Λ-module, isomorphic to $(\Lambda^* \otimes_k M)^\delta$. Similarly, for every DG Λ-module N the complex $\mathrm{Hom}_\Lambda(X^*, N)$ of k-vector spaces has a canonical structure of DG S-module, isomorphic to $\mathrm{Hom}_k(S^*, N)^\delta$.

7.6.4. The following functors preserve quasi-isomorphisms of DG S-modules

$$\mathrm{Hom}_S(X, -), \quad (X \otimes_S -), \quad \text{and} \quad \left(- \otimes_S \mathrm{Hom}_\Lambda\left(X^*, \Lambda\right)\right).$$

Indeed, by Theorem 4.8 and 3.1 it suffices to note that $X^\natural$ and $\mathrm{Hom}_\Lambda(X^*, \Lambda)^\natural$ are finite free over S. For the first module this is clear; for the second one has

$$\left(\mathrm{Hom}_\Lambda\left(X^*, \Lambda\right)\right)^\natural \cong \mathrm{Hom}_k\left(S^*, \Lambda\right) \cong \Lambda \otimes_k S.$$

7.6.5. The following functors preserve quasi-isomorphisms of DG Λ-modules:

$$\mathrm{Hom}_\Lambda\left(X^*, -\right) \quad \text{and} \quad \mathrm{Hom}_\Lambda(-, X).$$

Indeed, since $X^{*\natural} \cong \Lambda \otimes_k S^*$ as graded Λ-modules, X^* is a direct summand of a semi-free DG Λ-module by Proposition 4.4. Now apply 3.1 to X^* and $X \cong X^{**}$.

7.6.6. There exists an isomorphism of DG Λ-modules: $\Lambda^* \cong \Sigma^{-d}\Lambda$; in particular, the functor $\mathrm{Hom}_\Lambda(-, \Lambda)$ preserves quasi-isomorphisms of DG Λ-modules.

Indeed, $(\Lambda^*)_{-d}$ contains a unique k-linear map that sends $\xi_1 \cdots \xi_c$ to 1_k. It defines an element $\omega \in (\Sigma^d(\Lambda^*))_0$. The homomorphism $\Lambda \to \Sigma^d(\Lambda^*)$ of graded Λ-modules, given by $\lambda \mapsto \lambda\omega$, is easily seen to be injective, and hence is bijective.

Proof of Theorem 7.4. As the functors $(X \otimes_S -)$ and $\mathrm{Hom}_\Lambda(X^*, -)$ preserve quasi-isomorphisms, see 7.6.4 and 7.6.5, they induce exact functors $\mathsf{t} : \mathsf{D}(S) \to \mathsf{D}(\Lambda)$ and $\mathsf{h} : \mathsf{D}(\Lambda) \to \mathsf{D}(S)$ of the respective derived categories.

Using the isomorphism π from 7.6.1, in $\mathsf{D}(\Lambda)$ one gets

$$\mathsf{t}(S) = X \otimes_S S = X \simeq k.$$

Multiplication with δ induces the zero map on the DG module $(\Lambda^* \otimes_k S)^\natural \otimes_S k$. From this observation and 7.6.6 one obtains isomorphisms of DG Λ-modules

$$\mathsf{t}(k) = X \otimes_S k = \big(\Lambda^* \otimes_k S\big)^\delta \otimes_S k = \big(\Lambda^* \otimes_k S\big) \otimes_S k \cong \Lambda^* \cong \Sigma^{-d}\Lambda.$$

Multiplication with δ annihilates $\operatorname{Hom}_\Lambda(\Lambda \otimes_k (S^*), k)^\natural$. From this observation one obtains isomorphisms of DG S-modules

$$\mathsf{h}(k) = \operatorname{Hom}_\Lambda(X^*, k) \cong \operatorname{Hom}_\Lambda\big(\big(\Lambda \otimes_k S^*\big)^\delta, k\big) = \operatorname{Hom}_\Lambda\big(\Lambda \otimes_k S^*, k\big) \cong S.$$

Using the isomorphism ρ of DG Λ-modules from 7.6.2 and the exactness of the functor $\operatorname{Hom}_\Lambda(-, \Lambda)$, see 7.6.6, one gets quasi-isomorphisms

$$\mathsf{h}(\Lambda) = \operatorname{Hom}_\Lambda\big(X^*, \Lambda\big) \simeq \operatorname{Hom}_\Lambda(k, \Lambda) \cong \Sigma^d k$$

of complexes of vector spaces. It yields $\mathsf{h}(\Lambda) \simeq \Sigma^d k$ in $\mathsf{D}(S)$, see Proposition 3.12(1).

We have proved the isomorphisms in (7.4.2). They imply inclusions

$$\begin{matrix} \mathsf{h}\big(\mathsf{thick}_\Lambda(\Lambda)\big) \subseteq \mathsf{thick}_S(k), \\ \mathsf{h}\big(\mathsf{thick}_\Lambda(k)\big) \subseteq \mathsf{thick}_S(S) \end{matrix} \quad \text{and} \quad \begin{matrix} \mathsf{t}\big(\mathsf{thick}_S(k)\big) \subseteq \mathsf{thick}_\Lambda(\Lambda), \\ \mathsf{t}\big(\mathsf{thick}_S(S)\big) \subseteq \mathsf{thick}_\Lambda(k) \end{matrix}$$

of subcategories. The vertical inclusions in diagram (7.4.1) are clear.

The quasi-isomorphisms ρ from 7.6.2 and π from 7.6.1 are used again in the computations below. For each DG S-module M one has canonical morphisms

$$\begin{aligned} M &\cong \operatorname{Hom}_\Lambda\big(k, \operatorname{Hom}_k(\Lambda, M)\big) \\ &\cong \operatorname{Hom}_\Lambda\big(k, \Lambda^* \otimes_k M\big) \\ &= \operatorname{Hom}_\Lambda\big(k, \big(\Lambda^* \otimes_k M\big)^\delta\big) \\ &\cong \operatorname{Hom}_\Lambda(k, X \otimes_S M) \xrightarrow{\operatorname{Hom}_\Lambda(\rho, X \otimes_S M)} \operatorname{Hom}_\Lambda\big(X^*, X \otimes_S M\big) \end{aligned}$$

of DG S-modules; the equality holds because δ annihilates $\operatorname{Hom}_\Lambda(k, \Lambda^* \otimes_k M)^\natural$.

For each DG Λ-module N one has canonical morphisms

$$\begin{aligned} X \otimes_S \operatorname{Hom}_\Lambda\big(X^*, N\big) \xrightarrow{\pi \otimes_S \operatorname{Hom}_\Lambda(X^*, N)} k \otimes_S \operatorname{Hom}_\Lambda\big(X^*, N\big) & \\ \cong k \otimes_S \operatorname{Hom}_k\big(S^*, N\big)^\delta & \\ = k \otimes_S \operatorname{Hom}_k\big(S^*, N\big) & \\ \cong k \otimes_S (S \otimes_k N) & \\ \cong N & \end{aligned}$$

of DG Λ-modules, with equality due to the relation $\delta \cdot (k \otimes_S \operatorname{Hom}_k(S^*, N))^\natural = 0$.

The morphisms above yield natural transformations of exact functors

$$\sigma : \mathsf{Id}^S \to \mathsf{ht} \quad \text{and} \quad \lambda : \mathsf{th} \to \mathsf{Id}^\Lambda.$$

They are, respectively, the unit and counit exhibiting t and h as adjoint functors.

To see that σ and λ restrict to isomorphisms on $\mathsf{thick}_S(k)$ and $\mathsf{thick}_\Lambda(\Lambda)$ respectively, it suffices to show that the following maps are quasi-isomorphisms:

$$\mathrm{Hom}_\Lambda(\rho, X \otimes_S k) \quad \text{and} \quad \pi \otimes_S \mathrm{Hom}_\Lambda\big(X^*, \Lambda\big).$$

For the first one use the isomorphisms of DG Λ-modules $X \otimes_S k \cong \Lambda^* \cong \Sigma^{-d}\Lambda$, see 7.6.6. For the second one apply 7.6.4.

To prove that σ and λ are isomorphisms on $\mathsf{thick}_S(S)$ and $\mathsf{thick}_\Lambda(k)$ respectively, it suffices to show that the following maps are quasi-isomorphisms:

$$\mathrm{Hom}_\Lambda(\rho, X \otimes_S S) \quad \text{and} \quad \pi \otimes_S \mathrm{Hom}_\Lambda\big(X^*, k\big).$$

The one on the left follows from 7.6.5, while that on the right follows from the isomorphism $\mathrm{Hom}_\Lambda(X^*, k) \cong S$ of DG S-modules. □

Proof of Theorem 7.1. Recall the hypothesis: N is a DG Λ-module in $\mathsf{thick}_\Lambda(\Lambda)$, with $\mathrm{H}(N) \neq 0$. The equivalences of categories h from Theorem 7.4 imply that $\mathrm{rank}_k \mathrm{H}(\mathsf{h}(N))$ is finite and non-zero, and yields the first equality below:

$$\mathrm{level}_\Lambda^k(N) = \mathrm{level}_S^S\big(\mathsf{h}(N)\big) = c + 1.$$

The second one is given by Corollary 5.7. Finally, from Theorem 6.4(a) we get

$$\mathrm{card}\big\{n \in \mathbb{Z} \bigm| \mathrm{H}_n(N) \neq 0\big\} \geqslant \mathrm{level}_\Lambda^k(N). \qquad \square$$

Remark 7.7. Several publications deal with equivalences of subcategories of the derived categories of DG modules $\mathsf{D}(S)$ and $\mathsf{D}(\Lambda)$: see [14, II.7], [1, App.] and the bibliography of the latter. None of these is applicable to the present situation, for reasons having to do with specific choices of gradings, restrictions on the characteristic of k, focus on different subcategories, or reliance on tools from analysis.

Remark 7.8. Techniques in [18] allow for a different approach to Theorem 7.4.

The first step is to interpolate the endomorphism DG algebra $E = \mathrm{Hom}_\Lambda(X, X)$ between Λ and S. Using [18, 4.10] one can then prove that the functors

$$\big(- \otimes^{\mathbf{L}}_{\mathsf{E}} k\big) : \mathsf{D}(E) \to \mathsf{D}(\Lambda) \quad \text{and} \quad \mathbf{R}\mathrm{Hom}_\Lambda(k, -) : \mathsf{D}(\Lambda) \to \mathsf{D}(E)$$

induce an equivalence of triangulated categories between $\mathsf{D}(\Lambda)$ and a certain subcategory of $\mathsf{D}(E)$. It is well known that the homothety map $S \to E$ of DG algebras is a quasi-isomorphism. It remains to track this subcategory of $\mathsf{D}(E)$ under the equivalence $\mathsf{D}(S) \equiv \mathsf{D}(E)$. The necessary calculations are similar to those used above to prove Theorem 7.4, so we have presented a direct approach.

8. The conormal rank of a local ring

For each local ring we introduce a numerical invariant that corresponds to the maximal rank of a free direct summand of the conormal module of a graded algebra. We start by describing a language convenient for the discussion.

8.1. Let $(R, \mathfrak{m}, k)$ be a local ring.

A *local presentation* of R is simply an isomorphism of rings $R \cong Q/I$, where $(Q, \mathfrak{q}, k)$ is a local ring. We say that such a presentation is *minimal* if $\operatorname{edim} R = \operatorname{edim} Q$, that is, if the ideal I is contained in $\mathfrak{q}^2$.

A local presentation $R \cong Q/I$ is *regular* if the ring Q is regular. If it is not minimal, then choosing an element $x \in I \smallsetminus \mathfrak{q}^2$ one obtains a regular presentation $R \cong (Q/Qx)/(I/Qx)$ with $\operatorname{edim}(Q/Qx) = \operatorname{edim} Q - 1$. Iterating this procedure, one sees that every regular presentation can be factored through a minimal one.

Regular presentations exist when R is essentially of finite type over a field, or when R is complete: the latter case comes from Cohen's Structure Theorem, which provides a presentation $R \cong Q/I$ with a complete regular local ring Q.

Recall that for an R-module M the number $\operatorname{f-rank}_R M$, called the *free rank* of M over R, is defined to be the maximal rank of a free direct summand of M.

8.2. We define the *conormal free rank* of R to be the number

$$\operatorname{cf-rank} R = \sup\left\{ \operatorname{f-rank}_{\widehat{R}}\big(I/I^2\big) \;\middle|\; \begin{array}{l} \widehat{R} \cong Q/I \text{ is a minimal} \\ \text{regular presentation} \end{array} \right\}.$$

Some estimates on the new invariant are easy to come by:

Lemma 8.3. *Let $R \cong Q'/I'$ be a minimal presentation.*

(1) *The following equality holds:* $\operatorname{cf-rank} R = \operatorname{cf-rank} \widehat{R}$.
(2) *The following inequality holds:* $\operatorname{cf-rank} R \geqslant \operatorname{f-rank}_R(I'/I'^2)$.
(3) *If $I' = Q'\boldsymbol{x} + I''$ and $\boldsymbol{x}$ is a (Q'/I'')-regular sequence of length r, then*

$$\operatorname{cf-rank} R \geqslant r + \operatorname{f-rank}_{\widehat{R}}\big(\bar{I}/\bar{I}^2\big),$$

where $\bar{I}$ denotes the ideal $I'/(Q'\boldsymbol{x})$ in $\overline{Q} = Q'/(Q'\boldsymbol{x})$.

Proof. (1) The desired equality follows directly from the definition.

(2) We may assume R and Q' are complete, as the induced isomorphism $\widehat{R} \cong \widehat{Q'}/\widehat{I'}$ is a minimal presentation and one has $\operatorname{f-rank}_{\widehat{R}}(\widehat{I'}/\widehat{I'}^2) \geqslant \operatorname{f-rank}_R(I'/I'^2)$. Choose a minimal regular presentation $Q' \cong Q/J$. The composition $Q \to Q' \to \widehat{R}$ induces a minimal regular presentation $R \cong Q/I$ and a surjective homomorphism of R-modules $I/I^2 \to I'/I'^2$, which yields

$$\operatorname{f-rank}_R\big(I/I^2\big) \geqslant \operatorname{f-rank}_R\big(I'/I'^2\big).$$

(3) This follows from a well known fact recorded in 8.4 below. $\square$

8.4. Let $R \cong Q'/I'$, where $I' = Q'\boldsymbol{x} + I''$ and $\boldsymbol{x} = \{x_1, \dots, x_r\}$. Set $\overline{Q} = Q'/(Q'\boldsymbol{x})$ and $\bar{I} = I'/(Q'\boldsymbol{x})$. If $\boldsymbol{x}$ is Q'/I''-regular, then the sequence of R-modules

$$0 \longrightarrow R^r \xrightarrow{\varkappa} I'/I'^2 \longrightarrow \bar{I}/\bar{I}^2 \longrightarrow 0$$

where $\varkappa$ maps the ith element of the standard basis of R^r to $x_i + I'^2$ is split exact.

Indeed, for $r = 1$ this is in [21, Cor., p. 458]; the general case follows by iteration.

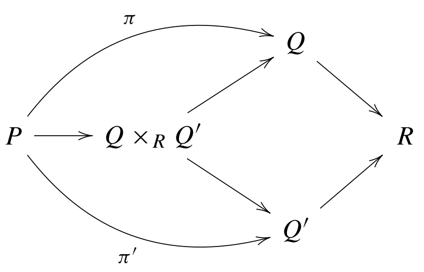
One says that R is *equicharacteristic* if $\operatorname{char}(k) \cdot R = 0$. We show that for such rings there exists a reasonable notion of *conormal module*. Whether this is so in general is related to Grothendieck's Lifting Problem, discussed by Hochster in [21].

Proposition 8.5. *Let $(R, \mathfrak{m}, k)$ be an equicharacteristic local ring.*

If $\widehat{R} \cong Q/I$ and $\widehat{R} \cong Q'/I'$ are minimal regular presentations with Q, Q' equicharacteristic, the $\widehat{R}$-modules I/I^2 and I'/I'^2 are isomorphic. In particular, one has

$$\text{cf-rank}\, R = \text{f-rank}_R\big(I/I^2\big).$$

Proof. Passing to completions, we may assume they are complete. The fiber product $Q \times_R Q'$ then is a complete equicharacteristic local ring. Choosing a minimal regular presentation of $Q \times_R Q'$, see 8.1, we get a commutative diagram of surjective homomorphisms of complete equicharacteristic local rings



As P and Q are regular, the ideal $\operatorname{Ker}(\pi)$ is generated by elements $x_1, \dots, x_r$ that form part of a regular system of parameters for P. One has

$$r = \dim P - \dim Q = \dim P - \operatorname{edim} Q = \dim P - \operatorname{edim} R.$$

By Cohen's Structure Theorem, the complete equicharacteristic local ring P is a ring of formal power series over k in $e = \dim P$ formal variables. It follows that one may find elements $x_{r+1}, \dots, x_e$ in P such that $P = k[\![x_1, \dots, x_e]\!]$; consequently, $Q = k[\![x_{r+1}, \dots, x_e]\!]$. There is thus an ideal $J'' \subseteq (x_{r+1}, \dots, x_e)$ in P such that $J''Q = I$ and $J = (x_1, \dots, x_r) + J''$ where $J = \operatorname{Ker}(P \to R)$. Now 8.4 yields

$$J/J^2 \cong I/I^2 \oplus R^r$$

as R-modules. By symmetry, a similar isomorphism holds with I' in place of I, so we get an isomorphism of finitely generated modules over the complete ring R:

$$I/I^2 \oplus R^r \cong I'/I'^2 \oplus R^r.$$

From the Krull–Remak–Schmidt Theorem one gets $I/I^2 \cong I'/I'^2$. □

9. DG algebra models for Koszul complexes

In this section $(R, \mathfrak{m}, k)$ denotes a local ring and K the Koszul complex on a minimal set of generators of $\mathfrak{m}$. We prove two results that play a significant role in the proof of our main theorem, given in the next section.

Theorem 9.1. *Let A be a non-negative DG algebra, linked to K by a sequence of quasi-isomorphisms of DG algebras, and let*

$$\mathsf{j} : \mathsf{D}(K) \xrightarrow{\equiv} \mathsf{D}(A)$$

be the induced equivalence of categories. In $\mathsf{D}(A)$ there is then an isomorphism

$$\mathsf{j}(K \otimes_R k) \simeq \bigoplus_{n \geqslant 0} \Sigma^n \mathrm{H}_0(A)^{\binom{e}{n}} \quad \text{where } e = \operatorname{edim} R.$$

A special case is worth noting: in $\mathsf{D}(K)$ one has $K \otimes_R k \simeq \bigoplus_{n \geqslant 0} \Sigma^n k^{\binom{e}{n}}$.

The second result involves the conormal free rank of R, an invariant of the ring denoted cf-rank R which is introduced and discussed in Section 8.

Theorem 9.2. *Let $(R, \mathfrak{m}, k)$ be a local ring, and set*

$$e = \operatorname{edim} R, \qquad d = \operatorname{depth} R, \qquad c = \operatorname{cf-rank} R.$$

Let Λ denote the exterior algebra $\bigwedge_k(\Sigma k^c)$ with $\partial(\Lambda) = 0$.

There exist quasi-isomorphisms of DG algebras linking K and $\Lambda \otimes_k B$, where B is a DG algebra with $B_0 = k$, $\operatorname{rank}_k B < \infty$, $\partial(B_1) = 0 = \partial(B_2)$, and

$$\sup\{i \in \mathbb{N} \mid \mathrm{H}_i(B) \neq 0\} = e - d - c.$$

To prove Theorems 9.1 and 9.2 we use DG algebra resolutions of a special type.

9.3. Let $(Q, \mathfrak{q}, k)$ be a local ring. A *semi-free Γ-extension* of Q is a DG algebra $Q\langle X\rangle$, where X is a set of divided powers indeterminates with X_n finite for each $n \geqslant 1$ and empty for $n \leqslant 0$. Thus, the graded algebra underlying $Q\langle X\rangle$ is a tensor product of the exterior algebra on the free Q-module with ordered basis X_{odd} and the divided powers algebra on the free Q-module with ordered basis X_{even}.

For $y \in X_{\mathrm{even}}$ we let $y^{(n)}$ denote the nth divided power of y. A Q-basis of $Q\langle X\rangle$ is given by all products $x_{i_1} \cdots x_{i_r} y_{j_1}^{(n_1)} \cdots y_{j_s}^{(n_s)}$ with $x_{i_u} \in X_{\mathrm{odd}}$ and $i_1 < \cdots < i_r$, with $y_{i_v} \in X_{\mathrm{even}}$ and $j_1 < \cdots < j_s$, and with $r, s, n_v \geqslant 0$.

We let $X^{(\geqslant 2)}$ denote the set of monomials with $r + n_1 + \cdots + n_s \geqslant 2$.

9.4. Let $R \cong Q/I$ be a regular presentation; see 8.1. An *acyclic closure* of R over Q is a quasi-isomorphism $\varphi : Q\langle X\rangle \to R$ of DG algebras satisfying

$$\partial(X) \subseteq I + \mathfrak{q}X + QX^{(\geqslant 2)}.$$

Every presentation can be extended to an acyclic closure, see [5, §6.3].

Lemma 9.5. *Let $\pi : R \to k$ denote the natural surjection, $\rho : R \to \widehat{R}$ the completion map, and $(Q, \mathfrak{q}) \to \widehat{R}$ a minimal regular presentation. Let E be the Koszul complex on a minimal generating set of $\mathfrak{q}$ and $\varepsilon : E \to k$ the canonical augmentation.*

If $\varphi : Q\langle X\rangle \to \widehat{R}$ is an acyclic closure, then there is a morphism $\varkappa : Q\langle X\rangle \to E$ of DG Γ-algebras making the following diagram commute.

$$\begin{array}{ccccc} R & \xrightarrow{\rho} & \widehat{R} & \xleftarrow[\simeq]{\varphi} & Q\langle X\rangle \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\widehat{\pi}}\downarrow & & \downarrow{\scriptstyle\varkappa} \\ k & = & k & \xleftarrow[\varepsilon]{\simeq} & E \end{array}$$

Furthermore, each such morphism satisfies $\varkappa(Q\langle X\rangle)_{\geqslant 1} \subseteq \mathfrak{q}E$.

Proof. The map ε is a quasi-isomorphism because Q is regular. The existence of $\varkappa$ follows from general principles: $Q\langle X\rangle$ is free as an algebra with divided powers over Q, the Q-algebra E has a system of divided powers, and $E \to k$ is a quasi-isomorphism. To finish the proof it suffices to show that $\varkappa(Q\langle X_{\leqslant n}\rangle)_{\geqslant 1} \subseteq \mathfrak{q}E$ holds for each integer $n \geqslant 0$. We proceed by induction on n.

The desired inclusion holds trivially for $n = 0$ because one has $Q\langle X_{\leqslant 0}\rangle = Q$ by definition. Assuming that the inclusion holds for some $n \geqslant 0$, one has

$$\begin{aligned} \partial\varkappa(X_{n+1}) &= \varkappa\partial(X_{n+1}) \\ &\subseteq \varkappa\bigl(IQ_0 + \mathfrak{q}X_n + (X_{\leqslant n})^{(\geqslant 2)}\bigr) \\ &\subseteq \mathfrak{q}^2E_0 + \mathfrak{q}\varkappa(X_n) + \bigl(\varkappa(X_{\leqslant n})\bigr)^{(\geqslant 2)} \\ &\subseteq \mathfrak{q}^2E. \end{aligned}$$

Indeed, the first equality holds because $\varkappa$ is a morphism of complexes. The first inclusion comes from the formula in 9.4. The second one is a consequence of the hypothesis that the presentation is minimal and the fact that $\varkappa$ is a homomorphism of algebras with divided powers. The last inclusion results from the induction hypothesis and the identity $(ay)^{(n)} = a^n y^{(n)}$ for $a \in Q$.

As an algebra with divided powers, $Q\langle X_{\leqslant n+1}\rangle$ is generated over $Q\langle X_{\leqslant n}\rangle$ by the set $X_{\leqslant n+1}$. Thus, the inclusion established above implies $\partial(\varkappa(Q\langle X_{\leqslant n+1}\rangle)_{\geqslant 1}) \subseteq \mathfrak{q}^2E$. A result of Serre [37, Ch. IV, App. I, Prop. 3], see [5, 4.1.6(2)], now yields $\varkappa(Q\langle X_{\leqslant n+1}\rangle)_{\geqslant 1} \subseteq \mathfrak{q}E$, and the induction step is complete. □

Proof of Theorem 9.1. Recall that K denotes the Koszul complex on a minimal set of generators of $\mathfrak{m}$. In view of Proposition 3.9 and 3.6.3, to prove the theorem it suffices to construct a specific non-negative DG algebra A with $\mathrm{H}_0(A) = k$ and to produce a chain of quasi-isomorphisms linking K and A, such that for the induced equivalence of categories $\mathsf{j}\colon \mathsf{D}(K) \to \mathsf{D}(A)$ there is an isomorphism

$$\mathsf{j}(K \otimes_R k) \cong \bigoplus_{n \geqslant 0} \Sigma^n k^{\binom{e}{n}}. \tag{9.5.6}$$

Let $\rho\colon R \to \widehat{R}$ be the completion map and $Q \to \widehat{R}$ a minimal regular presentation. Set $A = k \otimes_Q Q\langle X\rangle$, where $\varphi\colon Q\langle X\rangle \to \widehat{R}$ is an acyclic closure, see 9.4. Set $\widehat{K} = K \otimes_R \widehat{R}$, let E be the Koszul complex on a minimal generating set of the maximal ideal $\mathfrak{q}$ of Q, and let $\eta\colon k \to k \otimes_Q E$ denote the structure map. We claim that the following diagram of morphisms of DG Q-algebras commutes:

$$\begin{array}{ccccccccc}
k & = & k & = & k & = & k & = & k \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
K & \xrightarrow[\simeq]{K\otimes_R\rho} & \widehat{K} & = & E\otimes_Q \widehat{R} & \xleftarrow[\simeq]{E\otimes_Q\varphi} & E\otimes_Q Q\langle X\rangle & \xrightarrow[\simeq]{\varepsilon\otimes_Q Q\langle X\rangle} & A \\
\downarrow{\scriptstyle K\otimes_R\pi} & & \downarrow{\scriptstyle \widehat{K}\otimes_{\widehat{R}}\widehat{\pi}} & & \downarrow{\scriptstyle E\otimes_Q\widehat{\pi}} & & \downarrow{\scriptstyle E\otimes_Q\varkappa} & & \downarrow{\scriptstyle k\otimes_Q\varkappa} \\
K\otimes_R k & \xrightarrow{\simeq} & \widehat{K}\otimes_{\widehat{R}} k & = & E\otimes_Q k & \xleftarrow[E\otimes_Q\varepsilon]{\simeq} & E\otimes_Q E & \xrightarrow[\varepsilon\otimes_Q E]{\simeq} & k\otimes_Q E
\end{array}
\qquad \eta\colon k \to k\otimes_Q E$$

Indeed, Lemma 9.5 shows that the squares commute and yields $\varkappa(Q\langle X\rangle_{\geqslant 1}) \subseteq \mathfrak{q}E$; this implies $(k \otimes_Q \varkappa)(A_{\geqslant 1}) = 0$, so the triangle on the right commutes as well.

In the bottom left square one has quasi-isomorphisms because ρ is flat and the R-modules $\mathrm{H}(K)$ and $\mathrm{H}(K \otimes_R k)$ have finite length. The other quasi-isomorphisms hold because E and $Q\langle X\rangle$ are bounded below complexes of free Q-modules.

The middle row in the diagram is a sequence of quasi-isomorphism of DG algebras linking K with A. Let $\mathsf{j}\colon \mathsf{D}(K) \to \mathsf{D}(A)$ be the equivalence of triangulated categories induced by it. In view of the commutativity of the bottom part of the diagram, repeated application of 3.6.3 gives in $\mathsf{D}(A)$ an isomorphism

$$\mathsf{j}(K \otimes_R k) \simeq k \otimes_Q E.$$

The triangle containing η implies $A_{\geqslant 1} \cdot (k \otimes_Q E) = 0$, while the inclusion $\partial(E) \subseteq \mathfrak{q}E$ yields $\partial(k \otimes_Q E) = 0$. Thus, one has an isomorphism of DG A-modules

$$k \otimes_Q E \cong \bigoplus_{n \geqslant 0} \Sigma^n k^{\binom{e}{n}}.$$

Concatenating the isomorphisms above one obtains (9.5.6), as desired. □

The next proof is similar to that of [23, 2.1]; it uses an idea from [3].

Proof of Theorem 9.2. Let $\widehat{R} \cong Q/I$ be a minimal regular presentation such that one has $\text{f-rank}_{\widehat{R}}(I/I^2) = c$. Choose elements $\boldsymbol{a} = a_1, \ldots, a_c$ in I whose images form a basis for a free summand of I/I^2. The Koszul complex on the elements $\boldsymbol{a}$ is an exterior algebra $Q\langle x_1, \ldots, x_c\rangle$

with the $\boldsymbol{x}$ indeterminates in degree 1 and differential defined by $\partial(x_i) = a_i$. By Nakayama's Lemma $\boldsymbol{a}$ can be extended to a minimal generating set for the ideal I, so this Koszul complex can be extended to acyclic closure $Q\langle X\rangle \xrightarrow{\simeq} \widehat{R}$ with $\{x_1, \dots, x_c\}$ a subset of X_1.

We use the notation of Lemma 9.5. It follows from the proof of Theorem 9.1 that there exists a quasi-isomorphism of DG algebras

$$K \to \widehat{K} = E \otimes_Q \widehat{R} \xleftarrow{E\otimes_Q\varphi} E \otimes_Q Q\langle X\rangle.$$

We claim that one has an isomorphism of DG algebras

$$E \otimes_Q Q\langle X\rangle \cong Q\langle z_1, \dots, z_c\rangle \otimes_Q E\langle Y\rangle,$$

where $|z_i| = 1$ and $\partial(z_i) = 0$ for $i = 1, \dots, c$, and $Y_n = \varnothing$ for $n \leqslant 0$.

Indeed, there are derivations $\theta_1, \dots, \theta_c$ of the DG Γ-algebra $Q\langle X\rangle$ such that $\theta_i(x_1) = 1$ and $\theta_i(x) = 0$ for all $x \in X_1 \setminus \{x_1\}$; see [5, 6.2.7] and also [23, 1.4]. Replacing θ_i by $\theta_i - x_i\theta_i^2$ one may assume in addition $\theta_i^2 = 0$ holds. These derivations extend to E-linear derivations $\widetilde{\theta}_1, \dots, \widetilde{\theta}_c$ of the DG Γ-algebra $E\langle X\rangle = E \otimes_Q Q\langle X\rangle$ with $(\widetilde{\theta}_i)^2 = 0$. Since $I \subseteq \partial(E_1)$ holds there are elements $e_1, \dots, e_c$ in E_1 such that $\partial(e_i) = a_i$ for each i. Evidently, the elements $z_i = x_i - e_i$ are cycles of degree 1 in $E\langle X\rangle$ satisfying $\widetilde{\theta}_i(z_j) = \delta_{ij}$. Set $Y' = \{x - z_1\widetilde{\theta}_1(x) \mid x \in X \setminus \{x_1\}\}$. These are indeterminates over E and the induce a bijective morphism of DG Γ-algebras

$$Q\langle z_1\rangle \otimes_Q E\langle Y'\rangle \to E\langle X\rangle.$$

It is not hard to verify that the derivations $\widetilde{\theta}_2, \dots, \widetilde{\theta}_c$ restrict to derivations on $E\langle Y'\rangle$. Iteratiing the procedure one gets the desired isomorphism of DG algebras.

Since $E\langle Y\rangle$ is semi-free as a DG E-module, the quasi-isomorphism $\varepsilon : E \to k$ induces a quasi-isomorphism of DG algebras

$$E\langle Y\rangle = E\langle Y\rangle \otimes_E E \xrightarrow{E\langle Y\rangle\otimes_E\varepsilon} E\langle Y\rangle \otimes_E k = C$$

where $C = k\langle Y\rangle$. In turn, it induces the quasi-isomorphism below:

$$Q\langle z_1, \dots, z_c\rangle \otimes_Q E\langle Y\rangle \xrightarrow{\simeq} Q\langle z_1, \dots, z_c\rangle \otimes_Q C \cong \Lambda \otimes_k C.$$

The description of the acyclic closure $Q\langle X\rangle$, see 9.4, shows that C is non-negative, $\operatorname{rank}_k C_n$ is finite for each n, $C_0 = k$, and $\partial(C_1) = 0 = \partial(C_2)$ holds. Set $b = e - d - c$ and note the equalities

$$\sup\{i \in \mathbb{N} \mid \mathrm{H}_i(C) \neq 0\} = \sup\{i \in \mathbb{N} \mid \mathrm{H}_i(K) \neq 0\} - c = b.$$

The first one comes from the isomorphisms $\mathrm{H}(K) \cong \mathrm{H}(\Lambda \otimes_k C) \cong \Lambda \otimes_k \mathrm{H}(C)$. The second one is from the Auslander–Buchsbaum Formula. It is easy to check that

$$D = \cdots \to C_{b+2} \to C_{b+1} \to \partial(C_{b+1}) \to 0$$

is a DG ideal of C with $\mathrm{H}(D) = 0$. Thus, $B = C/D$ is a DG k-algebra with $B_n = 0$ for $n > b$ and the canonical surjection $C \to B$ is a quasi-isomorphism of DG algebras. It induces a quasi-isomorphism of DG algebras $\Lambda \otimes_k C \to \Lambda \otimes_k B$.

Now we have produced a DG algebra B that has the required properties, and we have linked K with $\Lambda \otimes_k B$ by a chain of quasi-isomorphisms of DG algebras. □

10. Loewy length of homology of perfect complexes

In this section we prove the following result; it contains Theorem 3.

Theorem 10.1. *Let $(R, \mathfrak{m}, k)$ be a local ring. Each finite free complex F of R-modules with $\mathrm{H}(F) \neq 0$ satisfies the following inequalities:*

$$\sum_{n\in\mathbb{Z}} \ell\ell_R \mathrm{H}_n(F) \geqslant \mathrm{level}_R^k(F) \geqslant \text{cf-rank}\, R + 1.$$

Recall that the *closed fiber* of a local homomorphism $(P, \mathfrak{p}) \to Q$ is the local ring $Q/\mathfrak{p}Q$. The following corollary contains Theorem 2 from the introduction.

Corollary 10.2. *If $\widehat{R}$ is the closed fiber of a flat local homomorphism $P \to Q$, then*

$$\sum_{n\in\mathbb{Z}} \ell\ell_R \mathrm{H}_n(F) \geqslant s + 1,$$

where $s = \mathrm{edim}\, P - \mathrm{edim}\, Q + \mathrm{edim}\, R$. In particular, one has $\mathfrak{m}^s \neq 0$.

Proof. Set $p = \mathrm{edim}\, P$, $q = \mathrm{edim}\, Q$, and $r = \mathrm{edim}\, R$. Let $\boldsymbol{x} = x_1, \ldots, x_p$ be a generating set for the maximal ideal $\mathfrak{p}$ of P; thus, $\widehat{R} = Q/\boldsymbol{x}Q$. Reindexing, if necessary, we may assume $x_i \in \mathfrak{q} \setminus \mathfrak{q}^2$ for $1 \leqslant i \leqslant q - r$ and $x_i \in \mathfrak{q}^2$ otherwise.

Set $Q' = Q/(x_1, \ldots, x_{q-r})$ and $I' = (x_{q-r+1}, \ldots, x_p)Q'$. One gets a minimal presentation $\widehat{R} \cong Q'/I'$ with $I'/I'^2 \cong (\mathfrak{p}/\mathfrak{p}'') \otimes_k R$, where $\mathfrak{p}'' = (x_1, \ldots, x_{q-r})P + \mathfrak{p}^2$. Thus the $\widehat{R}$-module I'/I'^2 is free of rank $p - q + r = s$, so the theorem applies. □

Theorem 10.1 has an analogue for complexes of finite injective dimension.

Corollary 10.3. *Let C be a bounded complex of injective R-modules with $\mathrm{H}(C)$ finitely generated and non-zero. The following inequalities then hold:*

$$\sum_{n\in\mathbb{Z}} \ell\ell_R \mathrm{H}_n(C) \geqslant \mathrm{level}_R^k(C) \geqslant \text{cf-rank}\, R + 1.$$

Proof. We may assume $\ell\ell_R \mathrm{H}(C)$ is finite. Since the R-module $\mathrm{H}(C)$ is finitely generated, this assumption implies that $\ell_R\, \mathrm{H}(C)$ is finite, see 6.1.2, so Theorem 6.2(3) yields the desired upper bound on $\mathrm{level}_R^k(C)$.

Let E be the injective hull of k. Since $\ell_R\, \mathrm{H}(C)$ is finite, one may replace C with a quasi-isomorphic complex if necessary, and assume that it is a finite complex with each C_i a finite direct sum of copies of E. Thus the complex $F = \mathrm{Hom}_R(C, E)$ is finite free over $\widehat{R}$ and the functor $\mathrm{Hom}_R(-, E) \colon \mathsf{D}(R)^{\mathrm{op}} \to \mathsf{D}(\widehat{R})$ is exact, so

$$\mathrm{level}_R^k(C) \geqslant \mathrm{level}_{\widehat{R}}^k(F)$$

holds by Lemma 2.4(6). Combining this with Theorem 10.1, we get the desired lower bound on $\operatorname{level}_R^k(C)$. □

Given this corollary, it is clear that one has also an analogue of Corollary 10.2 for complexes, and, in particular, for modules, of finite injective dimension.

The preceding results are optimal, in the sense that all the inequalities involved may become equalities, as the following example demonstrates.

Example 10.4. Let $f_1, \dots, f_p$ be a regular sequence in $k[\![y_1, \dots, y_q]\!]$. The local ring $R = k[\![y_1, \dots, y_q]\!]/(f_1, \dots, f_p)$ is the closed fiber of the flat homomorphism

$$k[\![x_1, \dots, x_p]\!] \longrightarrow k[\![y_1, \dots, y_q]\!]$$

of complete k-algebras that sends x_i to f_i for $i = 1, \dots, p$.

Take $f_i \in \mathfrak{q}^2$ for $i = 1, \dots, p$, and F the Koszul complex on the images in R of $y_1, \dots, y_q$. One then has $\mathrm{H}_1(F) \cong k^p$ and $\mathrm{H}(F) \cong \bigwedge \mathrm{H}_1(F)$, whence the equalities in the next display, while Theorems 10.1 and 8.2 give the inequalities:

$$p + 1 = \sum_{n \in \mathbb{Z}} \ell\ell_R \mathrm{H}_n(F) \geqslant \operatorname{level}_R^k(F) \geqslant \text{cf-rank}\, R + 1 \geqslant p + 1.$$

Thus, equalities hold throughout. As one has $p = \operatorname{edim} P$ and $\operatorname{edim} Q = q = \operatorname{edim} R$, the inequality in Corollary 10.2 is sharp.

Take $p = q$, let E be the injective hull of the R-module k, and form the complex $C = \operatorname{Hom}_R(F, E)$ of injective R-modules. The isomorphisms

$$\mathrm{H}_i(C) \cong \operatorname{Hom}_R\big(\mathrm{H}_{-i}(F), E\big) \cong \operatorname{Hom}_k\big(\textstyle\bigwedge_k^i(k^p), k\big)$$

show that all three quantities in the formula of Corollary 10.3 are equal to $p + 1$.

Proof of Theorem 10.1. Several DG algebras are used in the argument. Set

$$c = \text{cf-rank}\, R$$

and fix the following notation for the duration of the proof:

K is the Koszul complex on a minimal generating set for the ideal $\mathfrak{m}$,

Λ is the exterior algebra over k on c variables of degree 1, with $\partial^\Lambda = 0$.

By Theorem 9.2 there exists a DG k-algebra B with $\operatorname{rank}_k B < \infty$, such that

$$A = \Lambda \otimes_k B \quad \text{is linked to } K \text{ by a sequence of quasi-isomorphisms.}$$

The argument hinges on a sequence of exact functors of triangulated categories

$$\mathsf{D}(R) \xrightarrow{\ \mathsf{k}\ } \mathsf{D}(K) \xrightarrow[\equiv]{\ \mathsf{j}\ } \mathsf{D}(A) \xrightarrow{\ \mathsf{i}\ } \mathsf{D}(\Lambda)$$

where the first category is the derived category of R-modules, the rest are derived categories of DG modules over DG algebras. The functors are as follows:

$$\mathsf{k} = K \otimes_R -; \quad \text{it is exact because the } R\text{-module } K \text{ is flat,}$$

$$\mathsf{j} \quad \text{is the equivalence induced by the quasi-isomorphisms of Theorem 9.2,}$$

$$\mathsf{i} = \iota_* \quad \text{where } \iota : A \to A \otimes_k B \text{ is the inclusion of DG } k\text{-algebras.}$$

The flow chart below captures the structure of the argument:

The assertion of the theorem is obtained by concatenating the sequence of (in)equalities displayed in the zigzagging line in the middle.

The column on the left indicates the category in which a given numerical invariant is computed, and displays functors between such categories.

The column on the right tracks conditions needed for the final equality.

$$\begin{array}{lllll}
 & \sum_n \ell\ell_R \mathrm{H}_n(F) & & & \mathrm{H}(F) \neq 0 \ \&\ F \text{ perfect} \\
 & (1) \quad \rotatebox{90}{$\leqslant$} & & & \Downarrow \ (1') \\
\mathsf{D}(R) & \mathrm{level}_R^k(F) & & & \mathrm{level}_R^R(F) \neq 0, \infty \\
\ \downarrow \mathsf{k} & (2) \quad \rotatebox{90}{$\leqslant$} & & & \Downarrow \ (2') \\
\mathsf{D}(K) & \mathrm{level}_K^{\mathsf{k}(k)}(\mathsf{k}(F)) & & & \mathrm{level}_K^K(\mathsf{k}(F)) \neq 0, \infty \\
\ \downarrow \mathsf{j} \ \equiv & (3) \quad \Vert & & & \Downarrow \ (3') \\
\mathsf{D}(A) & \mathrm{level}_A^{\mathsf{jk}(k)}(\mathsf{jk}(F)) & \overset{(4)}{=\!=} & \mathrm{level}_A^k(\mathsf{jk}(F)) & \mathrm{level}_A^A(\mathsf{jk}(F)) \neq 0, \infty \\
\ \downarrow \mathsf{i} & & & (5) \quad \rotatebox{90}{$\leqslant$} & \Downarrow \ (5') \\
\mathsf{D}(\Lambda) & & & \mathrm{level}_\Lambda^k(\mathsf{ijk}(F)) & \mathrm{level}_\Lambda^\Lambda(\mathsf{ijk}(F)) \neq 0, \infty \\
 & & & \Vert \quad \overset{(7)}{\Leftarrow} & \Downarrow \ (6) \\
 & & & c+1 & \mathrm{H}(\mathsf{ijk}(F)) \neq 0 \ \&\ \mathsf{ijk}(F) \text{ perfect}
\end{array}$$

What follows are step by step directions for climbing down the chart.

(1) comes from Theorems 6.2(3) and 6.1.3;
(1′) holds by Example 2.5.3;
(2) holds because k is an exact functor; see Lemma 2.4(6);
(2′) is seen as follows: The exactness of k implies, as above, that $\mathrm{level}_K^K(\mathsf{k}(F))$ is finite; it is non-zero because one has $\mathrm{H}(K \otimes_R F) \neq 0$, see the end of the proof;
(3) and (3′) hold because j is an equivalence of categories; see Proposition 3.7(1);
(4) holds because one has $\mathsf{smd}(\mathsf{add}^{\Sigma}(\mathsf{jk}(k))) = \mathsf{smd}(\mathsf{add}^{\Sigma}(k))$, by Theorem 9.1;

(5) holds because i is an exact functor and $\mathsf{i}(k) = k$; see Lemma 2.4(6);
(5′) is seen as follows: $\operatorname{level}_A^A(\mathsf{ijk}(F))$ is non-zero since $\mathrm{H}(\mathsf{ijk}(F)) \cong \mathrm{H}(\mathsf{k}(F)) \neq 0$; it is finite because $\operatorname{level}_A^A(\mathsf{jk}(F))$ is and Corollary 4.6 applies;
(6) holds by definition;
(7) is Theorem 7.1.

It remains to show that $\mathrm{H}(K \otimes_R F)$ is not equal to zero. Since F is homologically bounded, the number $i = \inf\{n \in \mathbb{Z} \mid \mathrm{H}_n(F) \neq 0\}$ is finite and F is quasi-isomorphic to a complex L with $L_j = 0$ for all $j < i$. We then get

$$\mathrm{H}_i(K \otimes_R F) \cong \mathrm{H}_i(K \otimes_R L) \cong \mathrm{H}_0(K) \otimes_R \mathrm{H}_i(L) = k \otimes_R \mathrm{H}_i(F) \neq 0$$

by using flatness, right exactness of tensor products, and Nakayama's Lemma. □

11. Complete intersection local rings

We have thus far focused mainly on complexes admitting finite free resolutions. In this final section we turn to complete intersection local rings, over which we extend Theorem 10.1 to a statement applying to all homologically finite complexes.

11.1. *Complexity.* Let $(R, \mathfrak{m}, k)$ be a local ring and M a homologically finite complex of R-modules. A *minimal free resolution* of M is a quasi-isomorphism $F \xrightarrow{\simeq} M$ where F is a complex of free R-modules whose differential satisfies $\partial(F) \subseteq \mathfrak{m}F$. Such a resolution exists and is unique up to isomorphism of complexes; see [30].

The *complexity* of M over R is the number

$$\operatorname{cx}_R M = \inf\left\{ d \geqslant 0 \;\middle|\; \begin{array}{l} \text{there exists an integer } a > 0 \text{ such that} \\ \operatorname{rank}_R(F_n) \leqslant a n^{d-1} \text{ for all } n \gg 0 \end{array} \right\}.$$

Evidently, $\operatorname{cx}_R M = 0$ holds if and only if M is perfect.

11.2. Let $(R, \mathfrak{m}, k)$ be a local ring and $\widehat{R}$ its $\mathfrak{m}$-adic completion.

The ring R is said to be *complete intersection* if for some regular presentation $\widehat{R} \cong P/I$, see 8.1, the conormal module I/I^2 is free over $\widehat{R}$. This is equivalent to I being generated by a P-regular sequence; see [12, 2.2.8]. Such R satisfy

$$\operatorname{cf-rank} R = \operatorname{codim} R,$$

where $\operatorname{cf-rank} R$ is the conormal free rank of R; see Lemma 8.3(3), and $\operatorname{codim} R$ denotes the *codimension* of R, that is, the difference $\operatorname{edim} R - \dim R$.

The following result contains Theorem 7 from the Introduction. Over complete intersections it extends Theorem 10.1, from which it is deduced by using a result from [4]. A different approach to its proof is given in [8].

Theorem 11.3. *Let R be a complete intersection local ring and M a complex of R-modules with* $\mathrm{H}(M)$ *finite and non-zero. One then has inequalities*

$$\sum_{n\in\mathbb{Z}} \ell\ell_R \mathrm{H}_n(M) \geqslant \operatorname{level}_R^k(M) \geqslant \operatorname{codim} R - \operatorname{cx}_R M + 1.$$

The number $\operatorname{codim} R - \operatorname{cx}_R M$ in the inequality above is non-negative; see 11.4.2.

For the proofs we recall basic facts about complexity.

11.4. Let R be a local ring, M a complex of R-modules with $\mathrm{H}(M)$ finitely generated, and let F be a minimal free resolution of M.

11.4.1. Fix an integer s such that $\mathrm{H}_n(M) = 0$ for $n \geqslant s$ and set $C = \mathrm{H}_s(F_{\geqslant s})$. Evidently, $\Sigma^{-s} F_{\geqslant s}$ is a minimal free resolution of C, and hence $\operatorname{cx}_R C = \operatorname{cx}_R M$.

11.4.2. If there exists a local presentation $\widehat{R} \cong Q/I$ such that I is generated by a regular sequence and $\widehat{R} \otimes_R M$ is perfect over Q, then the following inequality holds:

$$\operatorname{cx}_R M \leqslant \dim Q - \dim R.$$

This follows from [4, 3.2(3)], in view of 11.4.1; see also [35, 3.10].

When the ring R is a complete intersection, [4, 3.6] provides a converse to 11.4.2 for modules. We extend that result to complexes.

Proposition 11.5. *Let R be a complete intersection with an infinite residue field and M a complex of R-modules with* $\mathrm{H}(M)$ *finitely generated.*

There exists then a minimal presentation $\widehat{R} \cong Q/J$, such that J is generated by a regular sequence of length $\operatorname{cx}_R M$ *and $\widehat{R} \otimes_R M$ is perfect over Q.*

Proof. Set $\operatorname{cx}_R M = d$, and let F be a minimal free resolution of M. The complex $\widehat{R} \otimes_R F$ is a minimal free resolution of $\widehat{R} \otimes_R M$ over $\widehat{R}$, and so $\operatorname{cx}_{\widehat{R}}(\widehat{R} \otimes M) = d$ holds. Thus, passing to $\widehat{R}$, one may assume that the ring R is complete.

Set $s = \max\{n \mid \mathrm{H}_n(M) \neq 0\}$ and $C = \mathrm{H}_s(F_{\geqslant s})$. Applying [4, 3.6] to a minimal regular presentation of R, see 8.1, one obtains a minimal presentation $R \cong Q/J$ with J generated by a Q-regular sequence of length d and C perfect over Q.

Since $\operatorname{cx}_R C = d$, see 11.4.1, it remains to verify that M is perfect over Q. The inclusion of complexes $F_{<s} \subseteq F$ yields an exact triangle

$$F_{<s} \to M \to \Sigma^s C \to \Sigma F_{<s}$$

in $\mathsf{D}(R)$. The Q-module R is perfect, because the Koszul complex on a minimal generating set for J is a free resolution. This implies that the bounded complex of free R-modules $F_{<s}$ is perfect over Q as well; see, for example, Lemma 2.5.2. The exact triangle above now implies that M is perfect over Q, as desired. □

Proof of Theorem 11.3. It suffices to verify the inequality on the right, by Theorem 6.2(3) and 6.1.3. One may assume that k is infinite. If not, for an indeterminate x over k one has flat local homomorphism $(R, \mathfrak{m}, k) \to (R', \mathfrak{m}', k')$ with $R' = R[x]_{\mathfrak{m}[x]}$ and $k' = k(x)$. The complex $R' \otimes_R M$ of R'-modules then has

$$\operatorname{level}_R^k(M) \geqslant \operatorname{level}_{R'}^{k'}(M') \quad \text{and} \quad \operatorname{cx}_R M = \operatorname{cx}_{R'} M'$$

with inequality given by Proposition 3.7, and equality by the observation that if F is a minimal free resolution of M over R, then $R' \otimes_R F$ is one of M' over R'.

Let $\widehat{R} \cong Q/J$ be a minimal presentation as in Proposition 11.5. Set $\widehat{M} = \widehat{R} \otimes_R M$; this is a complex of $\widehat{R}$-modules with $\mathrm{H}(\widehat{M})$ isomorphic to $\mathrm{H}(M)$, as the latter has finite length. One then has the following chain of (in)equalities:

$$\begin{aligned} \operatorname{level}_R^k(M) &\geqslant \operatorname{level}_{\widehat{R}}^k(\widehat{M}) \\ &\geqslant \operatorname{level}_Q^k(\widehat{M}) \\ &\geqslant \operatorname{codim} Q + 1 \\ &= \operatorname{codim} R - \operatorname{cx}_R M + 1. \end{aligned}$$

The first one holds because $\widehat{R} \otimes_R - \colon \mathsf{D}(R) \to \mathsf{D}(\widehat{R})$ is an exact functor; the second one holds because the restriction $\mathsf{D}(R) \to \mathsf{D}(Q)$ is an exact functor. The third inequality comes from Theorem 10.1, and the equality from the relations

$$\operatorname{edim} Q = \operatorname{edim} R \quad \text{and} \quad \dim Q = \dim R + \operatorname{cx}_R M,$$

which are implied by the construction of the ring Q. $\square$

References

[1] V. Alexeev, E. Meinrenken, Equivariant cohomology and the Maurer–Cartan equation, Duke Math. J. 130 (2005) 479–521.

[2] C. Allday, V. Puppe, Cohomological Methods in Transformation Groups, Cambridge Stud. Adv. Math., vol. 32, Cambridge Univ. Press, Cambridge, 1993.

[3] M. André, Le caractère additif des déviations des anneaux locaux, Comment. Math. Helv. 57 (1982) 648–675.

[4] L.L. Avramov, Modules of finite virtual projective dimension, Invent. Math. 96 (1989) 71–101.

[5] L.L. Avramov, Infinite free resolutions, in: Six Lectures on Commutative Algebra, Bellaterra, 1996, in: Progr. Math., vol. 166, Birkhäuser, Basel, 1998, pp. 1–118.

[6] L.L. Avramov, R.-O. Buchweitz, S. Iyengar, Class and rank of differential modules, Invent. Math. 169 (2007) 1–35.

[7] L.L. Avramov, H.-B. Foxby, S. Halperin, Differential graded homological algebra, preprint, 2009.

[8] L.L. Avramov, S.B. Iyengar, Cohomology over complete intersections via exterior algebras, in: Triangulated Categories, Leeds, 2006, in: London Math. Soc. Lecture Note Ser., Cambridge Univ. Press, Cambridge, in press.

[9] A. Beilinson, J. Bernstein, P. Deligne, Faisceaux pervers, Astérisque 100 (1983), Soc. Math. France.

[10] I.N. Bernstein, I.M. Gelfand, S.I. Gelfand, Algebraic vector bundles on P^n and problems of linear algebra, Funct. Anal. Appl. 12 (1978) 212–214.

[11] A. Bondal, M. Van den Bergh, Generators and representability of functors in commutative and non-commutative geometry, Mosc. Math. J. 3 (2003) 1–36.

[12] W. Bruns, J. Herzog, Cohen–Macaulay Rings, revised ed., Cambridge Stud. Adv. Math., vol. 39, Cambridge Univ. Press, Cambridge, 1998.

[13] G. Carlsson, On the homology of finite free $(\mathbb{Z}/2)^k$-complexes, Invent. Math. 74 (1983) 139–147.

[14] G. Carlsson, Free $(\mathbb{Z}/2)^k$-actions and a problem in commutative algebra, in: Transformation Groups, Poznań, 1985, in: Lecture Notes in Math., vol. 1217, Springer, Berlin, 1986, pp. 79–83.

[15] H. Cartan, S. Eilenberg, Homological Algebra, Princeton Univ. Press, Princeton, NJ, 1956.

[16] D.J. Christensen, Ideals in triangulated categories: Phantoms, ghosts and skeleta, Adv. Math. 136 (1998) 284–339.
[17] S. Ding, The associated graded ring and the index of a Gorenstein local ring, Proc. Amer. Math. Soc. 120 (1994) 1029–1033.
[18] W.G. Dwyer, J.P.C. Greenlees, S. Iyengar, Duality in algebra and topology, Adv. Math. 200 (2006) 357–402.
[19] W.G. Dwyer, J.P.C. Greenlees, S. Iyengar, Finiteness in derived categories of local rings, Comment. Math. Helv. 81 (2006) 383–432.
[20] S. Eilenberg, Homological dimension and syzygies, Ann. of Math. 64 (1956) 328–336.
[21] M. Hochster, An obstruction to lifting cyclic modules, Pacific J. Math. 61 (1975) 457–463.
[22] M. Hochster, Topics in the Homological Theory of Modules over Commutative Rings, Conf. Board Math. Sci., vol. 24, Amer. Math. Soc., Providence, RI, 1975.
[23] S. Iyengar, Free summands of conormal modules and central elements in homotopy Lie algebras of local rings, Proc. Amer. Math. Soc. 129 (2001) 1563–1572.
[24] B. Keller, Deriving DG categories, Ann. Sci. École Norm. Sup. 27 (4) (1994) 63–102.
[25] H. Krause, Derived categories, resolutions, and Brown representability, in: Interactions Between Homotopy Theory and Algebra, Chicago, 2004, in: Contemp. Math., vol. 436, Amer. Math. Soc., Providence, RI, 2007, pp. 101–139.
[26] H. Krause, D. Kussin, Rouquier's theorem on representation dimension, in: Representations of Algebras and Related Topics, Queretaro, 2004, in: Contemp. Math., vol. 406, Amer. Math. Soc., Providence, RI, 2007, pp. 95–103.
[27] S. MacLane, Homology, Grundlehren Math. Wiss., vol. 114, Springer, Berlin, 1963.
[28] A. Neeman, Triangulated Categories, Ann. of Math. Stud., vol. 148, Princeton Univ. Press, Princeton, NJ, 2001.
[29] C. Peskine, L. Szpiro, Dimension projective finie et cohomologie locale, Publ. Math. Inst. Hautes Études Sci. 42 (1973) 47–119.
[30] P. Roberts, Homological Invariants of Modules over Commutative Rings, Sem. Math. Sup., vol. 72, Presses Univ. Montréal, Montréal, 1980.
[31] P. Roberts, Multiplicities and Chern Classes in Local Algebra, Cambridge Tracts in Math., vol. 133, Cambridge Univ. Press, Cambridge, 1998.
[32] R. Rouquier, Representation dimension of exterior algebras, Invent. Math. 165 (2006) 357–367.
[33] R. Rouquier, Dimensions of triangulated categories, J. K-Theory 1 (2008) 193–256, 257–258.
[34] J.D. Sally, Superregular sequences, Pacific J. Math. 84 (1979) 465–481.
[35] S. Sather-Wagstaff, Complete intersection dimension for complexes, J. Pure Appl. Algebra 190 (2004) 267–290.
[36] L.M. Şega, Homological properties of powers of the maximal ideal of a local ring, J. Algebra 241 (2001) 827–858.
[37] J.-P. Serre, Local Algebra, Springer, Berlin, 2000.
[38] R.G. Swan, Algebraic K-Theory, Lecture Notes in Math., vol. 76, Springer, Berlin, 1968.