Distributional convolutors for Fourier transform

Jorge J. Betancor\textsuperscript{a,\,*},\textsuperscript{1}, Claudio Jerez\textsuperscript{a,1}, Sandra M. Molina\textsuperscript{b}, Lourdes Rodríguez-Mesa\textsuperscript{b,2}

\textsuperscript{a} Departamento de Análisis Matemático, Universidad de la Laguna, Campus de Anchieta, Avda. Astrofísico Francisco Sánchez, s/n, 38271 La Laguna ( Sta. Cruz de Tenerife), Spain

\textsuperscript{b} Departamento de Matemáticas. Facultad de Ciencias Exactas y Naturales, Universidad Nacional de Mar del Plata, Funes 3350 (7600), Mar del Plata, Argentina

Received 14 November 2005
Available online 24 February 2006
Submitted by Steven G. Krantz

Abstract

In this paper we complete a distributional Fourier analysis developed by Howell in a series of papers. We investigate convolution operators in the corresponding distribution spaces.

Keywords: Fourier analysis; Convolution operators; Distribution spaces

1. Introduction

In a series of papers [3–8] Howell introduced new spaces of functions and distributions to develop Fourier analysis. Our objective in this paper is to complete the results established by Howell concerning convolution operators in the considered spaces.

We collect some of the definitions and results of Howell’s theory. To simplify we consider only one dimension. Everything can be written in higher dimension without additional effort.
By \( G \) we denote the space of all those entire functions \( \Phi \) such that, for every \( k \in \mathbb{N} \),
\[
\gamma_k(\Phi) = \sup_{|\Im z| \leq k} e^{k|\Re z|} |\Phi(z)| < \infty.
\]

\( G \) is a Fréchet space when it is endowed with the topology associated with the family \( \{\gamma_k\}_{k \in \mathbb{N}} \) of norms. The space \( G \) is continuously contained in the Schwartz space \( S \) of rapidly decreasing functions on \( \mathbb{R} \) [3, Corollary 7.2]. In [3, Theorem 4.1] it was established that the Fourier transform \( \mathcal{F} \) defined by
\[
\mathcal{F}(\Phi)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ixy} \Phi(y) \, dy, \quad x \in \mathbb{R},
\]
is an automorphism of \( G \). Following a standard procedure the Fourier transform \( \mathcal{F}T \) of a distribution \( T \) in the dual space \( G' \) of \( G \) is defined by transposition, that is,
\[
\langle \mathcal{F}(T)(x), \Phi(x) \rangle = \langle T(x), \mathcal{F}(\Phi)(-x) \rangle, \quad \Phi \in G.
\]

Since the space of tempered distributions \( S' \) (the dual of \( S \)) is contained in \( G' \) the distributional Fourier analysis developed by Howell can be seen as an extension of the Schwartz distributional Fourier analysis [9].

The space \( \mathcal{M}_G \) consists of all those entire functions \( \Phi \) such that, for every \( k \in \mathbb{N} \), there exists \( m \in \mathbb{N} \) for which
\[
\sup_{|\Im z| \leq k} e^{-m|\Re z|} |\Phi(z)| < \infty.
\]

\( \mathcal{M}_G \) is the space of pointwise multipliers of \( G \) [8, Theorem 1.1]. It is clear that \( G \) is contained in \( \mathcal{M}_G \).

If \( \Phi, \Psi \in G \) the convolution \( \Phi \ast \Psi \) of \( \Phi \) and \( \Psi \) is defined as usual by
\[
(\Phi \ast \Psi)(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Phi(x)\Psi(z + x) \, dx, \quad z \in \mathbb{C}.
\]

The convolution \( T \ast \Phi \) of \( T \in G' \) and \( \Phi \in G \) is the function defined by
\[
(T \ast \Phi)(z) = \langle T(x), \Phi(z + x) \rangle, \quad z \in \mathbb{C}.
\]

In [6, Theorem 6.1] it was proved that \( T \ast \Phi \in \mathcal{M}_G \) for every \( T \in G' \) and \( \Phi \in G \). However, if \( T \in G' \) and \( \Phi \in G \), \( T \ast \Phi \) is not always in \( G \). Indeed, we define the functional \( T \) on \( G \) as follows:
\[
\langle T, \Phi \rangle = \int_{-\infty}^{+\infty} \Phi(x) \, dx, \quad \Phi \in G.
\]

It is clear that \( T \in G' \). Moreover, if \( \Phi \in G \) such that \( \int_{-\infty}^{+\infty} \Phi(x) \, dx \neq 0 \), we have
\[
(T \ast \Phi)(y) = \int_{-\infty}^{+\infty} \Phi(y + x) \, dx = \int_{-\infty}^{+\infty} \Phi(x) \, dx, \quad y \in \mathbb{R}.
\]

Then \( T \ast \Phi \notin G \).

In [6, Definition 7.1] Howell called elementary convolvers for \( G' \) to those \( T \in G' \) such that \( T \ast \Phi \in G \), for every \( \Phi \in G \). Note that as a consequence of the closed graph theorem \( T \in G' \)
is an elementary convolver for $G'$ if and only if the mapping $\Phi \rightarrow T \ast \Phi$ is continuous from $G$ into itself. It is not hard to see that $\mathcal{M}_G$ is contained in $G'$ (via (2.1)). Then we can define according to (1.1) the Fourier transform of multipliers in $\mathcal{M}_G$. Thus, in [8, Corollary 4.1] it was established that the Fourier transform $F(\mathcal{M}_G)$ of $\mathcal{M}_G$ coincides with the space of elementary convolvers for $G'$. One of the objectives in this paper is to obtain representations of the elementary convolvers for $G'$ and to determine the space of functions $Oc,G$ whose dual $O'c,G$ is the space of the elementary convolvers.

Let $m \in \mathbb{Z}$. We denote by $Oc,G,m$ the space constituted by all those entire functions $\Phi$ satisfying that

$$\gamma^m_k(\Phi) = \sup_{|\text{Im} z| \leq k} |e^{m|\text{Re} z|} \Phi(z)| < \infty, \quad k \in \mathbb{N}.$$ 

$Oc,G,m$ is equipped with the topology generated by the family $\{\gamma^m_k\}_{k \in \mathbb{N}}$ of norms. Thus $Oc,G,m$ is a Fréchet space. It is clear that $G$ is contained in $Oc,G,m$. We define $Oc,G,m$ as the closure of $G$ in $Oc,G,m$. Thus we can see that $Oc,G,m$ is constituted by all those entire functions $\Phi$ such that, for every $k \in \mathbb{N}$,

$$\lim_{|\text{Im} z| \leq k, |z| \to \infty} e^{m|\text{Re} z|} |\Phi(z)| = 0.$$ 

If $m_1, m_2 \in \mathbb{Z}, m_1 < m_2$, then $Oc,G,m_2 \subseteq Oc,G,m_1$ and the inclusion is continuous. We denote by $Oc,G$ the inductive limit space $\bigcup_{m \in \mathbb{Z}} Oc,G,m$. In Theorem 2.5 below we establish that the dual space $O'c,G$ of $Oc,G$ coincides with the space of elementary convolvers for $G'$. This characterization allows us to obtain representations for the elementary convolvers for $G'$.

2. Convolution operators on the spaces $G$ and $G'$

In this section we complete the results established in [6,8] about the convolution operators on the spaces $G$ and $G'$. We obtain new characterizations for the elementary convolvers for $G'$, that is, for those $T \in G'$ such that $T \ast \Phi \in G$, for every $\Phi \in G$. Previously it is necessary to comment some definitions and properties.

If $f$ is a Lebesgue measurable function on $\mathbb{R}$ such that, for a certain $m \in \mathbb{N}$, $\sup_{x \in \mathbb{R}} e^{-m|x|} \times |f(x)| < \infty$, then $f$ defines an element of the dual space $G'$ of $G$, that we continue denoting by $f$, through

$$\langle f, \Phi \rangle = \int_{-\infty}^{+\infty} f(x) \Phi(x) \, dx, \quad \Phi \in G. \quad (2.1)$$

According to [3, Theorem 3.2(iii)], for every $z \in \mathbb{C}$, the translation operator $\tau_z$ defined by $(\tau_z f)(\omega) = f(z + \omega), \omega \in \mathbb{C}$, is an isomorphism from $G$ into itself. Then if $T \in G'$ we define the translated $\tau_z T$ of $T$ by $z \in \mathbb{C}$ as follows:

$$\langle \tau_z T, \Phi \rangle = \langle T, \tau_{-z} \Phi \rangle, \quad \Phi \in G. \quad (2.2)$$
Note that definition (2.2) can be seen, according to (2.1), as an extension of the equality
\[
\int_{-\infty}^{+\infty} (\tau z \Phi)(\omega) \Psi(\omega) d\omega = \int_{-\infty}^{+\infty} \Phi(\omega)(\tau^{-z} \Psi)(\omega) d\omega,
\]
that holds for every \( z \in \mathbb{C} \) and \( \Phi, \Psi \in \mathcal{G} \).

Next we prove a useful representation for the elements of \( \mathcal{O}_c', \mathcal{G}, m, m \in \mathbb{Z} \).

**Lemma 2.3.** Let \( m \in \mathbb{Z} \) and \( T \in \mathcal{O}_c', \mathcal{G}, m \). Then, there exist \( k \in \mathbb{N} \) and a complex regular measure \( \mu \) on the strip \( \mathcal{C}_k = \{ w \in \mathbb{C} : |\text{Im} \, w| \leq k \} \) such that
\[
\langle T, \Phi \rangle = \int_{\mathcal{C}_k} e^{m|\text{Re} \, w|} \Phi(w) \, d\mu(w), \quad \Phi \in \mathcal{O}_c, \mathcal{G}, m.
\]

**Proof.** Since \( T \in \mathcal{O}_c', \mathcal{G}, m \), there exist \( C > 0 \) and \( k \in \mathbb{N} \) for which
\[
|\langle T, \Phi \rangle| \leq C \sup_{|\text{Im} \, w| \leq k} e^{m|\text{Re} \, w|} |\Phi(w)|, \quad \Phi \in \mathcal{O}_c, \mathcal{G}, m.
\]
By taking into account that \( \lim_{|\text{Im} \, w| \leq k, |w| \to \infty} e^{m|\text{Re} \, w|} |\Phi(w)| = 0 \), for every \( \Phi \in \mathcal{O}_c, \mathcal{G}, m \), a standard procedure by using Hahn–Banach and Riesz representation theorems [10, p. 259] allows us to obtain the desired representation for \( T \).

We establish now new characterizations for the elementary convolvers for \( \mathcal{G}' \).

**Theorem 2.5.** Let \( T \in \mathcal{G}' \). The following assertions are equivalent.

(a) \( T \) is an elementary convolver for \( \mathcal{G}' \), that is, \( T * \Phi \in \mathcal{G} \), for every \( \Phi \in \mathcal{G} \).
(b) \( \mathcal{F}(T) \in \mathcal{M}_G \).
(c) \( T \in \mathcal{O}_c', \mathcal{G} \).
(d) For every \( k \in \mathbb{N} \setminus \{0\} \) there exist \( d > 0 \) and \( \ell \in \mathbb{N} \) such that
\[
T = (\tau_{di} + \tau_{-di})^\ell f,
\]
for a certain analytic function \( f \) on the strip \( \{ z \in \mathbb{C} : |\text{Im} \, z| < k \} \) satisfying that
\[
\sup_{|\text{Im} \, z| \leq k} e^{k|\text{Re} \, z|} |f(z)| < \infty.
\]

**Proof.** In [8, Corollary 4.1] it was established that (a) is equivalent to (b). The plan of the proof is the following. We will prove that (c) \( \Rightarrow \) (b), (d) \( \Rightarrow \) (c) and (b) \( \Rightarrow \) (d).

(c) \( \Rightarrow \) (b). Assume that \( T \in \mathcal{O}_c', \mathcal{G} \). Then, for every \( m \in \mathbb{Z} \), \( T \in \mathcal{O}_c', \mathcal{G}, m \).

We are going to see that \( \mathcal{F}(T) \) is an entire function such that, for every \( k \in \mathbb{N} \), there exists \( m \in \mathbb{N} \) for which
\[
\sup_{|\text{Im} \, z| \leq k} e^{-m|\text{Re} \, z|} |\mathcal{F}(T)(z)| < \infty.
\]
Indeed, according to Lemma 2.3, for every \( m \in \mathbb{N} \) there exist \( \ell \in \mathbb{N} \) and a complex regular measure \( \mu \) on the strip \( \mathcal{C}_\ell = \{ z \in \mathbb{C} : |\text{Im} \, z| \leq \ell \} \) such that
\[
\langle T, \Phi \rangle = \int_{\mathcal{C}_\ell} e^{-m|\text{Re} \, z|} \Phi(z) \, d\mu(z), \quad \Phi \in \mathcal{G}.
\]
In particular, we can write
\[ \langle \mathcal{F}(T), \Phi \rangle = \langle T(z), \mathcal{F}(\Phi)(-z) \rangle = \int_{C_\ell} e^{-m|\text{Re}z|} \mathcal{F}(\Phi)(-z) d\mu(z), \quad \Phi \in \mathcal{G}. \]

Then, by interchanging the order of integration we get
\[ \langle \mathcal{F}(T), \Phi \rangle = \int_{C_\ell} e^{-m|\text{Re}z|} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ixz} \Phi(x) dx d\mu(z) \]
\[ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \Phi(x) \int_{C_\ell} e^{-m|\text{Re}z|+ixz} d\mu(z) dx, \quad \Phi \in \mathcal{G}. \]

Hence, according to (2.1),
\[ \mathcal{F}(T)(x) = \frac{1}{\sqrt{2\pi}} \int_{C_\ell} e^{-m|\text{Re}z|+ixz} d\mu(z), \quad x \in \mathbb{R}. \]

(2.6)

Note that, the right-hand side of (2.6) defines an analytic function on the strip \( \{x \in \mathbb{C} : |\text{Im}x| < m\} \).
Hence, the arbitrariness of \( m \) allows us to conclude that \( \mathcal{F}(T) \) is an entire function. Moreover, if \( k \in \mathbb{N} \), for certain \( \ell \in \mathbb{N} \) and a complex regular measure \( \mu \) on \( C_\ell \) we can write
\[ \mathcal{F}(T)(x) = \frac{1}{\sqrt{2\pi}} \int_{C_\ell} e^{-k|\text{Re}z|+ixz} d\mu(z), \quad |\text{Im}x| \leq k. \]

Then, if \( |\mu| \) denotes the total variation of \( \mu \),
\[ |\mathcal{F}(T)(x)| \leq \frac{1}{\sqrt{2\pi}} |\mu|(C_\ell)e^{\ell|\text{Re}x|}, \quad |\text{Im}x| \leq k. \]

Thus we conclude that \( \mathcal{F}(T) \) is a multiplier of \( \mathcal{G} \).

(d) \( \Rightarrow \) (c). Assume that (d) holds. To prove (c) it is sufficient to see that \( T \) can be extended to \( \mathcal{O}_{c,\mathcal{G},-m} \) as an element of \( \mathcal{O}_{c,\mathcal{G},-m}' \), for every \( m \in \mathbb{N} \). Let \( m \in \mathbb{N} \). Then there exist \( d > 0 \) and \( \ell \in \mathbb{N} \) such that
\[ T = (\tau_{di} + \tau_{-di})^\ell f, \]
where \( f \) is a Lebesgue measurable function such that \( \sup_{y \in \mathbb{R}} e^{(m+1)|y|} |f(y)| < \infty \).
For every \( \Phi \in \mathcal{G} \) we have
\[ \langle T, \Phi \rangle = \int_{-\infty}^{+\infty} f(x)(\tau_{di} + \tau_{-di})^\ell \Phi(x) dx. \]

(2.7)

Hence,
\[ |\langle T, \Phi \rangle| \leq C \sup_{|\text{Im}z| \leq \ell d} e^{-m|\text{Re}z|} |\Phi(z)|, \quad \Phi \in \mathcal{G}. \]

Thus we prove that \( T \) can be extended to \( \mathcal{O}_{c,\mathcal{G},-m} \) as an element of \( \mathcal{O}_{c,\mathcal{G},-m}' \) and this (unique) extension is defined by the right-hand side of (2.7).

(b) \( \Rightarrow \) (d). Assume that \( F = \mathcal{F}(T) \) is a multiplier of \( \mathcal{G} \). Let \( \gamma > 0 \) and \( n \in \mathbb{N} \). We consider the function \( h_{n,\gamma}(z) = (e^{\gamma z} + e^{-\gamma z})^n, \quad z \in \mathbb{C} \). The following properties of \( h_{n,\gamma}(z) \) are clear:
(i) \( h_{n,\gamma}(z) = 0 \) if and only if \( z = \frac{\pi}{2\gamma}(1 + 2k)i, \ k \in \mathbb{Z}; \)
(ii) \( |h_{n,\gamma}(z)| \leq Ce^{n\gamma|\text{Re}z|}, \ z \in \mathbb{C}; \)
(iii) \( |h_{n,\gamma}(z)| \geq Ce^{n\gamma|\text{Re}z|}, \ |\text{Im}z| \leq \frac{\pi}{2\gamma}. \)

Moreover, by using Cauchy’s theorem to interchange the path of integration, we can see that, for every \( \Phi \in \mathcal{G} \) and \( \sigma \in \mathbb{R}, \)
\[
e^{\sigma z} \mathcal{F}(\Phi)(z) = \mathcal{F}((\tau_{-i\gamma} + \tau_{i\gamma})^n \Phi)(z), \ z \in \mathbb{C}. \]

Hence the following formula holds for every \( \Phi \in \mathcal{G}: \)
\[
h_{n,\gamma}(z) \mathcal{F}(\Phi)(z) = \mathcal{F}((\tau_{-i\gamma} + \tau_{i\gamma})^n \Phi)(z), \ z \in \mathbb{C}. \quad (2.8)\]

We define the function \( G_{n,\gamma}(z) = \frac{\mathcal{F}(z)}{h_{n,\gamma}(z)}, \ |\text{Im}z| < \pi/(2\gamma). \) Thus \( G_{n,\gamma}(z) \) is analytic in the strip \( \{z \in \mathbb{C}: |\text{Im}z| < \pi/(2\gamma)\}. \) Moreover, since \( F \in \mathcal{M}, \) if \( k \in \mathbb{N} \) there exists \( \alpha_k \in \mathbb{N} \) for which
\[
|G_{n,\gamma}(z)| \leq Ce^{(\alpha_k - n\gamma)|\text{Re}z|}, \ |\text{Im}z| \leq k, \quad (2.9)\]
provided that \( k \leq \pi/(4\gamma). \)

Let now \( k \in \mathbb{N} \setminus \{0\}. \) We choose \( \gamma > 0 \) and \( n \in \mathbb{N} \) such that \( k \leq \pi/(4\gamma) \) and \( \alpha_k - n\gamma < -k, \) where \( \alpha_k \) is defined as above. Note that according to (2.9), \( G_{n,\gamma} \in L^1(\mathbb{R}) \) and we can define the function
\[
g(x) = \mathcal{F}(G_{n,\gamma})(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ixy} G_{n,\gamma}(y) \, dy, \ x \in \mathbb{R}. \]

For every \( \Phi \in \mathcal{G}, \) according to (2.8), we have
\[
\langle T, \Phi \rangle = \{F, \mathcal{F}(\Phi)\} = \int_{-\infty}^{+\infty} G_{n,\gamma}(z)h_{n,\gamma}(z)\mathcal{F}(\Phi)(z) \, dz
= \int_{-\infty}^{+\infty} G_{n,\gamma}(z)\mathcal{F}((\tau_{-i\gamma} + \tau_{i\gamma})^n \Phi)(z) \, dz
= \int_{-\infty}^{+\infty} g(x)(\tau_{-i\gamma} + \tau_{i\gamma})^n \Phi(x) \, dx
= \{((\tau_{-i\gamma} + \tau_{i\gamma})^n g, \Phi\}. \]

Then \( T = (\tau_{-i\gamma} + \tau_{i\gamma})^n g. \) Moreover, since \( \alpha_k - n\gamma < -k, \) from (2.9) we deduce that \( g \) is an analytic function on the strip \( \{z \in \mathbb{C}: |\text{Im}z| < k\}. \) Also, by using Cauchy’s theorem to interchange the line of integration, we can prove that
\[
\sup_{|\text{Im}z| \leq k} |g(z)|e^{k|\text{Re}z|} < \infty. \]

Thus the proof of theorem is finished. \( \Box \)
3. Fourier transform and convolution on the spaces $O'_{c,G,m}$, $m < 0$

Assume throughout this section that $m < 0$. If $T \in O'_{c,G,m}$, then $T \in G'$, and as it was proved in (2.6), $\mathcal{F}(T)$ is an analytic function on the strip $\{z \in \mathbb{C}: |\text{Im} z| < -m\}$ satisfying that

$$\mathcal{F}(T)(z) = \left\langle T(\omega), \frac{1}{\sqrt{2\pi}} e^{iz\omega} \right\rangle = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{m|\text{Re} w|+izw} d\mu(w), \quad |\text{Im} z| < -m,$$

for certain $\ell \in \mathbb{N}$ and a complex regular measure $\mu$ on $\{w \in \mathbb{C}: |\text{Im} w| \leq \ell\}$. Also,

$$|\mathcal{F}(T)(z)| \leq C e^{\ell |\text{Re} z|}, \quad |\text{Im} z| < -m.$$

Note that, for every $z \in \mathbb{C}$ with $|\text{Im} z| < -m$, the function $e_z(\omega) = e^{iz\omega}$, $\omega \in \mathbb{C}$, is in $O_{c,G,m}$.

Next we establish an inversion formula for the distributional Fourier transform on $O'_{c,G,m}$.

**Theorem 3.1.** Let $m \in \mathbb{Z}$, $m < 0$, and $T \in O'_{c,G,m}$. Then

$$T(x) = \lim_{r \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-r}^{+r} \mathcal{F}(T)(z)e^{-ixz} dz,$$

in the sense of convergence in $G'$.

**Proof.** According to Lemma 2.3 and (2.6) we can write

$$\mathcal{F}(T)(z) = \frac{1}{\sqrt{2\pi}} \int_{C_k} e^{m|\text{Re} \omega|+iz\omega} d\mu(\omega), \quad |\text{Im} z| < -m,$$

for certain $k \in \mathbb{N}$ and a complex regular measure $\mu$ on the strip $C_k = \{w \in \mathbb{C}: |\text{Im} w| \leq k\}$, where $T$ is given by (2.4).

Let $r > 0$. Note that the function $h_r(x) = \frac{1}{\sqrt{2\pi}} \int_{-r}^{+r} \mathcal{F}(T)(z)e^{-ixz} dz$, $x \in \mathbb{R}$, is bounded and continuous on $\mathbb{R}$. Then $h_r$ defines an element of $G'$ by (2.1) as follows:

$$\langle h_r, \Phi \rangle = \int_{-\infty}^{+\infty} h_r(x) \Phi(x) dx, \quad \Phi \in G.$$

Let $\Phi \in G$. By interchanging the order of integration we have that

$$\langle h_r, \Phi \rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \Phi(x) \int_{-r}^{+r} e^{-ixz} \left( \int_{C_k} e^{m|\text{Re} \omega|+iz\omega} d\mu(\omega) dz \right) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{C_k} e^{m|\text{Re} \omega|} \left( \int_{-r}^{+r} e^{iz\omega} \mathcal{F}(\Phi)(z) dz \right) d\mu(\omega)$$

$$= \int_{C_k} e^{m|\text{Re} \omega|} \Phi(\omega) d\mu(\omega) - \frac{1}{\sqrt{2\pi}} \int_{C_k} e^{m|\text{Re} \omega|} \int_{|z| > r, z \in \mathbb{R}} e^{iz\omega} \mathcal{F}(\Phi)(z) dz d\mu(\omega).$$

(3.2)
Moreover,
\[
\lim_{r \to \infty} \int_{C_k} e^{m|\text{Re } \omega|} \left( \int_{|z|>r, z \in \mathbb{R}} e^{iz\omega} \mathcal{F}(\Phi)(z) \, dz \right) \, d\mu(\omega) = 0. \tag{3.3}
\]
Indeed, since \( \mathcal{F}(\Phi) \in \mathcal{G} [3, \text{Theorem 4.1}] \), it has
\[
\left| \int_{|z|>r, z \in \mathbb{R}} e^{iz\omega} \mathcal{F}(\Phi)(z) \, dz \right| \leq \int_{|z|>r, z \in \mathbb{R}} e^{k|z|} |\mathcal{F}(\Phi)(z)| \, dz
\leq C \int_{|z|>r, z \in \mathbb{R}} e^{-|z|} \, dz, \quad |\text{Im } \omega| \leq k.
\]
Then
\[
\sup_{|\text{Im } w| \leq k} \left| \int_{|z|>r, z \in \mathbb{R}} e^{iz\omega} \mathcal{F}(\Phi)(z) \, dz \right| \to 0, \quad \text{as } r \to \infty.
\]
By taking into account that the measure \( \mu \) is finite, (3.3) now follows. Hence, from (3.2) we deduce that
\[
\lim_{r \to \infty} \langle h_r, \Phi \rangle = \langle T, \Phi \rangle.
\]
Since \( \mathcal{G} \) is a dense subspace of \( \mathcal{O}_{c,G,m} \), the following uniqueness result can be deduced immediately from Theorem 3.1.

**Corollary 3.4.** Let \( m \in \mathbb{Z}, m < 0 \) and \( T \in \mathcal{O}'_{c,G,m} \). If \( \mathcal{F}(T)(z) = 0, z \in \mathbb{R} \), then \( T = 0 \).

Our next objective is to study the convolution on the space \( \mathcal{O}'_{c,G,m} \). Note firstly that if \( z \in \mathbb{C} \) the \( z \)-translation mapping \( \Phi \to \tau_z \Phi \) is an isomorphism from \( \mathcal{O}_{c,G,m} \) onto itself. Indeed, let \( z \in \mathbb{C} \) and \( \Phi \in \mathcal{O}_{c,G,m} \). It is clear that \( \Phi(\omega + z) \) is an entire function. Also, if \( k \in \mathbb{N} \) we have that
\[
e^{m|\text{Re } \omega|} |\Phi(z + \omega)| \leq e^{m|\text{Re } (\omega + z)|} |\Phi(z + \omega)| e^{-m|\text{Re } z|} \to 0,
\]
as \( |\omega| \to \infty, |\text{Im } \omega| \leq k \). Moreover,
\[
\gamma_k^m(\tau_z \Phi) \leq e^{-m|\text{Re } z|} \gamma_k^m(\Phi).
\]
Then, if \( T \in \mathcal{O}'_{c,G,m} \) and \( \Phi \in \mathcal{O}_{c,G,m} \) we define the convolution \( T \# \Phi \) by
\[
(T \# \Phi)(z) = \langle T(\omega), \Phi(z + \omega) \rangle, \quad z \in \mathbb{C}.
\]

**Proposition 3.5.** Let \( m \in \mathbb{Z}, m < 0 \) and \( T \in \mathcal{O}'_{c,G,m} \). Then, the mapping \( \Phi \to T \# \Phi \) is continuous from \( \mathcal{O}_{c,G,m} \) into itself.

**Proof.** According to Lemma 2.3 there exist \( \ell \in \mathbb{N} \) and a complex regular measure \( \mu \) on the strip \( C_\ell = \{ \omega \in \mathbb{C}: |\text{Im } \omega| \leq \ell \} \) such that
\[
\langle T, \Phi \rangle = \int_{C_\ell} e^{m|\text{Re } \omega|} \Phi(\omega) \, d\mu(\omega), \quad \Phi \in \mathcal{O}_{c,G,m}.
\]
Let $\Phi \in \mathcal{O}_{c,G,m}$. By (3.6) we can write

$$(T \# \Phi)(z) = \int_{C_{\ell}} e^{m|\text{Re}\omega|} \Phi(\omega + z) \, d\mu(\omega), \quad z \in \mathbb{C}.$$ 

Thus, by [3, Corollary 7.1], $T \# \Phi$ is an entire function and, for every $k \in \mathbb{N}$,

$$\frac{d^k}{dz^k}(T \# \Phi)(z) = \left( T \# \frac{d^k}{d\omega^k} \Phi \right)(z), \quad z \in \mathbb{C}.$$ 

Let now $r \in \mathbb{N}$. Since $\mu$ is a bounded measure on $C_{\ell}$ we get

$$e^{m|\text{Re} \ z|} |(T \# \Phi)(z)| \leq \int_{C_{\ell}} e^{m|\text{Re}(\omega + z)|} |\Phi(\omega + z)| \, d|\mu|(\omega) \leq Cy_t^{m}(\Phi), \quad |\text{Im} \ z| \leq r.$$ 

Moreover, since $\lim_{|\text{Im} \ \omega| \to r+\ell} e^{m|\text{Re} \ \omega|} |\Phi(\omega)| = 0$, also $\lim_{|\text{Im} \ z| \to r, |z| \to \infty} e^{m|\text{Re} \ z|} \times |(T \# \Phi)(z)| = 0$. Hence $T \# \Phi \in \mathcal{O}_{c,G,m}$.

Suppose now that $(\Phi_n)_{n \in \mathbb{N}}$ is a sequence in $\mathcal{O}_{c,G,m}$ such that $\Phi_n \to \Phi$ and $T \# \Phi_n \to \Psi$, as $n \to \infty$, in $\mathcal{O}_{c,G,m}$, where $\Phi, \Psi \in \mathcal{O}_{c,G,m}$. Then, for every $z \in \mathbb{C}$, $\tau_z \Phi_n \to \tau_z \Phi$, as $n \to \infty$, in $\mathcal{O}_{c,G,m}$, and hence, $(T \# \Phi_n)(z) \to (T \# \Phi)(z)$, as $n \to \infty$. Since convergence in $\mathcal{O}_{c,G,m}$ implies pointwise convergence, we get that $T \# \Phi = \Psi$. Therefore, the closed graph theorem allows us to conclude that the mapping $\Phi \to T \# \Phi$ is continuous from $\mathcal{O}_{c,G,m}$ into itself. \hfill $\square$

According to Proposition 3.5, we define the convolution $T \# S$ of $T$ and $S \in \mathcal{O}'_{c,G,m}$ as the element of $\mathcal{O}'_{c,G,m}$ given by

$$(T \# S, \Phi) = (T, S \# \Phi), \quad \Phi \in \mathcal{O}_{c,G,m}.$$ 

Thus $T \# S \in \mathcal{O}'_{c,G,m}$.

Next we establish some algebraic properties of the convolution on the space $\mathcal{O}'_{c,G,m}$.

**Proposition 3.7.** Let $m \in \mathbb{Z}$, $m<0$, and $T, S, R \in \mathcal{O}'_{c,G,m}$. Then

(a) $\mathcal{F}(T \# S) = \sqrt{2\pi} \mathcal{F}(T) \mathcal{F}(S)$.
(b) $T \# S = S \# T$.
(c) $(T \# S) \# R = T \# (S \# R)$.
(d) $T \# \delta = T$, where $\delta$ denotes as usual the Dirac functional.

**Proof.** Property (a) follows immediately by (2.6). Then (b), (c) and (d) can be established by using (a) and Corollary 3.4. Note that $\mathcal{F}(\delta) = 1/\sqrt{2\pi}$. \hfill $\square$

**References**