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Combined Effects of Concave and Convex Nonlinearities in Some Elliptic Problems*

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This paper deals with a class of semilinear elliptic Dirichlet boundary value problems where the combined effects of a sublinear and a superlinear term allow us to establish some existence and multiplicity results. © 1994 Academic Press, Inc.

1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^N$ and consider the semilinear elliptic problem

$$\begin{cases} -\Delta u = f_\lambda(x, u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $f_\lambda : \Omega \times \mathbb{R} \to \mathbb{R}$ and λ is a real parameter.

The existence of, possibly multiple, solutions of (1.1) has been extensively investigated; see for example [1, 9] for a survey. According to the behaviour of f_λ and to the kind of results one wants to prove, topological or variational methods turn out to be more appropriate.

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When f_λ is sublinear, for example, $f_\lambda = \lambda u^q$, $0 < q < 1$, sub- and supersolutions easily provide the existence of a unique positive solution of (1.1), for all $\lambda > 0$; while, if $f_\lambda = \lambda\, |u|^{q-1} u$, variational methods yield the existence of infinitely many solutions of (1.1); see for example [2].

Variational tools, such as min–max arguments, are quite convenient when f_λ is superlinear, say $f_\lambda = \lambda u + |u|^{p-1} u$, $1 < p < (N+2)/(N-2)$. Indeed, one can show that in this case, (1.1) possesses infinitely many solutions at positive energy for all λ, and at least one positive solution provided $\lambda < \lambda_1$, the first eigenvalue of $-\Delta$ on Ω with zero Dirichlet boundary conditions; see [3]. When p equals the critical Sobolev exponent, $p = (N+2)/(N-2)$, the problem becomes delicate because of the lack of compactness. However, the existence of one positive solution of (1.1) for all $0 < \lambda < \lambda_1$, respectively $(0 <)\lambda^* < \lambda < \lambda_1$, and $N > 3$, respectively $N = 3$, has been proved in [8], still by variational arguments. See also [12] for some multiplicity results. Moreover, if Ω is a ball, the positive solution is unique; see [19, 22].

The purpose of the present paper is to study (1.1) when f_λ is, roughly, the sum of a sublinear and a superlinear term. The combined effects of these two nonlinearities change considerably the structure of the solution set.

To be specific, we first look for solutions of

$$\begin{cases} -\Delta u = \lambda u^q + u^p, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

with $0 < q < 1 < p$. In Theorem 2.1 we show that there exists a positive constant $\Lambda \in \mathbb{R}$ such that a solution of (1.2) exists iff $0 < \lambda \leqslant \Lambda$. To find such a solution we use sub- and supersolutions. The essential term is here u^q and p can be arbitrary.

In contrast with the pure concave case, a second (positive) solution of (1.2) is found in Theorem 2.3 by variational arguments. Here, the term u^p plays its role and one has to take $p \leqslant (N+2)/(N-2)$.

The behaviour of $\|u_\lambda\|_\infty$ (u_λ solution of (1.2)) as $\lambda \to 0$ is investigated in Theorems 2.2 and 2.4.

Finally, in Theorem 2.5 we prove, for $\lambda > 0$ and small, the existence of infinitely many solutions of

$$\begin{cases} -\Delta u = \lambda\, |u|^{q-1} u + |u|^{p-1} u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

by taking advantage of the oddness of the nonlinearity. To be slightly more precise, the presence of the sublinear term $|u|^{q-1} u$ yields the existence of infinitely many solutions having negative energy, provided $\lambda > 0$ is close to zero. On the other hand, the term $|u|^{p-1}u$ plays an important role in proving the existence of infinitely many solutions with positive energy, provided $p < (N+2)/(N-2)$.

The paper is organized as follows. Section 2 contains the statements of Theorems 2.1–2.5; Section 3, respectively 4, contains the proofs of Theorems 2.1 and 2.2, respectively 2.3 and 2.4; Section 5 deals with the proof of Theorem 2.5; finally, in Section 6 we list some open problems.

Notation. In the rest of the paper we make use of the following notation

$L^p(\Omega)$, $1 \leqslant p \leqslant \infty$, denote Lebesgue spaces; the norm in L^p is denoted by $\|\cdot\|_p$;

$W^{k,p}(\Omega)$ denote Sobolev spaces; the norm in $W^{k,p}$ is denoted by $\|\cdot\|_{k,p}$;

H denotes $W_0^{1,2}(\Omega)$, endowed with the norm $\|u\|^2 = \int_\Omega |\nabla u|^2\, dx$;

$C, C_1, C_2, \ldots$ denote (possibly different) positive constants;

The smallest eigenvalue of

$$-\Delta\varphi = \lambda\varphi,\ x \in \Omega; \qquad \varphi = 0,\ x \in \partial\Omega$$

is denoted by λ_1; φ_1 denotes the corresponding eigenfunction satisfying $\varphi_1 > 0$ in Ω and such that $\|\varphi_1\|_2 = 1$.

2. Statements of the Results

We consider below the problem of finding solutions of the boundary value problem (1.2), namely

$$\begin{cases} -\Delta u = \lambda u^q + u^p, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $N \geqslant 1$, Δ is the Laplace operator and λ is a real parameter. To emphasize the dependence on λ, problem (1.2) often referred to as problem $(1.2)_\lambda$ (the subscript λ is omitted if no confusion arises). By a solution of (1.2) we mean, unless specifically stated, a classical solution, which satisfies (1.2) pointwise. If $u \in H$ is a solution of (1.2), we let

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{q+1}\int_\Omega |u|^{q+1}\, dx - \frac{1}{p+1}\int_\Omega |u|^{p+1}\, dx$$

denote the energy of u. Our first result is the following.

THEOREM 2.1. *For all* $0 < q < 1 < p$ *there exists* $\Lambda \in R$, $\Lambda > 0$, *such that*

1. *for all* $\lambda \in (0, \Lambda)$ *problem* $(1.2)_\lambda$ *has a minimal solution* u_λ *such that* $I_\lambda(u_\lambda) < 0$. *Moreover* u_λ *is increasing with respect to* λ;
2. *for* $\lambda = \Lambda$ *problem* $(1.2)_\lambda$ *has at least one weak solution* $u \in H \cap L^{p+1}$;
3. *for all* $\lambda > \Lambda$ *problem* $(1.2)_\lambda$ *has no solution.*

Remark 2.1. If $N \leqslant 10$ the solution found for $\lambda = \Lambda$ is in fact a classical solution, for any p. If $N \geqslant 11$ it is also a classical solution provided $p < p_N$, for some p_N. The proof of this fact follows from the arguments of [15]. See Remark 3.4 below.

Remark 2.2. The existence of one solution of $(1.2)_\lambda$, with $\lambda > 0$ sufficiently small and the Laplace operator substituted by the p-Laplacian Δ_p, has been proved in [11].

The proof of Theorem 2.1, based on the method of sub- and supersolutions, shows that $\|u_\lambda\|_\infty \to 0$ as $\lambda \downarrow 0$. Actually, for solutions with small L^∞ norm one can prove a uniqueness result.

THEOREM 2.2. *There exists* $A > 0$ *such that for all* $\lambda \in (0, \Lambda)$ *problem* $(1.2)_\lambda$ *has at most one solution* u *such that* $\|u\|_\infty \leqslant A$.

Using the variational structure of the problem (1.2) and taking advantage of the superlinear term u^p, it is possible to improve Theorem 2.1, at the expense of the standard growth restriction $p \leqslant (N+2)/(N-2)$, when $N \geqslant 3$. Henceforth we take $N \geqslant 3$. As usual, in the cases $N = 1, 2$ there is no restriction.

THEOREM 2.3. *Let* $0 < q < 1 < p \leqslant (N+2)/(N-2)$. *Then for all* $\lambda \in (0, \Lambda)$ *problem* $(1.2)_\lambda$ *has a second solution* $v_\lambda > u_\lambda$.

Roughly, one shows that $(1.2)_\lambda$ has a first solution which is a local minimum of the corresponding energy functional; a second solution is then found by means of a variant of the Mountain Pass Theorem, see [14]. In addition, one can show

THEOREM 2.4. *Let* $p = (N+2)/(N-2)$ *and suppose that* Ω *is starshaped. Then* $\|w_\lambda\|_\infty \to \infty$ *as* $\lambda \downarrow 0$, *where* w_λ *is any solution of* $(1.2)_\lambda$ *distinct from the minimal solution* u_λ.

Remark 2.3. When Ω is a ball Theorem 2.4 was proved by Peletier [16] by shooting arguments.

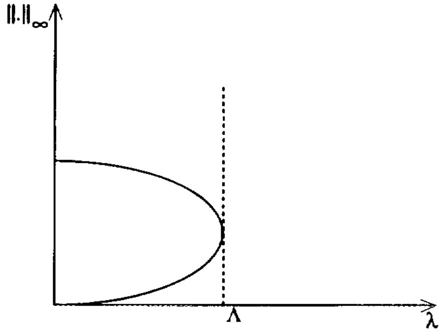

FIG. 1. Case $p < (N+2)/(N-2)$.

The above results suggest that the structure of the set of positive solutions of $(1.2)_\lambda$ looks as follows, see Figs. 1 and 2 below.

Our last result deals with the existence of infinitely many solutions (possibly not positive) of the problem (1.3). Indeed, taking advantage of the fact that the nonlinearity is now odd, one can use the stronger results of the Lusternik–Schnirelman theory for $\mathbf{Z}_2$-invariant functionals to show

THEOREM 2.5. 1. *Let* $0 < q < 1 < p \leqslant (N+2)/(N-2)$. *Then there exists* $\lambda^* > 0$ *such that for all* $\lambda \in (0, \lambda^*)$ *problem* (1.3) *has infinitely many solutions such that* $I_\lambda(u) < 0$;

2. *if* $0 < q < 1 < p < (N+2)/(N-2)$ *then for all* $\lambda \in (0, \lambda^*)$ *problem* (1.3) *has also infinitely many solutions such that* $I_\lambda(u) > 0$.

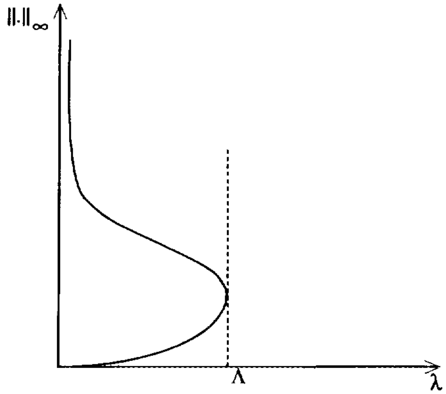

FIG. 2. Case $p = (N+2)/(N-2)$.

Remark 2.4. A result similar to the one stated in point 1 of the preceding theorem has been proved in [13] for the equation $-\Delta_p u = \lambda |u|^{q-1}u + |u|^{s-1}u$, where Δ_p denotes the p-Laplacian, $1<p<N$, $s = pN/(N-p) - 1$ and $0<q<1<s$.

Remark 2.5. It is clear from the proofs that in Theorems 2.1, 2.3, and 2.5 one can substitute u^q with any concave function that behaves like u^q near $u=0$. Similarly, when $p<(N+2)/(N-2)$, u^p can be substituted by any superlinear function that behaves like u^p near $u=0$ and near $u=+\infty$.

3. Existence of Positive Solutions

In this section we prove Theorems 2.1 and 2.2. Let us define

$$\Lambda = \sup\{\lambda > 0 : (1.2)_\lambda \text{ has a solution}\}.$$

LEMMA 3.1. $0<\Lambda<\infty$.

Proof. Let e denote the solution of

$$\begin{cases} -\Delta e = 1, & x \in \Omega, \\ \quad e = 0, & x \in \partial\Omega. \end{cases}$$

Since $0<q<1<p$, we can find $\lambda_0 > 0$ such that for all $0<\lambda \leqslant \lambda_0$ there exists $M = M(\lambda) > 0$ satisfying

$$M \geqslant \lambda M^q \|e\|_\infty^q + M^p \|e\|_\infty^p.$$

As a consequence, the function Me verifies

$$M = -\Delta(Me) \geqslant \lambda(Me)^q + (Me)^p,$$

and hence it is a supersolution of $(1.2)_\lambda$. Moreover, any $\varepsilon\varphi_1$ is a subsolution of $(1.2)_\lambda$, provided

$$\varepsilon\lambda_1\varphi_1 = -\Delta(\varepsilon\varphi_1) \leqslant \lambda\varepsilon^q\varphi_1^q + \varepsilon^p\varphi_1^p,$$

which is satisfied for all $\varepsilon > 0$ small enough and all λ. Taking ε possibly smaller, we also have

$$\varepsilon\varphi_1 < Me.$$

It follows that $(1.2)_\lambda$ has a solution $\varepsilon\varphi_1 \leqslant u \leqslant Me$ whenever $\lambda \leqslant \lambda_0$ and thus $\Lambda \geqslant \lambda_0$. Next, let $\bar{\lambda}$ be such that

$$\bar{\lambda} t^q + t^p > \lambda_1 t, \qquad \forall t \in \mathbb{R},\ t > 0. \tag{3.1}$$

If λ is such that $(1.2)_\lambda$ has a solution u, multiplying $(1.2)_\lambda$ by φ_1 and integrating over Ω we find

$$\lambda_1 \int_\Omega u\varphi_1 \, dx = \lambda \int_\Omega u^q \varphi_1 \, dx + \int_\Omega u^p \varphi_1 \, dx.$$

This and (3.1) immediately implies that $\lambda < \bar{\lambda}$ and shows that $\Lambda \leqslant \bar{\lambda}$. ▮

LEMMA 3.2. *For all* $0 < \lambda < \Lambda$ *problem* $(1.2)_\lambda$ *has a solution.*

Proof. Given $\lambda < \Lambda$, let u_μ be a solution of $(1.2)_\mu$ with $\lambda < \mu < \Lambda$. Plainly, such a u_μ is a supersolution for $(1.2)_\lambda$. Since $\varepsilon\varphi_1 < u_\mu$ provided $\varepsilon > 0$ is sufficiently small, it follows that $(1.2)_\lambda$ has a solution. ▮

We next prove that $(1.2)_\lambda$ possesses a *minimal* solution. For this we need the following lemma.

LEMMA 3.3. *Assume that* $f(t)$ *is a function such that* $t^{-1}f(t)$ *is decreasing for* $t > 0$. *Let* v *and* w *satisfy*

$$\begin{cases} -\Delta v \leqslant f(v), & x \in \Omega \\ v > 0, & x \in \Omega \\ v = 0, & x \in \partial\Omega \end{cases} \tag{3.2}$$

and

$$\begin{cases} -\Delta w \geqslant f(w), & x \in \Omega \\ w > 0, & x \in \Omega \\ w = 0, & x \in \partial\Omega. \end{cases} \tag{3.3}$$

Then $w \geqslant v$ *in* Ω.

Proof. The proof of the lemma is inspired by Method II in [6, p. 103]. From (3.2) and (3.3) we infer

$$\begin{aligned} -v\,\Delta w + w\,\Delta v &\geqslant f(w)v - f(v)w \\ &= vw\left(\frac{f(w)}{w} - \frac{f(v)}{v}\right). \end{aligned} \tag{3.4}$$

Let $\theta(t)$ be a smooth nondecreasing function such that $\theta(0) = 0$, $\theta(t) \equiv 1$ for $t \geqslant 1$ and $\theta(t) \equiv 0$ for $t \leqslant 0$. Set

$$\theta_\varepsilon(t) = \theta\left(\frac{t}{\varepsilon}\right),$$

so that $\theta_\varepsilon(t) \geqslant 0$, for all $t \in \mathbb{R}$. We multiply (3.4) by $\theta_\varepsilon(v-w)$ and integrate over Ω; this yields

$$\int_\Omega [-v\,\Delta w + w\,\Delta v]\,\theta_\varepsilon(v-w)\,dx = \int_\Omega vw\left[\frac{f(w)}{w} - \frac{f(v)}{v}\right]\theta_\varepsilon(v-w)\,dx. \qquad (3.5)$$

Observe that

$$\begin{aligned}
&\int_\Omega [-v\,\Delta w + w\,\Delta v]\,\theta_\varepsilon(v-w)\,dx \\
&\quad= \int_\Omega v\theta'_\varepsilon(v-w)\,\nabla w \cdot (\nabla v - \nabla w)\,dx \\
&\qquad- \int_\Omega w\theta'_\varepsilon(v-w)\,\nabla v \cdot (\nabla v - \nabla w)\,dx \\
&\quad= \int_\Omega v\theta'_\varepsilon(v-w)(\nabla w - \nabla v) \cdot (\nabla v - \nabla w)\,dx \\
&\qquad+ \int_\Omega (v-w)\,\theta'_\varepsilon(v-w)\,\nabla v \cdot (\nabla v - \nabla w)\,dx \\
&\quad\leqslant \int_\Omega (v-w)\,\theta'_\varepsilon(v-w)\,\nabla v \cdot (\nabla v - \nabla w)\,dx \\
&\quad= \int_\Omega \nabla v \cdot \nabla[\gamma_\varepsilon(v-w)]\,dx = -\int_\Omega \Delta v \gamma_\varepsilon(v-w)\,dx,
\end{aligned}$$

where $\gamma_\varepsilon(t) = \int_0^t s\theta'_\varepsilon(s)\,ds$. Since

$$0 \leqslant \gamma_\varepsilon(t) \leqslant \varepsilon, \qquad \forall t \in \mathbb{R},$$

then it follows

$$\int_\Omega [-v\,\Delta w + w\,\Delta v]\,\theta_\varepsilon(v-w)\,dx \leqslant \varepsilon.$$

Inserting this into (3.5) we find

$$\int_\Omega vw\left[\frac{f(w)}{w} - \frac{f(v)}{v}\right]\theta_\varepsilon(v-w)\,dx \leqslant \varepsilon.$$

As $\varepsilon \to 0$ we are led to

$$\int_{[v>w]} vw\left[\frac{f(w)}{w} - \frac{f(v)}{v}\right]dx \leqslant 0.$$

But $f(v)/v < f(w)/w$ on $[v > w]$ and therefore $\text{meas}[v > w] = 0$; thus $v \leqslant w$. This completes the proof of the lemma. ∎

LEMMA 3.4. *For all* $0 < \lambda < \Lambda$, *Problem* $(1.2)_\lambda$ *has a minimal solution* u_λ.

Proof. Let v_λ be the unique positive solution of

$$\begin{cases} -\Delta v = \lambda v^q, & x \in \Omega, \\ \quad v = 0, & x \in \partial\Omega. \end{cases}$$

We already know that there exists a solution $u > 0$ of $(1.2)_\lambda$ for every $\lambda \in (0, \Lambda)$. Since $-\Delta u \geqslant \lambda u^q$ we can use Lemma 3.3 with $w = u$ to deduce that any solution u of $(1.2)_\lambda$ must satisfy $u \geqslant v_\lambda$. Clearly, v_λ is a subsolution of $(1.2)_\lambda$. The monotone iteration

$$-\Delta u_{n+1} = \lambda u_n^q + u_n^p, \qquad u_0 = v_\lambda,$$

satisfies $u_n \uparrow u_\lambda$, with u_λ solution of $(1.2)_\lambda$. It is easy to check that u_λ is a minimal solution of $(1.2)_\lambda$. Indeed, if u is any solution of $(1.2)_\lambda$, then $u \geqslant v_\lambda$ and u is a supersolution of $(1.2)_\lambda$. Thus $u_n \leqslant u$, $\forall n$, by induction, and $u_\lambda \leqslant u$. ∎

Henceforth we use the symbol u_λ, $\lambda \in (0, \Lambda)$, to denote the minimal solution of $(1.2)_\lambda$.

Before proving Theorem 2.1 a further lemma is in order.

LEMMA 3.5. *Let* $\psi < \Psi$ *be a subsolution, respectively a supersolution, of* $(1.2)_\lambda$, *and suppose* ψ *is not a solution. Let* u *be the minimal solution such that* $\psi \leqslant u \leqslant \Psi$. *Then* $v_1 := \lambda_1[-\Delta - a(x)] \geqslant 0$, *where* $a = a(x) = \lambda q u^{q-1} + p u^{p-1}$ *and* $\lambda_1[-\Delta - a(x)]$ *denotes the first eigenvalue of* $-\Delta - a(x)$ *with zero Dirichlet boundary conditions on* $\partial\Omega$ (*see Remark* 3.1 *below*).

Remark 3.1. It is worth pointing out that the spectral theory for $-\Delta - a(x)$ can still be carried over in H, even if $a(x) = +\infty$ on $\partial\Omega$. Indeed,

$$\int_\Omega a\phi^2\, dx \leqslant C_1 \|\phi\|^2, \qquad \forall \phi \in H.$$

To see this fact, we note that

$$\int_\Omega u^{q-1}\phi^2\, dx = \int_\Omega u^q \left(\frac{1}{u}\phi\right)\phi\, dx \leqslant \|u\|_\infty^q \cdot \left\|\frac{\phi}{\delta}\right\|_2 \cdot \|\phi\|_2,$$

where $\delta(x) = \text{dist}(x, \partial\Omega)$. By the Hardy inequality,

$$\left\|\frac{\phi}{\delta}\right\|_2 \leqslant C_2 \|\nabla\phi\|_2 = C_2 \|\phi\|, \qquad \forall \phi \in H,$$

it follows that

$$\int_\Omega u^{q-1}\phi^2\,dx \leqslant C_3\,\|\phi\|^2.$$

Moreover, the map $\phi \mapsto \int_\Omega a\phi^2\,dx$ is sequentially continuous for the weak $H^{1,2}$ topology because

$$\left|\int_\Omega u^{q-1}(\phi_n^2-\phi^2)\,dx\right| \leqslant \|u\|_\infty^q \cdot \left\|\frac{\phi_n+\phi}{\delta}\right\|_2 \cdot \|\phi_n-\phi\|_2 \to 0.$$

Proof of Lemma 3.5. We follow the same method of [10] except that here the nonlinearity f_λ is not C^1 at $u=0$. By contradiction, suppose that $\nu_1<0$ and let $\bar\phi>0$ denote a corresponding eigenfunction:

$$\begin{cases} -\Delta\bar\phi - a\bar\phi = \nu_1\bar\phi, & x\in\Omega \\ \bar\phi = 0, & x\in\partial\Omega. \end{cases}$$

We claim that $u-\alpha\bar\phi$ is a supersolution of $(1.2)_\lambda$ for $\alpha>0$ small enough. Indeed, let us compute

$$\begin{aligned} &-\Delta(u-\alpha\bar\phi) - [\lambda(u-\alpha\bar\phi)^q + (u-\alpha\bar\phi)^p] \\ &\qquad = \lambda u^q + u^p - \alpha\nu_1\bar\phi - \alpha(\lambda q u^{q-1} + p u^{p-1})\,\bar\phi - \lambda(u-\alpha\bar\phi)^q - (u-\alpha\bar\phi)^p. \end{aligned}$$

Since $t\mapsto t^q$ is concave then

$$(u-\alpha\bar\phi)^q \leqslant u^q - \alpha q u^{q-1}\bar\phi,$$

and hence

$$\begin{aligned} &-\Delta(u-\alpha\bar\phi) - [\lambda(u-\alpha\bar\phi)^q + (u-\alpha\bar\phi)^p] \\ &\qquad \geqslant u^p - \alpha\nu_1\bar\phi - \alpha p u^{p-1}\bar\phi - (u-\alpha\bar\phi)^p = -\alpha\nu_1\bar\phi + o(\alpha\bar\phi) > 0, \end{aligned}$$

for $\alpha>0$ small, because $\nu_1<0$ and $\bar\phi>0$. As a consequence, $u-\alpha\bar\phi$ is a supersolution of $(1.2)_\lambda$. Moreover, since ψ is not a solution, then $u>\psi$ and, taking α possibly smaller, we can also assume that $u-\alpha\bar\phi\geqslant\psi$. Then $(1.2)_\lambda$ has a solution $\tilde u$, with $\psi\leqslant\tilde u\leqslant u-\alpha\bar\phi$, a contradiction because u is minimal. This proves the lemma. ∎

For future reference, it is worth recalling that $\lambda_1[-\Delta - a(x)]\geqslant 0$ iff

$$\int_\Omega (|\nabla\phi|^2 - a\phi^2)\,dx \geqslant 0, \qquad \forall\phi\in H. \tag{3.6}$$

Remark 3.2. Plainly, Lemma 3.5 applies to the minimal solution u_λ. In particular, (3.6) holds with $a = a_\lambda = \lambda q u_\lambda^{q-1} + p u_\lambda^{p-1}$ yielding

$$\int_\Omega (|\nabla \phi|^2 - a_\lambda \phi^2)\, dx \geqslant 0, \qquad \forall \phi \in H. \tag{3.7}$$

We are now ready to give the

Proof of Theorem 2.1. 1. From Lemmas 3.1, 3.2, and 3.4 it follows that $(1.2)_\lambda$ has a minimal solution u_λ for all $\lambda \in (0, \Lambda)$. Recall that

$$I_\lambda(u_\lambda) = \frac{1}{2} \|u_\lambda\|^2 - \frac{\lambda}{q+1} \|u_\lambda\|_{q+1}^{q+1} - \frac{1}{p+1} \|u_\lambda\|_{p+1}^{p+1}.$$

Since u_λ is a solution of $(1.2)_\lambda$ one also has

$$\|u_\lambda\|^2 = \lambda \|u_\lambda\|_{q+1}^{q+1} + \|u_\lambda\|_{p+1}^{p+1}.$$

By Lemma 3.5 and Remark 3.2, in particular from (3.7) with $\phi = u_\lambda$, we infer

$$\|u_\lambda\|^2 - \lambda q \|u_\lambda\|_{q+1}^{q+1} - p \|_\lambda\|_{p+1}^{p+1} \geqslant 0.$$

Putting together these relations one finds that $I_\lambda(u_\lambda) < 0$. To complete the proof of point 1 it remains to show that

$$u_\lambda < u_{\lambda_1} \qquad \text{whenever} \quad \lambda < \lambda_1.$$

Indeed, if $\lambda < \lambda_1$ then u_{λ_1} is a supersolution of $(1.2)_\lambda$. Since, for $\varepsilon > 0$ small, $\varepsilon\varphi_1$ is a subsolution of $(1.2)_\lambda$ and $\varepsilon\varphi_1 < u_{\lambda_1}$, then $(1.2)_\lambda$ possesses a solution v, with

$$(\varepsilon\varphi_1 \leqslant) v \leqslant u_{\lambda_1}.$$

Since u_λ is the minimal solution of $(1.2)_\lambda$, we infer that $u_\lambda \leqslant v \leqslant u_{\lambda_1}$. The strict inequality follows from the strong maximum principle, since u_λ is not identically equal to u_{λ_1}. This completes the proof of point 1.

2. Let λ_n be a sequence such that $\lambda_n \uparrow \Lambda$. Since the solutions $u_n = u_{\lambda_n}$ satisfy $I_{\lambda_n}(u_n) < 0$, it follows there exists $C > 0$ such that

$$\|\nabla u_n\|^2 \leqslant C,$$
$$\|u_n\|_{p+1}^{p+1} \leqslant C.$$

Then there exists $u^* \in H$ such that $u_n \to u^* > 0$ a.e. in Ω, strongly in L^{p+1} and weakly in H. Such a u^* is thus a weak solution of $(1.2)_\lambda$ for $\lambda = \Lambda$.

3. This follows from the definition of Λ. ∎

Remark 3.3. Since $M(\lambda)\to 0$ as $\lambda\to 0$ (see the proof of Lemma 3.1), it follows that $\|u_\lambda\|_\infty\to 0$ as $\lambda\to 0$.

Remark 3.4. Completing Remark 2.1 we recall that to prove the regularity of the solution found for $\lambda=\Lambda$ one follows the arguments of [15, Théorème 4] and uses the inequality (3.7) with $\phi=u^\gamma$, for a suitable $\gamma>0$.

To prove Theorem 2.2 we use the following

LEMMA 3.6. *Let z denote the unique solution satisfying*

$$\begin{cases} -\Delta z=z^q, & x\in\Omega,\\ z>0, & x\in\Omega,\\ z=0, & x\in\partial\Omega. \end{cases} \tag{3.8}$$

Then there exists $\beta>0$ *such that*

$$\int_\Omega [|\nabla\phi|^2-qz^{q-1}\phi^2]\,dx\geqslant\beta\,\|\phi\|_2^2,\qquad \forall\phi\in H. \tag{3.9}$$

Proof of Lemma 3.6. Let us recall that z can be obtained by

$$\min\left\{\frac{1}{2}\|u\|^2-\frac{1}{q+1}\|u\|_{q+1}^{q+1},\, u\in H\right\}.$$

As a consequence, we have that

$$\int_\Omega [|\nabla\phi|^2-qz^{q-1}\phi^2]\,dx\geqslant 0,\qquad \forall\phi\in H,$$

namely $\lambda_1[-\Delta-qz^{q-1}]\geqslant 0$. Suppose that $\lambda_1[-\Delta-qz^{q-1}]=0$. Then there exists $\phi\in H$, $\phi>0$, such that

$$-\Delta\phi-qz^{q-1}\phi=0,$$

and hence

$$\int_\Omega \nabla\phi\cdot\nabla z\,dx=q\int_\Omega z^q\phi\,dx. \tag{3.10}$$

On the other hand, using (3.8), one also has

$$\int_\Omega \nabla\phi\cdot\nabla z\,dx=\int_\Omega z^q\phi\,dx,$$

a contradiction with (3.10), because $q<1$. Thus $\lambda_1[-\Delta - qz^{q-1}]>0$, i.e. (3.9) holds. ▮

Proof of Theorem 2.2. Let $A>0$ be such that

$$pA^{p-1}<\beta,$$

where β is the value found in Lemma 3.6. We show that for every $\lambda \in (0, \Lambda)$ $(1.2)_\lambda$ has at most one solution u satisfying

$$\|u\|_\infty \leqslant A.$$

Suppose, by contradiction, that $(1.2)_\lambda$ has a second solution $w=u_\lambda+v$ such that

$$\|w\|_\infty \leqslant A. \tag{3.11}$$

Note that, since u_λ is the minimal solution of $(1.2)_\lambda$, then $v>0$. Letting $\zeta(x)=\lambda^{1/(1-q)}z(x)$ we obtain

$$-\Delta\zeta=\lambda\zeta^q.$$

Moreover, one also has

$$-\Delta u_\lambda \geqslant \lambda u_\lambda^q,$$

and therefore, using Lemma 3.5 with $f(t)=\lambda t^q$, $v=\zeta$, and $w=u_\lambda$, it follows that

$$u_\lambda \geqslant \lambda^{1/(1-q)}z. \tag{3.12}$$

Since $w=u_\lambda+v$ is a solution of $(1.2)_\lambda$ we have

$$-\Delta(u_\lambda+v)=\lambda(u_\lambda+v)^q+(u_\lambda+v)^p.$$

By concavity,

$$\lambda(u_\lambda+v)^q \leqslant \lambda u_\lambda^q+\lambda q u_\lambda^{q-1}v$$

and thus

$$-\Delta v \leqslant \lambda q u_\lambda^{q-1}v+(u_\lambda+v)^p-u_\lambda^p. \tag{3.13}$$

Moreover (3.12) implies

$$u_\lambda^{q-1} \geqslant \lambda^{-1}z^{q-1}. \tag{3.14}$$

From (3.13) and (3.14) we deduce that

$$-\Delta v \leqslant qz^{q-1}v + (u_\lambda + v)^p - u_\lambda^p.$$

On the other hand, since $w = u_\lambda + v \leqslant A$, see (3.11), we have

$$(u_\lambda + v)^p - u_\lambda^p \leqslant pA^{p-1}v$$

and hence

$$-\Delta v - qz^{q-1}v \leqslant pA^{p-1}v.$$

Multiplying this inequality by v and using (3.9) with $\phi = v$, we infer that

$$\beta \int_\Omega v^2\, dx \leqslant pA^{p-1} \int_\Omega v^2\, dx.$$

Since $pA^{p-1} < \beta$ it follows that $v = 0$. ∎

Remark 3.5. The behaviour of u_λ near $\lambda = 0$ must clearly be of the form $u_\lambda \simeq \lambda^{1/(1-q)} z$.

4. Existence of a Second Positive Solution

In this section we always assume that

$$q < 1 < p \leqslant \frac{N+2}{N-2}.$$

In particular, letting

$$f_\lambda(s) = \begin{cases} \lambda s^q + s^p & s \geqslant 0 \\ 0 & s < 0 \end{cases}$$

and

$$F_\lambda(u) = \int_0^u f_\lambda(s)\, ds,$$

we may define the functional $\bar{I}_\lambda : H \to R$ by setting

$$\bar{I}_\lambda(u) = \tfrac{1}{2}\|u\|^2 - \int_\Omega F_\lambda(u)\, dx.$$

Of course, $I_\lambda(u) = \bar{I}_\lambda(u)$ whenever $u > 0$. Moreover it is well known that critical points of $\bar{I}_\lambda$ correspond to solutions of $(1.2)_\lambda$.

In the preceding section (see in particular Lemma 3.5 and Remark 3.2) we have established the existene of a minimal solution u_λ of $(1.2)_\lambda$ such that $\nu_1 := \lambda_1[-\Delta - a_\lambda] \geqslant 0$. If $\nu_1 > 0$ this solution is a local minimum of $\bar{I}_\lambda$, but if $\nu_1 = 0$ this need not be the case. For the purpose of finding a second solution by variational methods it is essential to have a first solution which is also a local minimum. To carry over this program we begin by proving a preliminary result.

LEMMA 4.1. *For all $\lambda \in (0, \Lambda)$ Problem $(1.2)_\lambda$ has a solution u which is in addition a local minimum of $\bar{I}_\lambda$ in the C^1 topology.*

Proof. We fix $\lambda_1 < \lambda < \lambda_2 < \Lambda$ and consider the minimal solutions $u_1 := u_{\lambda_1}$ and $u_2 := u_{\lambda_2}$ defined in Theorem 2.1. Thus $u_1 \leqslant u_2$ and u_1, respectively u_2, is a subsolution, respectively supersolution, of $(1.2)_\lambda$. Moreover,

$$\begin{aligned}-\Delta(u_2 - u_1) &= \lambda_2 u_2^q + u_2^p - (\lambda_1 u_1^q + u_1^p)\\ &\geqslant \lambda_1 u_2^q + u_2^p - \lambda_1 u_1^q - u_1^p \geqslant 0 \text{ in } \Omega.\end{aligned}$$

Since $u_1 \not\equiv u_2$ (because $\lambda_1 < \lambda_2$), then the Hopf Maximum Principle yields

$$u_1 < u_2 \qquad \text{and} \qquad \frac{\partial}{\partial \nu}(u_2 - u_1) < 0,$$

where ν is the outer unit normal at $\partial\Omega$. We now follow the arguments of Theorem 9, Part IV, Section 2 of [9], see also [10], and we will therefore be brief. We set

$$\tilde{f}_\lambda(x, s) = \begin{cases} f_\lambda(u_1(x)) & s \leqslant u_1 \\ f_\lambda(s) & u_1 < s < u_2 \\ f_\lambda(u_2(x)) & s \geqslant u_2 \end{cases}$$

$$\tilde{F}_\lambda(x, s) = \int_0^u \tilde{f}_\lambda(x, s)\, ds$$

and

$$\tilde{I}_\lambda(u) = \tfrac{1}{2}\|u\|^2 - \int_\Omega \tilde{F}_\lambda(x, u)\, dx.$$

One verifies in a standard way that $\tilde{I}_\lambda$ achieves its (global) minimum at some $u \in W^{2,p}(\Omega)$, $\forall p < \infty$, and moreover,

$$-\Delta u = \tilde{f}_\lambda(x, u), \qquad x \in \Omega.$$

Using once more the Hopf Maximum Principle as above, we find that

$$\begin{cases} u_1 < u < u_2 & \text{in } \Omega \\ \dfrac{\partial}{\partial \nu}(u - u_1) < 0 & \text{on } \partial\Omega \\ \dfrac{\partial}{\partial \nu}(u - u_2) > 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

From (4.1) it follows that if

$$\|v - u\|_{C^1} = \varepsilon$$

with ε small, then $u_1 \leqslant v \leqslant u_2$. Moreover $\bar{I}_\lambda(v) - \tilde{I}_\lambda(v)$ is constant for $u_1 \leqslant v \leqslant u_2$ and therefore u is also a local minimum for $\bar{I}_\lambda$ in the C^1 topology. ∎

In the sequel λ is fixed. We look for a second solution of $(1.2)_\lambda$ of the form $u = u_0 + v$, where u_0 denotes the solution found in Lemma 4.1 above and $v > 0$. The corresponding equation for v becomes

$$-\Delta v = \lambda(u_0 + v)^q - \lambda u_0^q + (u_0 + v)^p - u_0^p. \tag{4.2}$$

Let us define

$$g(x, s) = g_\lambda(x, s) = \begin{cases} \lambda(u_0 + s)^q - \lambda u_0^q + (u_0 + s)^p - u_0^p & s \geqslant 0 \\ 0 & s < 0, \end{cases}$$

$$G(v) = G_\lambda(\lambda) = \int_0^v g(x, s)\, ds,$$

and

$$J(v) = J_\lambda(v) = \tfrac{1}{2}\|v\|^2 - \int_\Omega G(v)\, dx.$$

If $v \in H$, $v \not\equiv 0$ is a critical point of J then v is a solution of (4.2) and, by the Maximum Principle, $v > 0$ in Ω. Here $u = u_0 + v$ is a solution of $(1.2)_\lambda$ and $u \neq u_0$. We argue by contradiction and assume that $v = 0$ is the only critical point of J.

LEMMA 4.2. *$v = 0$ is a local minimum of J in H.*

Proof. In view of Theorem 8 of [9] (see also [10]) it suffices to prove that $v = 0$ is a local minimum of J in the C^1 topology. Let v^+ denote the positive part of v. Since

$$G(v^+) - F(u_0 + v^+) = -\frac{\lambda}{q+1} u_0^{q+1} - \lambda u_0^q v^+ - \frac{1}{p+1} u_0^{p+1} - u_0^p v^+,$$

then

$$
\begin{aligned}
J(v) &= \tfrac{1}{2}\|v^+\|^2 + \tfrac{1}{2}\|v^-\|^2 - \int_\Omega G(v^+)\,dx \\
&= \tfrac{1}{2}\|v^+\|^2 + \tfrac{1}{2}\|v^-\|^2 - \int_\Omega F(u_0+v^+)\,dx \\
&\quad + \int_\Omega \left[\frac{\lambda}{q+1}u_0^{q+1} + \lambda u_0^q v^+ + \frac{1}{p+1}u_0^{p+1} + u_0^p v^+\right] dx \\
&= \tfrac{1}{2}\|v^+\|^2 + \tfrac{1}{2}\|v^-\|^2 - \int_\Omega F(u_0+v^+)\,dx \\
&\quad + \int_\Omega F(u_0)\,dx + \int_\Omega (\lambda u_0^q + u_0^p)v^+\,dx.
\end{aligned}
$$

By a straightforward computation one finds

$$
\begin{aligned}
\bar{I}(u_0+v^+) &= \tfrac{1}{2}\|u_0\|^2 + \tfrac{1}{2}\|v^+\|^2 + \int_\Omega \nabla u_0\cdot\nabla v^+\,dx - \int_\Omega F(u_0+v^+)\,dx \\
&= \tfrac{1}{2}\|u_0\|^2 + \tfrac{1}{2}\|v^+\|^2 + \int_\Omega (\lambda u_0^q + u_0^p)v^+\,dx - \int_\Omega F(u_0+v^+)\,dx.
\end{aligned}
$$

Hence

$$
\begin{aligned}
J(v) &= \tfrac{1}{2}\|v^-\|^2 + \bar{I}(u_0+v^+) - \tfrac{1}{2}\|u_0\|^2 + \int_\Omega F(u_0)\,dx \\
&= \tfrac{1}{2}\|v^-\|^2 + \bar{I}(u_0+v^+) - \bar{I}(u_0).
\end{aligned}
$$

Using Lemma 4.1 it follows that

$$J(v) \geqslant \tfrac{1}{2}\|v^-\|^2 \geqslant 0$$

provided $\|v\|_{C^1} < \varepsilon$. ∎

Recall that J satisfies the Palais–Smale condition at level c, $(PS)_c$ for short, whenever any sequence $u_n \in H$ such that $J(u_n) \to c$ and $J'(u_n) \to 0$ has a convergent subsequence.

LEMMA 4.3. (i) *If* $p < (N+2)/(N-2)$ *then* J *satisfies* $(PS)_c$ *for all* c;

(ii) *if* $p = (N+2)/(N-2)$ *and if* 0 *is the only critical point of* J, *then* J *satisfies* $(PS)_c$ *for all* $c < (1/N)\,S^{N/2}$, *where* S *denotes the best Sobolev constant.*

Proof. (i) is standard, see e.g. [3].

(ii) The result follows in the same way as in [8]. ∎

Proof of Theorem 2.3. We first deal with the case $p<(N+2)/(N-2)$, when $(PS)_c$ holds for all c. Since $p>1$, it follows that for every $v>0$, $J(tv)\to-\infty$ as $t\to+\infty$ and there exists $v_1\in H$ such that $J(v_1)<0$. Then the existence of a critical point $v\neq 0$ follows from the Ghoussoub–Preiss version of the Mountain–Pass Theorem [14]; see also [17]. More precisely, letting

$$\Gamma=\{\gamma\in C([0,1],H):\gamma(0)=0,\gamma(1)=v_1\}$$

and

$$c=\inf_{\gamma\in\Gamma}\max\{J(\gamma(t)):t\in[0,1]\},$$

Lemma 4.2 implies that $c\geqslant 0$. Then one applies the Mountain–Pass Theorem whenever $c>0$ and the Ghoussoub–Preiss version whenever $c=0$.

To prove Theorem 2.3 when $p=(N+2)/(N-2)$ one has to show that $c<(1/N)\,S^{N/2}$. For this some preliminaries are in order.

Following the method of [8], we consider test functions of the form $v=t\Psi_\mu$, with

$$\Psi_\mu(x)=a\zeta(x)\left(\frac{\mu}{\mu^2+|x|^2}\right)^{(N-2)/2},$$

and where $\zeta(x)\equiv 1$ near $x=0$ (for the sake of simplicity, we assume henceforth that $0\in\Omega$), ζ has compact support in Ω and a is chosen in such a way that for $\mu=1$, Ψ_1 satisfies

$$-\Delta\Psi_1=\Psi_1^{(N+2)/(N-2)},\qquad \text{near } 0.$$

Note that there still exists $t_\mu>0$ such that $J(t_\mu\Psi_\mu)<0$. Letting

$$\Gamma_\mu=\{\gamma\in C([0,1],H):\gamma(0)=0,\gamma(1)=t_\mu\Psi_\mu\}$$

and

$$c_\mu=\inf_{\gamma\in\Gamma_\mu}\max\{J(\gamma(t)):t\in[0,1]\},$$

we claim

LEMMA 4.4. *We have*

$$c_\mu\leqslant\sup_{t\geqslant 0}J(t\Psi_\mu)<\frac{1}{N}S^{N/2}$$

provided $\mu>0$ *is small enough.*

Proof. (i) *Case* $N \geqslant 4$. First of all we claim that for all $p > 1$ there exists $\alpha = \alpha(p) > 0$ such that

$$(a+b)^p \geqslant a^p + b^p + \alpha a^{p-1} b, \qquad \forall a, b \geqslant 0. \tag{4.3}$$

Indeed, by scaling it suffices to show that

$$(1+t)^p \geqslant 1 + t^p + \alpha t, \qquad \forall 0 < t < 1,$$

which follows from the fact that

$$\lim_{t \downarrow 0} \frac{(1+t)^p - 1 - t^p}{t} = p > 0.$$

From (4.3) it follows that

$$\begin{aligned} g(x, s) &\geqslant s^p + \alpha s u_0^{p-1} && (s > 0), \\ G(x, s) &\geqslant \frac{1}{p+1} s^{p+1} + \frac{1}{2} \alpha s^2 u_0^{p-1} && (s > 0). \end{aligned} \tag{4.4}$$

Using (4.4) we infer

$$\begin{aligned} J(t\Psi_\mu) &= \tfrac{1}{2} t^2 \|\Psi_\mu\|^2 - \int_\Omega G(x, t\Psi_\mu)\, dx \\ &\leqslant \frac{1}{2} t^2 \|\Psi_\mu\|^2 - \frac{t^{p+1}}{p+1} \int_\Omega \Psi_\mu^{p+1} - \frac{t^2}{2} \alpha \int_\Omega u_0^{p-1} \Psi_\mu^2\, dx. \end{aligned}$$

Since $u_0 \geqslant a_0 > 0$ on the support of Ψ_μ, we deduce

$$J(t\Psi_\mu) \leqslant \frac{1}{2} t^2 \|\Psi_\mu\|^2 - \frac{t^{p+1}}{p+1} \int_\Omega \Psi_\mu^{p+1} - \frac{t^2}{2} \alpha a_0 \int_\Omega \Psi_\mu^2\, dx.$$

The conclusion now follows as in [8].

(ii) *Case* $N = 3$. We do not use here (4.4) but instead (recall that $p = (N+2)/(N-2) = 5$)

$$(a+b)^5 \geqslant a^5 + b^5 + 5ab^4 \qquad (a, b > 0).$$

Hence

$$G(x, s) \geqslant \tfrac{1}{6} s^6 + u_0 s^5 \qquad (s \geqslant 0)$$

and

$$J(t\Psi_\mu) \leqslant \tfrac{1}{2} t^2 \|\Psi_\mu\|^2 - \tfrac{1}{6} t^6 \int_\Omega \Psi_\mu^6\, dx - C_0 t^5 \int_\Omega \Psi_\mu^5\, dx. \tag{4.5}$$

Recall that

$$\begin{cases} B := \|\Psi_\mu\|^2 = S^{3/2} + O(\mu) \\ B' := \|\Psi_\mu\|_6^6 = S^{3/2} + O(\mu^3) \\ B'' := 5C_0 \|\Psi_\mu\|_5^5 = k\mu^{1/2} + O(\mu^{5/2}) \end{cases} \tag{4.6}$$

with $k > 0$. Let

$$d(\mu) = \max_{t \geqslant 0} \{\tfrac{1}{2}Bt^2 - \tfrac{1}{6}B't^6 - \tfrac{1}{5}B''t^5\}.$$

Since $d(\mu)$ is achieved for $t = t(\mu)$ satisfying

$$B = B't^4 + B''t^3,$$

then, by (4.6) it follows that

$$t(\mu) = 1 - \frac{k}{4S^{3/2}}\mu^{1/2} + o(\mu^{1/2}).$$

As a consequence we infer that

$$d(\mu) = \frac{1}{3}S^{3/2} - \frac{k}{5}\mu^{1/2} + o(\mu^{1/2})$$

and therefore $c_\mu \leqslant d(\mu) < \frac{1}{3}S^{3/2}$ for $\mu > 0$ small enough. This completes the proof of the lemma. ∎

Proof of Theorem 2.3 *completed.* By Lemma 4.4 we know that $c \leqslant c_\mu < (1/N)\, S^{1/N}$. Then, according to Lemma 4.3, $(PS)_c$ holds and the conclusion follows as in the case when $p < (N+2)/(N-2)$. ∎

Remark 4.1. The solutions we have found are in H. But one knows by [7] that they belong to every L^α and thus they are classical solutions.

Proof of Theorem 2.4. We argue by contradiction. Suppose that there exist $\lambda \downarrow 0$ and solutions v_n of $(1.2)_{\lambda_n}$, different from the minimal solutions u_{λ_n} and such that $\|v_n\|_\infty \leqslant C$. Since the minimal solution u_λ satisfies $\|u_\lambda\|_\infty < A$ as $\lambda \downarrow 0$ (see Remark 3.3), then by Theorem 2.2 it follows that $\|v_n\|_\infty > A$, for n large. On the other hand, elliptic estimates imply that, up to a subsequence, v_n converges uniformly to some $w \in H$ and

$$-\Delta w = w^{(N+2)/(N-2)}.$$

Since Ω is star-shaped, $w = 0$; a contradiction. ∎

Remark 4.2. In the case $q=0$ Problem $(1.2)_\lambda$ becomes

$$\begin{cases} -\Delta u=\lambda+u^p, & x\in\Omega,\\ u=0, & x\in\partial\Omega,\end{cases}$$

and the existence of at least two positive solutions when $p=(N+2)/(N-2)$ has been established in [21] for every $\lambda>0$ sufficiently small.

5. Existence of Infinitely Many Solutions

In this section we prove Theorem 2.5. We are brief because the arguments to prove part 1 are similar to those of [13], while part 2 is closely related to Section 2 of [3]. We let $p\leqslant(N+2)/(N-2)$ and for $u\in H$ define

$$I(u)=I_\lambda(u)=\frac{1}{2}\|u\|^2-\frac{\lambda}{q+1}\|u\|_{q+1}^{q+1}-\frac{1}{p+1}\|u\|_{p+1}^{p+1}.$$

The critical points of I_λ on H give rise to solutions of $(1.3)_\lambda$. Let $B_r=\{u\in H:\|u\|\leqslant r\}$. Using the Sobolev and Hölder inequalities one has

$$I_\lambda(u)\geqslant\tfrac{1}{2}\|u\|^2-\lambda C_1\|u\|^{q+1}-C_2\|u\|^{p+1}. \tag{5.1}$$

From this one readily finds that there exists $\lambda^*>0$ such that for all $\lambda\in(0,\lambda^*]$ there are $r, a>0$ such that

(I.1) $I_\lambda(u)\geqslant a$ for all $\|u\|=r$;

(I.2) I_λ is bounded from below on B_r.

Moreover, one easily shows that

(I.3) I_λ satisfies (PS) on B_r.

Henceforth we fix $\lambda\in(0,\lambda^*]$ and drop the subscript λ.

After these preliminaries, let us give the

Proof of Theorem 2.5 (1). We set

$$\Sigma=\{A\subset H:0\notin A, u\in A\Rightarrow -u\in A\}.$$

For $A\in\Sigma$ the $\mathbf{Z}_2$-genus of A is denoted by $\gamma(A)$ (see, for example, [3]). We set also

$$\mathscr{A}_{n,r}=\{A\in\Sigma: A\text{ compact}, A\subset B_r, \gamma(A)\geqslant n\}.$$

Clearly, $\mathscr{A}_{n,r} \neq \emptyset$ for all $n = 1, 2, \ldots$, because

$$S_{n,\varepsilon} := \partial(H_n \cap B_\varepsilon) \in \mathscr{A}_{n,r}.$$

Here H_n denotes an n-dimensional subspace of H. Let

$$b_{n,r} = \inf_{A \in \mathscr{A}_{n,r}} \max_{u \in A} I(u).$$

Each $b_{n,r}$ is finite because of (I.2). Moreover, one has

$$b_{n,r} < 0, \qquad \forall n \in \mathbf{N}. \tag{5.2}$$

Indeed, let $w \in H_n$ be such that $\|w\| = \varepsilon$. From

$$I(w) \leqslant \tfrac{1}{2}\varepsilon^2 - \lambda C_1 \varepsilon^{q+1},$$

it follows that $I(w) < 0$ provided $\varepsilon > 0$ is small enough, and this suffices to prove (5.2).

Next, let us note that for all $u \in B_r \cap \{I \leqslant 0\}$ the steepest descent flow η_t (defined through the pseudo-gradient vector field, see e.g. the Deformation Lemma in [3]) is well defined for $t \in [0, \infty)$ and

$$\eta_t(u) \in B_r \cap \{I \leqslant 0\} \qquad \forall t \geqslant 0,$$

because of (I.1). Since, by (5.2), $b_{n,r} < 0$ and (PS) holds in B_r, see (I.3), we can make use of the Lusternik–Schnirelman theory to find infinitely many critical points of I in B_r such that $I(u) < 0$. This proves point 1 of Theorem 2.5.

Proof of Theorem 2.5(2). Here we take $p < (N+2)/(N-2)$. As mentioned before, we adapt the arguments of [3], where we refer for more details. First of all, let us remark that (I.1) is nothing but the assumption (I_1) of [3]. Moreover, letting

$$\hat{A}_0 = B_r \cup \{I \geqslant 0\},$$

one clearly has that $H_n \cap \hat{A}_0$ is bounded for all $n \in \mathbf{N}$; i.e., (I_5) of [3] holds true. Next we set

$$\Gamma^* = \{h \in C(H, H) : h \text{ is an odd homeomorphism and } h(B_1) \subset \hat{A}_0\},$$

$$\Gamma_n = \{K \in \Sigma : \gamma(K \cap h(\partial B_1)) \geqslant n, \forall h \in \Gamma^*\},$$

and

$$c_n = \inf_{K \in \Gamma_n} \max_{u \in K} I(u).$$

After noting that Lemma 2.7 of [3] still holds in the present setting, we remark that, choosing

$$h(u) = ru,$$

r defined in (I.1 − 3), one has that $h \in \Gamma^*$. Therefore one finds again that $K \cap B_r \neq \emptyset$, $\forall K \in \Gamma_n$, and it follows that

$$c_n \geqslant a > 0.$$

Moreover, letting $\phi = \eta_1$ we obtain

$$\phi^{-1}(\hat{A}_0) \subset \hat{A}_0. \tag{5.3}$$

To prove this fact, it suffices to take $u \in B_r$, otherwise if $I(u) > 0$ then $I(\phi^{-1}(u)) > I(u)$ and (5.3) holds trivially. Now, if $u \in B_r$ and $w := \phi^{-1} \notin B_r$ (otherwise there is again nothing to prove), then there exists $\tau < 1$ such that $\eta_\tau(w) \in \partial B_r$. By (I.1) one finds that $I(\eta_\tau(w)) \geqslant a > 0$. Hence $I(w) \geqslant I(\eta_\tau(w)) > 0$ and $w \in \hat{A}_0$, proving (5.3).

Finally, as for J, see Lemma 4.3(i), I satisfies (PS), too. Hence Theorem 2.8 of [3] applies and yields the existence of infinitely many critical points of I such that $I(u) > 0$. ∎

6. Some Open Problems

In this final section we indicate some open questions.

(a) Suppose that $p > (N+2)/(N-2)$, Ω is a ball and $N \geqslant 3$. Does $(1.2)_\lambda$ have two positive solutions for $\lambda > 0$ small enough?

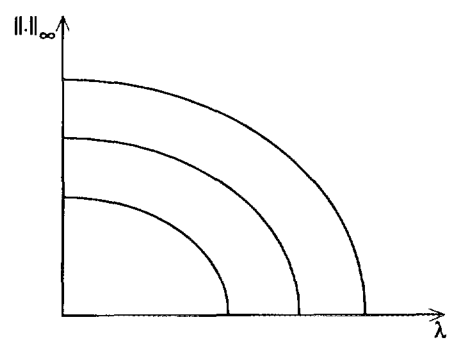

Fig. 3. Solution set of when $q = 1$ and $p < (N+2)/(N-2)$.

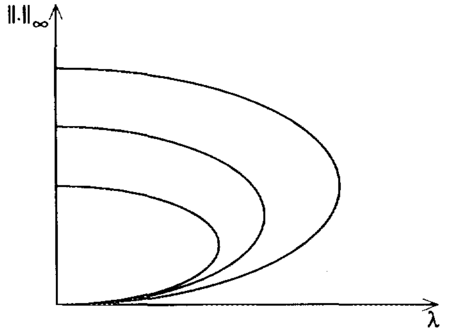

FIG. 4. Expected solution set when $0<q<1$ and $p<(N+2)/(N-2)$.

(b) Does $(1.3)_\lambda$ have infinitely many solutions with positive energy when $p=(N+2)/(N-2)$, for $\lambda>0$ small enough?

(c) Does $(1.3)_\lambda$ have infinitely many solutions at negative and/or positive energy for *all* $\lambda>0$? If the answer to question (c) is positive, then the diagram of the solution set looks as follows, see Figs. 3 and 4 below.

It is perhaps worth recalling that the answer is indeed positive in the following two limiting cases:

1. when $q=1$, $p<(N+2)/(N-2)$; and
2. when $q=0$, $p<N/(N-2)$.

In the former $(1.3)_\lambda$ becomes

$$-\Delta u=\lambda u+|u|^{p-1}u,\ x\in\Omega, \qquad u=0,\ x\in\partial\Omega,$$

that can be handled by Theorem 3.32 of [3]. In the latter $(1.3)_\lambda$ becomes

$$-\Delta u=|u|^{p-1}u+\lambda,\ x\in\Omega, \qquad u=0,\ x\in\partial\Omega$$

that can be handled by the results of [5], see also [4, 18, 20].

(d) It could be interesting to study in detail the structure of all solutions of $(1.2)_\lambda$ and $(1.3)_\lambda$ in the special case when $N=1$ and $\Omega=[a,b]$.

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Note added in proof. We were informed by M. Willem that M. Ramos has observed that problem (c) can be answered positively when p is subcritical and λ is any real number, using a version of the Symmetric Mountain Pass Theorem given in the lecture notes of M. Willem, "Un lemme de déformation quantitatif en calcul des variations", Univ. Catholique de Louvain, 1992; see pp. 76–77.

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