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# Expressions for the g-Drazin inverse of additive perturbed elements in a Banach algebra ☆

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ABSTRACT

We study additive properties of the $g$-Drazin inverse in a Banach algebra $\mathcal{A}$. In our development we derive a representation of the resolvent of a $2 \times 2$ matrix with entries in $\mathcal{A}$, which is then used to find explicit expressions for the $g$-Drazin inverse of the sum $a + b$, under new conditions on $a, b \in \mathcal{A}$. As an application of our results we obtain a representation for the Drazin inverse of a $2 \times 2$ complex block matrix in terms of the individual blocks, under certain conditions.



## 1. Introduction

Let $\mathcal{A}$ be a complex Banach algebra with unit 1. We write $\rho(a)$ and $\sigma(a)$ for the resolvent set and the spectrum of $a \in \mathcal{A}$, respectively. The set of all *quasinilpotent* elements of $\mathcal{A}$ will be denoted by $\mathcal{A}^{qnil}$. Banach algebras basic definitions can be found in [24, Chapter 10].

An element $a \in \mathcal{A}$ is said to have a *g-Drazin inverse* if there exists $x \in \mathcal{A}$ such that

$$xa = ax, \quad x = ax^2, \quad a - a^2x \in \mathcal{A}^{qnil}. \tag{1.1}$$

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If such element $x$ exists, then it is unique and is denoted by $a^D$. We denote by $\mathcal{A}^D$ the set of all $g$-Drazin invertible elements of $\mathcal{A}$. We observe that, if $a \in \mathcal{A}^{qnil}$ then $a^D = 0$ and if $a$ is nonsingular then $a^D = a^{-1}$. We recall that if $a, b \in \mathcal{A}^{qnil}$ and $ab = ba = 0$ or $ab = 0$, then $a + b \in \mathcal{A}^{qnil}$.

The $g$-Drazin inverse was studied by Koliha [18] in Banach algebras and by Koliha and Patrício [20] in rings. Harte gave an alternative definition of a generalized Drazin inverse in a ring [14], which coincide with the $g$-Drazin in the context of Banach algebras. A link between Drazin and Moore-Penrose properties in $C^*$-algebras and rings can be found in [19,23].

It is known that $a \in \mathcal{A}$ has a $g$-inverse if and only if 0 is not an accumulation point of $\sigma(a)$.

When we have in (1.1) that $a - a^2x$ is $k$-nilpotent, which is equivalent to have $a^{k+1}x = a^k$, then $a^D$ is the conventional *Drazin inverse* of $a$ and the integer $k$ is the *Drazin index* of $a$, denoted by $\operatorname{ind}(a)$. For a development of the theory of the Drazin inverse and its applications we refer an interested reader to the books [1,25].

For $a, b \in \mathcal{A}$, $ab$ is $g$-Drazin invertible if and only if $ba$ is $g$-Drazin invertible and

$$(ab)^D = a((ba)^2)^D b = a((ba)^D)^2 b. \tag{1.2}$$

When 0 is an isolated point of the spectrum of an element $a \in \mathcal{A}$, we write $a^\pi$ for the *spectral idempotent* of $a$ corresponding to $\{0\}$. In this case, the resolvent, $R(\lambda, a) = (\lambda 1 - a)^{-1}$, has a Laurent series [18]

$$R(\lambda, a) = \sum_{n=1}^{\infty} \lambda^{-n} a^{n-1} a^\pi - \sum_{n=0}^{\infty} \lambda^n (a^D)^{n+1}, \tag{1.3}$$

on some punctured disc $\{\lambda : 0 < |\lambda| < r\}$, $r > 0$.

Some recent papers deal with the problem of finding an explicit expression for the $g$-Drazin inverse of $a + b$ in terms of $a$, $b$ $a^D$, and $b^D$. In [12], it was solved under assumption that $ab = 0$ in the context of the Banach algebra of all bounded linear operators on a complex Banach space, and for complex matrices in [16]. In [5,10], expressions for $(a + b)^D$ were given under different sets of conditions, relaxing condition $ab = 0$. Formulas for the Drazin inverse of the sum of four complex square matrices were obtained in [8]. The papers [6,7] deal with the subject of perturbation analysis of the Drazin inverse, which is connected with additive results.

Our aim in this paper is to investigate the existence of the $g$-Drazin inverse of the sum $a + b$ and to give explicit expression for $(a + b)^D$. We pay special attention to the case in which either $a$ or $b$ is quasinilpotent and the case in which both of them are quasinilpotent. We note that if $a, b \in \mathcal{A}^{qnil}$, then $a^D = b^D = 0$, and, thus, we would need to introduce an element different from $a$, $b$, and its $g$-Drazin inverses, in the desired formula for $(a + b)^D$. One goal was to give conditions under which the $g$-Drazin inverse of $a + b$ with $a \in \mathcal{A}^{qnil}$ could be expressed in terms of $a, b, ab, b^D$, and $(ab)^D$. With this in mind, we first assume that $b$ and $ab$ are $g$-Drazin invertible elements, and conditions $a^2b = ab^2 = 0$, and we conclude that $a + b$ is $g$-Drazin invertible and obtain an explicit expression for $(a + b)^D$. This result appears in Section 3 and it extends to the setting of Banach algebras the main result of [3] established for bounded linear operators. Theorem 3.1 was used to prove Theorem 3.2, in which the weaker conditions $a^Db = 0$, $a^2ba^\pi = ab^2a^\pi = 0$ are assumed. The auxiliary results used in our development involve the resolvent of a $2 \times 2$ matrix with entries in a Banach algebra, and the square of its $g$-Drazin inverse and they are presented in Section 2. Several specializations of our main result are given in Section 4 and, finally, we show in Section 5 an application of our results to obtain representations of the Drazin inverse of a $2 \times 2$ complex block matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ in terms of the individual blocks, under some conditions. In respect of this, Theorem 5.2 generalizes the result given in [15] for the case $BC = 0$, $BD = 0$ and $D$ nilpotent, and it is an extension of the well known result for block triangular matrices [22]. Formulas for the Drazin inverse of $2 \times 2$ block matrices under other conditions are given in [9,26].

In the following, we introduce a class of matrices with elements in a Banach algebra which will be use to proof the main result of this paper. We consider the system of complementary idempotents $\mathcal{P} = (p_1, p_2)$ where $p_1 = 1 - a^\pi = aa^D$ and $p_2 = a^\pi$. The set of matrices

$$\mathcal{M}_2(\mathcal{A}, \mathcal{P}) = \left\{ \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}_{\mathcal{P}} : x_{ij} \in p_i \mathcal{A} p_j, \text{ for all } i, j \in \{1, 2\} \right\}$$

is a Banach algebra with the usual matrix operations, with the unit $\begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix}_{\mathcal{P}}$. For brevity, we denote $\mathcal{A}_i = p_i \mathcal{A} p_i$ which is an algebra with unit $p_i$, $i = 1, 2$.

We may work with matrix representations of elements in $\mathcal{A}$ by identifying $a \in \mathcal{A}$ with the matrix in $\mathcal{M}_2(\mathcal{A}, \mathcal{P})$ defined as $\begin{bmatrix} p_1 a p_1 & p_1 a p_2 \\ p_2 a p_1 & p_2 a p_2 \end{bmatrix}_{\mathcal{P}}$.

It is known that $a \in \mathcal{A}^D$ has the following matrix representation

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_{\mathcal{P}}, \quad a_1 \in \mathcal{A}_1^{-1} \quad \text{and} \quad a_2 \in \mathcal{A}_2^{qnil},$$

and the generalized Drazin inverse of $a$ is given by

$$a^D = \begin{bmatrix} a_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{\mathcal{P}},$$

where the inverse $a_1^{-1}$ is taken in $\mathcal{A}_1$.

The proof of the following result can be found in [5, Theorem 2.3], and will be needed in Section 3.

**Theorem 1.1.** *Let $a \in \mathcal{A}$ be represented as $a = \begin{bmatrix} x & 0 \\ z & y \end{bmatrix}_{\mathcal{P}}$. If $x \in \mathcal{A}_1^D$, $y \in \mathcal{A}_2^D$, then $a \in \mathcal{A}^D$ and*

$$a^D = \begin{bmatrix} x^D & 0 \\ u & y^D \end{bmatrix}_{\mathcal{P}}, \tag{1.4}$$

*where*

$$u = \sum_{i=0}^{\infty} (y^D)^{i+2} z x^i x^\pi + \sum_{i=0}^{\infty} y^\pi y^i z (x^D)^{i+2} - y^D z x^D. \tag{1.5}$$

*Moreover, if $a \in \mathcal{A}^D$ and $x \in \mathcal{A}_1^D$, then $y \in \mathcal{A}_2^D$ and $a^D$ is given by (1.4) and (1.5).*

## 2. Preleminary results

By $\mathcal{M}_2(\mathcal{A})$ we denote the set of all $2 \times 2$ matrices with entries in $\mathcal{A}$ which is a complex Banach algebra (usual matrix operations) with unit $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, under the operator norm. Throughout this paper whenever the lower limit of a sum is bigger than its upper limit, it is assumed that the sum is equal 0.

The proof of the following theorem is similar to the proof of an analogous result for operator matrix [17, Proposition 1.1].

**Theorem 2.1.** *Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{M}_2(\mathcal{A})$. The following conditions are equivalent*

(i) $\lambda \in \rho(A) \cap \rho(a)$.
(ii) $\lambda \in \rho(d + cR(\lambda, a)b) \cap \rho(a)$.

*Moreover, in this case, by denoting $S(\lambda) = \lambda 1 - d - cR(\lambda, a)b$, we have*

$$R(\lambda, A) = \begin{bmatrix} R(\lambda, a)(1 + bS(\lambda)^{-1} cR(\lambda, a)) & R(\lambda, a)bS(\lambda)^{-1} \\ S(\lambda)^{-1} cR(\lambda, a) & S(\lambda)^{-1} \end{bmatrix}. \tag{2.1}$$

Now, we establish a crucial auxiliary result.

**Theorem 2.2.** *Let $A = \begin{bmatrix} a & c \\ 1 & b \end{bmatrix}$, where $a, b, c \in \mathcal{A}$ are such that $ac = 0$ and $cb = 0$. Then $\Gamma := \{\lambda \neq 0 : \lambda \in \rho(a) \cap \rho(b)$ and $\lambda^2 \in \rho(c)\} \subseteq \rho(A)$, and for $\lambda \in \Gamma$ we have*

$$R(\lambda,A)=\begin{bmatrix} {}^2R(\lambda^2,c)R(\lambda,a) & cR(\lambda^2,c) \\ {}^2R(\lambda,b)R(\lambda^2,c)R(\lambda,a) & {}^2R(\lambda,b)R(\lambda^2,c)\end{bmatrix}. \tag{2.2}$$

**Proof.** First, since $ac=0$ it follows that $R(\lambda,a)c=\lambda^{-1}c$ for $\lambda\in\rho(a)\setminus\{0\}$. Analogously, since $cb=0$ we get $cR(\lambda,b)=\lambda^{-1}c$ for $\lambda\in\rho(b)\setminus\{0\}$. Now, let $S(\lambda)=\lambda 1-b-R(\lambda,a)c$. Then $S(\lambda)=\lambda 1-b-c\lambda^{-1}=\lambda^{-2}(\lambda^2 1-c)(\lambda 1-b)$ for $\lambda\in\rho(a)\setminus\{0\}$. Hence, we have that $S(\lambda)^{-1}=\lambda^2R(\lambda,b)R(\lambda^2,c)$ on the set $\Gamma$.

Applying Theorem 2.1, we obtain that if $\lambda\in\Gamma$ then $\lambda\in\rho(A)$ and, in this case, (2.1) holds. Finally, substituting $S(\lambda)^{-1}$ by $\lambda^2R(\lambda,b)R(\lambda^2,c)$ in (2.1) after simplifications we get (2.2). $\square$

Using the previous result we obtain the following lemma.

**Lemma 2.3.** *Let $A=\begin{bmatrix} a & c\\ 1 & b\end{bmatrix}$, where $a\in\mathcal{A}^{qnil}$, $b,c\in\mathcal{A}$, such that $ac=0$, $cb=0$. If $b^D,c^D$ exist then $A^D$ exists and*

$$(A^D)^2=\begin{bmatrix} y & 0\\ \sum_{n=1}^{\infty} z_{n-1}a^{n-1} & z_{-1}\end{bmatrix},$$

*where*

$$y=\sum_{k=0}^{\infty}(c^D)^{k+1}a^{2k}, \tag{2.3}$$

*and for any $n\geqslant 0$, by denoting for $\nu_n$ the integer part of $n/2$,*

$$z_{n-1}=\sum_{k=1}^{\infty}(b^D)^{n+2k}c^{k-1}c^{\pi}-\sum_{k=0}^{\nu_{n-1}}(b^D)^{n-2k}(c^D)^{k+1}+\sum_{k=\nu_{n-1}+1}^{\infty}b^{\pi}b^{2k-n}(c^D)^{k+1}.$$

**Proof.** Since $a\in\mathcal{A}^{qnil}$, then for any $\lambda\neq 0$ we have

$$R(\lambda,a)=\sum_{n=1}^{\infty}\lambda^{-n}a^{n-1}. \tag{2.4}$$

Since $0\in\operatorname{iso}\sigma(b)$ we have that $R(\lambda,b)$ has a Laurent series

$$R(\lambda,b)=\sum_{n=1}^{\infty}\lambda^{-n}b^{n-1}b^{\pi}-\sum_{n=0}^{\infty}\lambda^{n}(b^D)^{n+1},$$

on some punctured disc $\{\lambda:0<|\lambda|<r_1\}$. Analogously, since $0\in\operatorname{iso}\sigma(c)$ we have

$$R(\lambda^2,c)=\sum_{n=1}^{\infty}\lambda^{-2n}c^{n-1}c^{\pi}-\sum_{n=0}^{\infty}\lambda^{2n}(c^D)^{n+1}, \tag{2.5}$$

on some punctured disc $\{\lambda:0<|\lambda|^2<r_2\}$. From Theorem 2.2 it follows that $\Omega=\{\lambda:0<|\lambda|<\min\{r_1,\sqrt{r_2}\}\}$ satisfies $\Omega\subset\rho(A)$ and identity (2.2) holds for $\lambda\in\Omega$. Consequently, $A\in\mathcal{M}_2(\mathcal{A})$ has a Laurent series as in (1.3) on some punctured disc $\{\lambda:0<|\lambda|<r_3\}$. We observe that the coefficient of $\lambda$ in $R(\lambda,A)$ is given by $-(A^D)^2$.

Next, we shall obtain the coefficient of $\lambda$ on the right hand side of (2.2). In view of expansions (2.4) and (2.5), we easily see that the coefficient $\lambda^{-1}$ in $R(\lambda^2,c)R(\lambda,a)$ is given by $-y$, where $y$ is defined as in (2.3).

On the other hand, the coefficient $\lambda^{-1}$ in $R(\lambda,b)R(\lambda^2,c)R(\lambda,a)$ is given by

$$-\sum_{n=1}^{\infty}z_{n-1}a^{n-1},$$

where $-z_{n-1}$ is the coefficient $\lambda^{n-1}$ in $R(\lambda,b)R(\lambda^2,c)$. For any $n\geqslant 0$, $-z_{n-1}$ is obtained either multiplying the coefficient of $\lambda^{n-1+2k}$ in $R(\lambda,b)$ and the coefficient of $\lambda^{-2k}$ in $R(\lambda^2,c)$ for all $k\geqslant 1$ or multiplying the coefficient of $\lambda^{n-1-2k}$ in $R(\lambda, b)$ and the coefficient of $\lambda^{2k}$ in $R(\lambda^2, c)$ for any $k \geqslant 0$. Therefore, $z_{n-1}$ is as in (2.3). $\square$

## 3. Main result

Throughout, for any integer $n$, by $\nu_n$ we denote the integer part of $n/2$. First, we prove an important special case of our main result.

**Theorem 3.1.** *If $a \in \mathcal{A}^{qnil}$, $b, ab \in \mathcal{A}^D$, and $a^2b = ab^2 = 0$, then $(a+b)^D$ exists and*

$$(a+b)^D = \sum_{n=0}^{\infty} x_n a^n,$$

*where, for all $n \geqslant 0$*

$$\begin{aligned} x_n &= \sum_{k=0}^{\infty} (b^D)^{n+1+2k}(ab)^k(ab)^\pi - \sum_{k=0}^{\nu_n - 1}(b^D)^{n-2k-1}((ab)^D)^{k+1} \\ &\quad + \sum_{k=\nu_n}^{\infty} b^\pi b^{2k+1-n}((ab)^D)^{k+1}. \end{aligned} \tag{3.1}$$

**Proof.** In the Banach algebra $\mathcal{M}_2(\mathcal{A})$, applying formula (1.2) and Lemma 2.3, we have

$$\begin{aligned} \begin{bmatrix} a+b & 0 \\ 0 & 0 \end{bmatrix}^D &= \left( \begin{bmatrix} 1 & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ 1 & 0 \end{bmatrix} \right)^D \\ &= \begin{bmatrix} 1 & b \\ 0 & 0 \end{bmatrix} \left( \begin{bmatrix} a & ab \\ 1 & b \end{bmatrix}^D \right)^2 \begin{bmatrix} a & 0 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y & 0 \\ \sum_{n=1}^{\infty} z_{n-1}a^{n-1} & z_{-1} \end{bmatrix} \begin{bmatrix} a & 0 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} b \sum_{n=0}^{\infty} z_{n-1}a^n + ya & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

where $y$ and $z_{n-1}$ are defined as in (2.3), with $c = ab$. Next, we note that the integers $\nu_{n-1}$, for $n \geqslant 0$, satisfy $\nu_{n-1} = \begin{cases} \nu_n, & n \text{ odd} \\ \nu_n - 1, & n \text{ even} \end{cases}$. By denoting $\chi_n := \begin{cases} 1, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$ and, using that $b(b^D)^j = (b^D)^{j-1}$ for all $j \geqslant 2$, we get

$$\begin{aligned} bz_{n-1} &= \sum_{k=1}^{\infty} (b^D)^{n+2k-1}c^{k-1}c^\pi - \sum_{k=0}^{\nu_n - 1}(b^D)^{n-2k-1}(c^D)^{k+1} - \chi_n bb^D(c^D)^{\nu_n+1} \\ &\quad + \sum_{k=\nu_n}^{\infty} b^\pi b^{2k+1-n}(c^D)^{k+1} - \chi_n b^\pi (c^D)^{\nu_n+1}. \end{aligned}$$

Finally, in view of the relation above and taking into account the following identity

$$ya - \sum_{n=0}^{\infty} \chi_n bb^D(c^D)^{\nu_n+1}a^n = \sum_{n=0}^{\infty} \chi_n b^\pi (c^D)^{\nu_n+1}a^n,$$

we conclude, after replacing $c$ by $ab$, that

$$(a+b)^D = b\sum_{n=0}^{\infty} z_{n-1}a^n + ya = \sum_{n=0}^{\infty} x_n a^n,$$

where $x_n$ is defined as in (3.1) for all $n \geqslant 0$. $\square$

Next we present our main theorem.

**Theorem 3.2.** *Assume that $a, b, ab \in \mathcal{A}^D$. If $a^D b = 0$, $a^2 b a^\pi = ab^2 a^\pi = 0$ then*

$$\begin{aligned}(a+b)^D &= a^D + (b^\pi - x_1 ab - x_0 a) \sum_{i=0}^{\infty} (a+b)^i b (a^D)^{i+2} \\ &\quad + x_0(a^\pi - ba^D) + \sum_{n=1}^{\infty} x_n a^n \left( a^\pi - \sum_{i=0}^{\infty} a^i b (a^D)^{i+1} \right),\end{aligned} \tag{3.2}$$

*where, for all $n \geqslant 0$, $x_n$ is defined as in (3.1).*

**Proof.** Taking into account the identification of elements in $\mathcal{A}$ with matrices in $\mathcal{M}_2(\mathcal{A}, \mathcal{P})$ as mentioned in the introduction, we may write

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_{\mathcal{P}}, \quad a_1 \in \mathcal{A}_1^{-1}, \quad a_2 \in \mathcal{A}_2^{qnil}, \quad b = \begin{bmatrix} b_1 & b_{12} \\ b_{21} & b_2 \end{bmatrix}_{\mathcal{P}}.$$

Then,

$$a^D b = \begin{bmatrix} a_1^{-1} b_1 & a_1^{-1} b_{12} \\ 0 & 0 \end{bmatrix}_{\mathcal{P}}.$$

Hence, from $a^D b = 0$ it follows that $b_1 = 0$ and $b_{12} = 0$ and, thus,

$$b = \begin{bmatrix} 0 & 0 \\ b_{21} & b_2 \end{bmatrix}_{\mathcal{P}}, \quad ab = \begin{bmatrix} 0 & 0 \\ a_2 b_{21} & a_2 b_2 \end{bmatrix}_{\mathcal{P}}. \tag{3.3}$$

Since $b^D$ and $(ab)^D$ exist, from Theorem 1.1 we obtain that $b_2, a_2 b_2 \in \mathcal{A}_2^D$ and

$$\begin{aligned} b^D &= \begin{bmatrix} 0 & 0 \\ (b_2^D)^2 b_{21} & b_2^D \end{bmatrix}_{\mathcal{P}}, \quad (ab)^D = \begin{bmatrix} 0 & 0 \\ ((a_2 b_2)^D)^2 a_2 b_{21} & (a_2 b_2)^D \end{bmatrix}_{\mathcal{P}}, \\ b^\pi &= \begin{bmatrix} p_1 & 0 \\ -b_2^D b_{21} & b_2^\pi \end{bmatrix}_{\mathcal{P}}, \quad (ab)^\pi = \begin{bmatrix} p_1 & 0 \\ -(a_2 b_2)^D a_2 b_{21} & (a_2 b_2)^\pi \end{bmatrix}_{\mathcal{P}}. \end{aligned} \tag{3.4}$$

Expressing the conditions $a^2 b a^\pi = ab^2 a^\pi = 0$ in matrix form, we get $a_2^2 b_2 = a_2 b_2^2 = 0$. Now, we apply Theorem 3.1 to obtain that $a_2 + b_2 \in \mathcal{A}_2^D$ and

$$(a_2 + b_2)^D = \sum_{n=0}^{\infty} \tilde{x}_n a_2^n, \tag{3.5}$$

where, for all $n \geqslant 0$,

$$\begin{aligned} \tilde{x}_n &= \sum_{k=0}^{\infty} (b_2^D)^{n+1+2k} (a_2 b_2)^k (a_2 b_2)^\pi - \sum_{k=0}^{\nu_n - 1} (b_2^D)^{n-2k-1} ((a_2 b_2)^D)^{k+1} \\ &\quad + \sum_{k=\nu_n}^{\infty} b_2^\pi b_2^{2k+1-n} ((a_2 b_2)^D)^{k+1}. \end{aligned} \tag{3.6}$$

On the other hand, from Theorem 1.1 it follows that $(a+b)^D$ exists and

$$(a+b)^D = \begin{bmatrix} a_1 & 0 \\ b_{21} & a_2 + b_2 \end{bmatrix}_{\mathcal{P}}^D = \begin{bmatrix} a_1^{-1} & \\ u & (a_2 + b_2)^D \end{bmatrix}_{\mathcal{P}}, \tag{3.7}$$

where

$$u = (a_2 + b_2)^\pi \sum_{i=0}^{\infty} (a_2 + b_2)^i b_{21} (a_1^{-1})^{i+2} - (a_2 + b_2)^D b_{21} a_1^{-1}. \tag{3.8}$$

Using (3.5), condition $a_2^2 b_2 = 0$, and taking into account that $p_2 = a^\pi$ is the unit in $\mathcal{A}_2$, we get

$$(a_2 + b_2)^\pi = p_2 - (a_2 + b_2)^D (a_2 + b_2) = p_2 - \tilde{x}_0 b_2 - \tilde{x}_0 a_2 - \tilde{x}_1 a_2 b_2 - \sum_{n=1}^{\infty} \tilde{x}_n a_2^{n+1}.$$

With $\tilde{x}_0$ defined as in (3.6), we have that $\tilde{x}_0 b_2 = b_2^D b_2$ because $a_2 b_2^2 = 0$ and, thus, $(a_2 b_2)^D b_2 = 0$ also holds. Moreover, $a_2^2(a_2+b_2)^i = a_2^{i+2}$ for all $i \geqslant 0$. Using the preceding relations together with (3.8), we get

$$\begin{aligned} u = {} & (b_2^\pi - \tilde{x}_0 a_2 - \tilde{x}_1 a_2 b_2) \sum_{i=0}^{\infty} (a_2+b_2)^i b_{21} (a_1^{-1})^{i+2} \\ & - \tilde{x}_0 b_{21} a_1^{-1} - \sum_{n=1}^{\infty} \tilde{x}_n a_2^n \left( \sum_{i=0}^{\infty} a_2^i b_{21} (a_1^{-1})^{i+1} \right). \end{aligned} \tag{3.9}$$

Finally, let $x_n$ be defined as in (3.1) for all $n \geqslant 0$, and let us introduce the notation

$$\Sigma_1 = (b^\pi - x_1 ab - x_0 a) \sum_{i=0}^{\infty} (a+b)^i b (a^D)^{i+2}, \quad \Sigma_2 = \sum_{n=1}^{\infty} x_n a^n \left( a^\pi - \sum_{i=0}^{\infty} a^i b (a^D)^{i+1} \right).$$

Then, using the matrix representations (3.3) and (3.4), by a straightforward computation we obtain

$$a^D + \Sigma_1 = \begin{bmatrix} a_1^{-1} & 0 \\ (b_2^\pi - \tilde{x}_0 a_2 - \tilde{x}_1 a_2 b_2) \sum_{i=0}^{\infty} (a_2+b_2)^i b_{21} (a_1^{-1})^{i+2} & 0 \end{bmatrix}_{\mathcal{P}},$$

$$x_0(a^\pi - b a^D) = \begin{bmatrix} 0 & 0 \\ -\tilde{x}_0 b_{21} a_1^{-1} & \tilde{x}_0 \end{bmatrix}_{\mathcal{P}},$$

$$\Sigma_2 = \begin{bmatrix} 0 & 0 \\ -\sum_{n=1}^{\infty} \tilde{x}_n a_2^n \left( \sum_{i=0}^{\infty} a_2^i b_{21} (a_1^{-1})^{i+1} \right) & \sum_{n=1}^{\infty} \tilde{x}_n a_2^n \end{bmatrix}_{\mathcal{P}}.$$

Hence, in view of (3.7), (3.9), and (3.5), Eq. (3.2) follows. $\square$

## 4. Special cases

In this section, we consider some specializations of our main result.

**Corollary 4.1.** *With the conditions in Theorem 3.2, we have*

(*i*) *If $b \in \mathcal{A}^{qnil}$, then $x_n$ defined in (3.1) has the form*

$$x_n = \sum_{k=\nu_n}^{\infty} b^{2k+1-n} ((ab)^D)^{k+1}, \quad n \geqslant 0.$$

(*ii*) *If $ab \in \mathcal{A}^{qnil}$, then $x_n$ defined in (3.1) has the form*

$$x_n = \sum_{k=0}^{\infty} \left(b^D\right)^{n+1+2k} (ab)^k, \quad n \geqslant 0.$$

(*iii*) *If $b, ab \in \mathcal{A}^{qnil}$, then*

$$(a+b)^D = a^D + \sum_{i=0}^{\infty} (a+b)^i b (a^D)^{i+2}.$$

From Theorem 3.2 we can obtain the following result which was also derived in [5, Theorem 4.1] from a different additive result.

**Corollary 4.2.** *Let $a, b \in \mathcal{A}^{\mathrm{D}}$. If $a^D b = 0$ and $aba^\pi = 0$, then*

$$\begin{aligned} (a+b)^D = {} & a^D + \sum_{i=0}^{\infty} \left( \sum_{k=0}^{i} b^\pi b^k a^{i-k} \right) b (a^D)^{i+2} \\ & + \sum_{n=0}^{\infty} (b^D)^{n+1} a^n \left( a^\pi - \sum_{i=0}^{\infty} a^i b (a^D)^{i+1} \right). \end{aligned}$$

**Proof.** Firstly, we note that with the conditions of this theorem, condition $ab^2 = ab(a^\pi b + aa^D b) = 0$ also holds. Furthermore, $(ab)^2 = abaa^D ab = 0$, i.e., $ab$ is 2-nilpotent and, thus, $(ab)^D = 0$. Therefore, we can apply Theorem 3.2 and thus $(a+b)^D$ admits the representation (3.2). Now, $x_n$ defined in (3.1) has the form

$$x_n = (b^D)^{n+1} + (b^D)^{n+3} ab, \quad n \geqslant 0. \tag{4.1}$$

On the other hand, taking into account that $ab^2 = 0$ and $aba^\pi = 0$ we easily prove by induction that

$$(a+b)^i a^\pi = \sum_{k=0}^{i} b^k a^{i-k} a^\pi, \quad i \geqslant 0$$

and, thus, $a(a+b)^i b = a(a+b)^i a^\pi b = a^{i+1} b$ and $ab(a+b)^i b = 0$ for all $i \geqslant 0$. Consequently,

$$(b^\pi - x_1 ab - x_0 a)(a+b)^i b = \sum_{k=0}^{i} b^\pi b^k a^{i-k} b - b^D a^{i+1} b, \quad i \geqslant 0.$$

By using the latter relation and (4.1) in (3.2) one gets, after regrouping terms, the formula established in this corollary. $\square$

The following corollary is an important case by itself because it reflects a compact formula for $(a+b)^D$ under conditions $a^2 b = ab^2 = 0$.

**Corollary 4.3.** *Let $a, b, ab \in \mathcal{A}^{\mathrm{D}}$. If $a^2 b = ab^2 = 0$, then*

$$\begin{aligned}(a+b)^D = {} & \sum_{i=0}^{\infty} \sum_{j=0}^{\nu_i} b^\pi b^{i-2j} (ab)^j (a^D)^{i+1} + \sum_{n=0}^{\infty} x_n a^n a^\pi \\ & - \sum_{j=0}^{\infty} (b x_1 (ab)^{j+1} a^D + x_1 (ab)^{j+1}) (a^D)^{2j+1},\end{aligned}$$

*where, for all $n \geqslant 0$, $x_n$ are defined as in (3.1).*

**Proof.** By Theorem 3.2 for the case $a^2 b = ab^2 = 0$, using that $(ab)^D b = 0$ and, thus, $x_0 b = b^D b$, we have

$$\begin{aligned}(a+b)^D = {} & b^\pi a^D + (b^\pi - x_1 ab - x_0 a) \sum_{i=1}^{\infty} (a+b)^{i-1} b (a^D)^{i+1} \\ & + \sum_{n=0}^{\infty} x_n a^n a^\pi - x_1 aba^D.\end{aligned} \tag{4.2}$$

On the other hand, we can prove by induction that

$$(a+b)^{i-1} b = \sum_{j=0}^{\nu_i} b^{i-2j} (ab)^j, \quad i \geqslant 1. \tag{4.3}$$

Hence, it follows that, for all $j \geqslant 0$,

$$\begin{aligned} & ab(a+b)^{2j} b = 0, && a(a+b)^{2j} b = (ab)^{j+1}, \\ & ab(a+b)^{2j+1} b = (ab)^{j+2}, && a(a+b)^{2j+1} b = 0. \end{aligned}$$

Further, using the relation $x_0 = b x_1$, we have

$$\begin{aligned} (x_1 ab + x_0 a)(a+b)^{2j} b &= b x_1 (ab)^{j+1} \\ (x_1 ab + x_0 a)(a+b)^{2j+1} b &= x_1 (ab)^{j+2}. \end{aligned}$$

By using these relations and (4.3) in (4.2) one gets, after regrouping terms, the formula established in this corollary. $\square$

The following two corollaries can be derived either from the previous one or from Theorem 3.1.

**Corollary 4.4.** *If $a, b \in \mathcal{A}^{qnil}$, $ab \in \mathcal{A}^D$, and $a^2b = ab^2 = 0$ then*

$$(a+b)^D = \sum_{n=0}^{\infty} \sum_{k=\nu_n}^{\infty} b^{2k+1-n}((ab)^D)^{k+1}a^n.$$

**Corollary 4.5.** *Let $b \in \mathcal{A}^D$, $a, ab \in \mathcal{A}^{qnil}$, and $a^2b = ab^2 = 0$ then*

$$(a+b)^D = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (b^D)^{n+1+2k}(ab)^k a^n.$$

With Corollary 4.3 we recover the case $ab = 0$ studied in [12] for bounded linear operators.

**Corollary 4.6.** *Let $a, b \in \mathcal{A}^D$. If $ab = 0$, then*

$$(a+b)^D = \sum_{n=0}^{\infty} b^\pi b^n (a^D)^{n+1} + \sum_{n=0}^{\infty} (b^D)^{n+1} a^n a^\pi.$$

**Proof.** The result follows from Corollary 4.3 taking into account that under condition $ab = 0$ we have $x_n = (b^D)^{n+1}$. □

## 5. Application to complex block matrices

In this section, we get a representation for the Drazin inverse of $2 \times 2$ complex block matrices with application of our previous results. Let $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{C}^{n \times n}$, where $A$ and $D$ are square. For a block triangular matrix we have the following well-known result [22]. Specific information about the index of block triangular matrices can be found in [2].

**Lemma 5.1.** *If $M = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix}$, $r = \operatorname{ind}(A)$, and $s = \operatorname{ind}(D)$, then*

$$M^D = \begin{bmatrix} A^D & 0 \\ Z_1 & D^D \end{bmatrix},$$

*where*

$$Z_1 = \sum_{i=0}^{r-1} (D^D)^{i+2} C A^i A^\pi + D^\pi \sum_{i=0}^{s-1} D^i C (A^D)^{i+2} - D^D C A^D. \tag{5.1}$$

*Moreover*, $\max\{r, s\} \leqslant \operatorname{ind}(M) \leqslant r + s$.

An open problem is to find an explicit representation for $M^D$ with arbitrary $A, B, C$, and $D$. Some special cases were studied in [4,11,13,15,21]. In [15, Lemma 2.2, Corollary 2.3], the case $BC = 0$, $BD = 0$ and $D$ nilpotent was studied. The following result is an extension of the one given therein.

**Theorem 5.2.** *If $B(CA + DC) = 0$, $BD^2 = 0$, and $BC$ is t-nilpotent, then*

$$M^D = \sum_{k=0}^{t-1} \begin{bmatrix} (A^D)^{2k+1}(BC)^k & (A^D)^{2k+2}(BC)^k B + (A^D)^{2k+3}(BC)^k BD \\ Z_{2k+1}(BC)^k & D^D + Z_{2k+2}(BC)^k B + Z_{2k+3}(BC)^k BD \end{bmatrix},$$

*where $Z_1$ is defined as in (5.1) and $Z_l = \sum_{j=0}^{l-1} (D^D)^{l-1-j} Z_1 (A^D)^j$ for all $l \geqslant 1$.*

**Proof.** Consider $M = M_1 + M_2$, where $M_1 = \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}$ and $M_2 = \begin{bmatrix} A & 0 \\ C & D \end{bmatrix}$. We have

$$M_1M_2^2 = \begin{bmatrix} B(CA+DC) & BD^2 \\ 0 & 0 \end{bmatrix}, \quad (M_1M_2)^k = \begin{bmatrix} (BC)^k & (BC)^{k-1}BD \\ 0 & 0 \end{bmatrix}, \; k \geqslant 1.$$

Since $M_1M_2^2 = 0$ and $M_1$ is 2-nilpotent, then we can apply Theorem 3.1 in the Banach algebra of complex matrices of order $n$ with $M_1$ in the place of $a$ and $M_2$ in the place of $b$ to obtain

$$M^D = X_0 + X_1M_1, \tag{5.2}$$

where, taking into account that $M_1M_2$ is nilpotent of index either $t$ or $t+1$ because $BC$ is $t$-nilpotent,

$$X_n = \sum_{k=0}^{t+1} (M_2^D)^{n+1+2k}(M_1M_2)^k, \quad n = 0, 1. \tag{5.3}$$

From Lemma 5.1 we get

$$M_2^D = \begin{bmatrix} A^D & 0 \\ Z_1 & D^D \end{bmatrix}, \quad (M_2^D)^l = \begin{bmatrix} (A^D)^l & 0 \\ Z_l & (D^D)^l \end{bmatrix},$$

where $Z_1$ is defined as in (5.1) and $Z_l = \sum_{j=0}^{l-1} (D^D)^{l-1-j} Z_1 (A^D)^j$ for all $l \geqslant 1$. By using the block representations of the powers of $M_2^D$ and $M_1M_2$ in (5.3) we obtain for $n = 0, 1$,

$$X_n = \begin{bmatrix} (A^D)^{n+1} & 0 \\ Z_{n+1} & (D^D)^{n+1} \end{bmatrix} + \sum_{k=1}^{t} \begin{bmatrix} (A^D)^{n+1+2k}(BC)^k & (A^D)^{n+1+2k}(BC)^{k-1}BD \\ Z_{n+1+2k}(BC)^k & Z_{n+1+2k}(BC)^{k-1}BD \end{bmatrix}.$$

Hence

$$X_1M_1 = \sum_{k=0}^{t-1} \begin{bmatrix} 0 & (A^D)^{2+2k}(BC)^kB \\ 0 & Z_{2+2k}(BC)^kB \end{bmatrix}$$

and by substituting the latter identity and the former for the case $n = 0$ in (5.2) we get the formula given in the statement of this theorem. $\square$

We note that we might obtain other representations for complex block matrices, under new conditions, taking a different splitting and applying our results like in the previous theorem.

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