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On convergence in fuzzy metric spaces

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ABSTRACT

The concept of p-convergence in fuzzy metric spaces, in George and Veeramani's sense, has been recently given by D. Mihet in [D. Mihet, On fuzzy contractive mappings in fuzzy metric spaces, Fuzzy Sets and Systems 158 (2007) 915–921]. In this note we study some aspects relative to this concept and characterize those fuzzy metric spaces, that we call principal, in which the family of p-convergent sequences agrees with the family of convergent sequences. Also a non-completable fuzzy metric space, which is not principal, is given.

1. Introduction

The problem of constructing a satisfactory theory of fuzzy metric spaces has been investigated by several authors from different points of view. Here we use the concept of fuzzy metric space that George and Veeramani [2,4] introduced and studied with the help of continuous t-norms. In [3,5], it is proved that the class of topological spaces which are fuzzy metrizable agrees with the class of metrizable spaces. This result allows to restate some classical theorems on metrics in the realm of fuzzy metric spaces. Nevertheless, the theory of fuzzy metric completion is, in this context, very different from the classical theory of metric completion. Indeed, there exist fuzzy metric spaces which are non-completable [6,7].

This class of fuzzy metrics can be easily included within fuzzy systems since the value given by them can be directly interpreted as a fuzzy certainty degree of nearness, and in particular, recently, they have been applied to colour image filtering, improving some filters when replacing classical metrics [1,9,10].

Since this class of fuzzy metrics includes in its definition a parameter t, it allows to introduce novel (fuzzy metric) concepts with respect to the classical metric concepts. In this sense, in [8] the author modified the definition of convergence and obtained a more general concept which is called p-convergence. In this paper we characterize those fuzzy metric spaces, that we call *principal*, in which both concepts agree. Most of the well-known fuzzy metrics of the literature are principal. Later, following the suggestions of the author in [8], we introduce and study the concept of p-Cauchy sequence and we

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show that p-Cauchy sequence and Cauchy sequence are two different concepts even in principal fuzzy metric spaces. Some illustrative examples, including a non-principal fuzzy metric space, which is non-completable, are given.

2. Preliminaries

Definition 1. ([2]) A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a (nonempty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times]0, +\infty[$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$:

(GV1) $M(x, y, t) > 0$;
(GV2) $M(x, y, t) = 1$ if and only if $x = y$;
(GV3) $M(x, y, t) = M(y, x, t)$;
(GV4) $M(x, y, t) * M(y, z, s) \leqslant M(x, z, t + s)$;
(GV5) $M(x, y, \_) :]0, +\infty[\to]0, 1]$ is continuous.

If $(X, M, *)$ is a fuzzy metric space we say that $(M, *)$ or M is a fuzzy metric on X. Also, we say that (X, M) or, simply, X is a fuzzy metric space.

Let (X, d) be a metric space. Denote by $a \cdot b$ the usual multiplication for all $a, b \in [0, 1]$, and let M_d be the fuzzy set defined on $X \times X \times]0, +\infty[$ by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}.$$

Then $(M_d, \cdot)$ is a fuzzy metric on X called *standard fuzzy metric space* [2].

George and Veeramani proved in [2] that every fuzzy metric M on X generates a topology τ_M on X which has as a base the family of open sets of the form $\{B_M(x, \varepsilon, t)\colon\ x \in X,\ 0 < \varepsilon < 1,\ t > 0\}$, where $B_M(x, \varepsilon, t) = \{y \in X\colon\ M(x, y, t) > 1 - \varepsilon\}$ for all $x \in X$, $\varepsilon \in]0, 1[$ and $t > 0$.

A sequence (x_n) in X converges to x if and only if $\lim_n M(x_n, x, t) = 1$, for all $t > 0$.

Definition 2. ([4]) A sequence $(x_n)_{n\in\mathbb{N}}$ in a fuzzy metric space (X, M) is said to be *Cauchy* if for each $\varepsilon \in]0, 1[$ and each $t > 0$ there is $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geqslant n_0$. X is called *complete* if every Cauchy sequence in X is convergent with respect to τ_M. In such a case M is called complete.

Definition 3. ([6]) Let (X, M) and (Y, N) be two fuzzy metric spaces. A mapping f from X to Y is called an *isometry* if for each $x, y \in X$ and $t > 0$, $M(x, y, t) = N(f(x), f(y), t)$ and, in this case, X and Y are called isometric. A fuzzy metric completion of (X, M) is a complete fuzzy metric space (X^*, M^*) such that (X, M) is isometric to a dense subspace of X^*. X is called completable if it admits a fuzzy metric completion.

Definition 4. A fuzzy metric M on X is said to be *stationary*, [7], if M does not depend on t, i.e. if for each $x, y \in X$, the function $M_{x,y}(t) = M(x, y, t)$ is constant. In this case we write $M(x, y)$ instead of $M(x, y, t)$.

From now on, $\mathbb{R}^+$, $\mathbb{N}$ and $\mathbb{Q}$ will denote the sets of positive real numbers, positive integers and rational numbers, respectively.

3. The results

In [8] the author give the following definition.

Definition 5. ([8]) Let (X, M) be a fuzzy metric space. A sequence (x_n) in X is said to be *point convergent* to $x_0 \in X$ if $\lim_n M(x_n, x_0, t_0) = 1$ for some $t_0 > 0$.

In such a case we say that (x_n) is p-convergent to x_0 for $t_0 > 0$, or, simply, (x_n) is p-convergent.

Equivalently, (x_n) is p-convergent if there exist $x_0 \in X$ and $t_0 > 0$ such that (x_n) is eventually in $B(x_0, r, t_0)$ for each $r \in]0, 1[$ (or, without lost of generality, in $B(x_0, \frac{1}{n}, t_0)$ for each $n \in \mathbb{N}$).

Clearly (x_n) is convergent to x_0 if and only if (x_n) is p-convergent to x_0 for all $t > 0$. The following properties hold [8]:

(1) If $\lim_n M(x_n, x, t_1) = 1$ and $\lim_n M(x_n, y, t_2) = 1$ then $x = y$.
(2) If $\lim_n M(x_n, x_0, t_0) = 1$ then $\lim_k M(x_{n_k}, x_0, t_0) = 1$ for each subsequence (x_{n_k}) of (x_n).

By property (1) the next corollary is obtained:

Corollary 6. *If (x_n) is p-convergent to x_0 and it is convergent, then (x_n) converges to x_0.*

Corollary 7. *Let (X, M) be a completable fuzzy metric space. If (x_n) is a Cauchy sequence in X, and it is p-convergent to $x_0 \in X$, then (x_n) converges to x_0.*

Proof. Let (X^*, M^*) be the completion of (X, M). Then there exists $x^* \in X^*$ such that (x_n) converges to x^*.

Now, suppose that $\lim_n M(x_n, x_0, t_0) = 1$ for some $t_0 > 0$. Then, $\lim_n M^*(x_n, x_0, t_0) = \lim_n M(x_n, x_0, t_0) = 1$, so (x_n) is M^*-p-convergent to x_0, and by the previous corollary $x^* = x_0$. $\square$

An example of a p-convergent sequence which is not convergent is given in the next example.

Example 8. ([8]) Let $(x_n) \subset]0, 1[$ be a strictly increasing sequence convergent to 1 respect to the usual topology of $\mathbb{R}$ and $X = (x_n) \cup \{1\}$. Define on $X^2 \times \mathbb{R}^+$ the function M given by $M(x, x, t) = 1$ for each $x \in X$, $t > 0$, $M(x_n, x_m, t) = \min\{x_n, x_m\}$, for all $m, n \in \mathbb{N}$, $t > 0$, and $M(x_n, 1, t) = M(1, x_n, t) = \min\{x_n, t\}$ for all $n \in \mathbb{N}$, $t > 0$. Then $(M, *)$ is a fuzzy metric on X, where $a * b = \min\{a, b\}$. The sequence (x_n) is not convergent since $\lim_n M(x_n, 1, \frac{1}{2}) = \frac{1}{2}$. Nevertheless it is p-convergent to 1, since $\lim_n M(x_n, 1, 1) = 1$.

Notice that in the above example $\{1\}$ is open in τ_M since $B(1, \frac{1}{2}, \frac{1}{2}) = \{1\}$. On the other hand for $r \in]0, 1[$ we have that $B(1, r, 1) = \{x_m, x_{m+1}, \ldots\} \cup \{1\}$ where x_m is the first element of (x_n) such that $0 < 1 - r < x_m$. Hence, the family of open balls $\{B(1, r, 1)\colon r \in]0, 1[\}$ is not a local base at 1. This fact motivates our next definition.

Definition 9. We say that the fuzzy metric space $(X, M, *)$ is *principal* (or simply, M is principal) if $\{B(x, r, t)\colon r \in]0, 1[\}$ is a local base at $x \in X$, for each $x \in X$ and each $t > 0$.

As we have just seen, the fuzzy metric of Example 8 is not principal. Next we see some examples of principal fuzzy metrics.

Example 10.

(a) Stationary fuzzy metrics are, obviously, principal.
(b) The well-known standard fuzzy metric is principal.
(c) $M(x, y, t) = e^{-\frac{d(x,y)}{t}}$, where d is a metric on X, [2], is principal.
(d) $M(x, y, t) = \frac{\min\{x,y\}+t}{\max\{x,y\}+t}$ is a fuzzy metric on $\mathbb{R}^+$ [11], which is principal.

Theorem 11. *The fuzzy metric space (X, M) is principal if and only if all p-convergent sequences are convergent.*

Proof. Suppose that M is principal and that (x_n) is a sequence which is p-convergent to x_0, for $t_0 > 0$.

Let $\varepsilon \in]0, 1[$ and $t > 0$. Since M is principal then $\{B(x_0, \frac{1}{n}, t_0)\colon n \in \mathbb{N}\}$ is a local base at x_0. Hence, we can find $m \in \mathbb{N}$ such that $B(x_0, \frac{1}{m}, t_0) \subset B(x_0, \varepsilon, t)$.

Since $\lim_n M(x_n, x_0, t_0) = 1$ we can find $\delta \in]0, 1[$, with $\delta < \frac{1}{m}$, and $n_1 \in \mathbb{N}$ such that $x_n \in B(x_0, \delta, t_0)$ for all $n \geqslant n_1$, and thus $x_n \in B(x_0, \varepsilon, t)$ for all $n \geqslant n_1$.

Hence, $M(x_n, x_0, t) > 1 - \varepsilon$ for all $n \geqslant n_1$, and so $\lim_n M(x_n, x_0, t) = 1$. The above argument is valid for all $t > 0$, then (x_n) converges to x_0.

For the converse, assume that M is not principal. We will construct a p-convergent sequence which is not convergent.

If M is not principal we can find $x_0 \in X$ and $t > 0$ such that $\{B(x_0, \frac{1}{n}, t_0)\colon n \in \mathbb{N}\}$ is not a local base at x_0. Then, we can find $t > 0$ and $r \in]0, 1[$ such that $B(x_0, \frac{1}{n}, t_0) \not\subseteq B(x_0, r, t)$ for all $n \in \mathbb{N}$.

Now, by induction, we form the sequence (x_n) as follows. For each $n \in \mathbb{N}$ we take $x_n \in B(x_0, \frac{1}{n}, t_0) \setminus B(x_0, r, t)$. Now, given $\varepsilon \in]0, 1[$ we choose $n_1 \in \mathbb{N}$ with $\frac{1}{n_1} < \varepsilon$. Hence, for $m \geqslant n_1$ we have

$$M(x_m, x_0, t_0) > 1 - \frac{1}{m} > 1 - \frac{1}{n_1} > 1 - \varepsilon$$

and, since ε is arbitrary, $\lim_n M(x_n, x_0, t_0) = 1$, and so (x_n) is p-convergent to x_0.

On the other hand, by construction $x_n \in X \setminus B(x_0, r, t)$ for all $n \in \mathbb{N}$, and so (x_n) does not converge to x_0 and, by Corollary 6, (x_n) is not convergent. $\square$

Next examples illustrate the last theorem.

Example 12. Let $\varphi : \mathbb{R}^+ \to]0, 1[$ be an increasing continuous function. Define the function M on $X^2 \times \mathbb{R}^+$ by

$$M(x, y, t) = \begin{cases} 1, & x = y, \\ \varphi(t), & x \neq y. \end{cases}$$

It is easy to verify that $(M, \cdot)$ is a fuzzy metric on X. Now, for each $x \in X$ and $t > 0$ we have $B(x, 1 - \varphi(t), t) = \{x\}$ and so M is principal. Further, τ_M is the discrete topology, and then only the constant sequences are convergent, so they are the only p-convergent sequences in X.

Next we give an example of a complete fuzzy metric space which is not principal.

Example 13. Let $X = \mathbb{R}^+$ and let $\varphi : \mathbb{R}^+ \to]0, 1]$ be a function given by $\varphi(t) = t$ if $t \leqslant 1$ and $\varphi(t) = 1$ elsewhere. Define the function M on $X^2 \times \mathbb{R}^+$ by

$$M(x, y, t) = \begin{cases} 1, & x = y, \\ \frac{\min\{x,y\}}{\max\{x,y\}} \cdot \varphi(t), & x \neq y. \end{cases}$$

It is easy to verify that $(M, \cdot)$ is a fuzzy metric on X and, since $M(x, y, t) < t$, whenever $t \in]0, 1[$ and $x \neq y$, it is obvious that the only Cauchy sequences in X are the constant sequences and so, X is complete.

This fuzzy metric is not principal. In fact, notice that $B(x, \frac{1}{2}, \frac{1}{2}) = \{x\}$ for each $x \in X$ and so τ_M is the discrete topology. Now, if we set $x = 1$ and $t = 1$ we have $B(1, r, 1) =]1 - r, \frac{1}{1-r}[$ for all $r \in]0, 1[$ and so $\{B(1, r, 1)\colon r \in]0, 1[\}$ is not a local base at $x = 1$, since $\{1\}$ is open.

Now, consider the sequence (x_n) in X given by $x_n = 1 - \frac{1}{n}$, $n \in \mathbb{N}$. We have $M(x_n, 1, 1) = 1 - \frac{1}{n}$ for all $n \in \mathbb{N}$, so $\lim_n M(x_n, 1, 1) = 1$ and (x_n) is p-convergent to 1, but (x_n) is not convergent.

In the next example, (X, M) is a fuzzy metric space which is not principal and non-completable.

Example 14. Let $X =]0, 1]$, $A = X \cap \mathbb{Q}$, $B = X \setminus A$. Define the function M on $X^2 \times \mathbb{R}^+$ by

$$M(x, y, t) = \begin{cases} \frac{\min\{x,y\}}{\max\{x,y\}}, & x, y \in A \text{ or } x, y \in B,\ t > 0, \\ \frac{\min\{x,y\}}{\max\{x,y\}}, & x \in A,\ y \in B \text{ or } x \in B,\ y \in A,\ t \geqslant 1, \\ \frac{\min\{x,y\}}{\max\{x,y\}} \cdot t, & \text{elsewhere}. \end{cases}$$

It is easy to verify that $(M, \cdot)$ is a fuzzy metric on X. Now, we will show that M is not principal. For it, we will see that $\{B(1, r, 1)\colon r \in]0, 1[\}$ is not a local base at $x = 1$.

In fact, $B(1, r, 1) =]1 - r, 1]$ for all $r \in]0, 1[$. Now, since $1 \in A$ we have $\{y \in B\colon M(1, y, \frac{1}{2}) > 1 - \frac{1}{2}\} = \emptyset$, so $B(1, \frac{1}{2}, \frac{1}{2}) =]\frac{1}{2}, 1] \cap \mathbb{Q}$ and clearly $B(1, r, 1) \nsubseteq B(1, \frac{1}{2}, \frac{1}{2})$ for all $r \in]0, 1[$, and then M is not principal.

Now, it is easy to verify that for $n \geqslant 2$,

$$B\left(x, \frac{1}{n}, \frac{1}{n}\right) = \begin{cases}]x - \frac{x}{n}, \frac{nx}{n-1}[\cap A, & x \in A, \\ ]x - \frac{x}{n}, \frac{nx}{n-1}[\cap B, & x \in B, \end{cases}$$

and clearly τ_M is not the discrete topology, although it is finer than the usual topology of $\mathbb{R}$ relative to X.

Finally, let (x_n) be an increasing sequence contained in B which converges to 1 for the usual topology of $\mathbb{R}$ relative to X. Thus, for each $r \in]0, 1[$ there exists $n_1 \in \mathbb{N}$ such that $x_n \in]1 - r, 1[$ for all $n \geqslant n_1$, and by the definition of M we have $M(x_n, 1, 1) > 1 - r$, for all $n \geqslant n_1$, and so $\lim_n M(x_n, 1, 1) = 1$, since r is arbitrary. Then, (x_n) is p-convergent to 1. Further, (x_n) is a Cauchy sequence. Indeed, since $M(x_n, x_m, t) = \frac{\min\{x_n, x_m\}}{\max\{x_n, x_m\}}$ for all $t > 0$, we have that $M(x_n, x_m, t) \geqslant x_{n_1} > 1 - r$, for all $t > 0$, and so (x_n) is a Cauchy sequence. Now, (x_n) is not convergent since $x_n \notin B(1, \frac{1}{2}, \frac{1}{2})$, for all $n \in \mathbb{N}$ and then by Corollary 7, (X, M) is non-completable.

Continuing the above study we give the next definition.

Definition 15. Let (X, M) be a fuzzy metric space. A sequence (x_n) in X is said to be p-Cauchy if for each $\varepsilon \in]0, 1[$ there are $n_0 \in \mathbb{N}$ and $t_0 > 0$ such that $M(x_n, x_m, t_0) > 1 - \varepsilon$ for all $n, m \geqslant n_0$, i.e. $\lim_{m,n} M(x_n, x_m, t_0) = 1$ for some $t_0 > 0$.

In such a case we say that (x_n) is p-Cauchy for $t_0 > 0$, or, simply, (x_n) is p-Cauchy.

Clearly (x_n) is a Cauchy sequence if and only if (x_n) is p-Cauchy for all $t > 0$ and, obviously, p-convergent sequences are p-Cauchy.

Definition 16. The fuzzy metric space (X, M) is called p-complete if every p-Cauchy sequence in X is p-convergent to some point of X. In such a case M is called p-complete.

Obviously, p-completeness and completeness are equivalent concepts in stationary fuzzy metrics, and it is easy to verify that the standard fuzzy metric M_d is p-complete if and only if M_d is complete.

Proposition 17. *Let (X, M) be a principal fuzzy metric space. If X is p-complete then X is complete.*

Proof. Let (x_n) be a Cauchy sequence in X. Then (x_n) is p-Cauchy and so (x_n) p-converges to some point $x_0 \in X$, and, since X is principal, (x_n) converges to x_0. □

The assumption that X is principal cannot be removed in the last proposition as shows the next example.

Example 18. Consider the fuzzy metric space $(X, M, *)$ of Example 8. The sequence (x_n) satisfies $\lim_{m,n} M(x_n, x_m, t) = 1$ for all $t > 0$, so (x_n) is a Cauchy sequence and in consequence X is not complete, since (x_n) is not convergent.

Next we show that X is p-complete.

Let (x_n) be a p-Cauchy sequence in X. Then, with an easy argument one can verify that (x_n) must be a convergent sequence to 1 with respect to the usual topology of $\mathbb{R}$ relative to X. Now, $\lim_n M(x_n, 1, 1) = \lim_n x_n = 1$ and hence (x_n) is p-convergent to 1.

One could expect p-Cauchy sequences to be Cauchy sequences in principal fuzzy metric spaces. In fact, this property is satisfied by all examples of Example 10. Nevertheless, as shows the next example, it is not true, in general, for any principal fuzzy metric and, in consequence, the converse of the above proposition is not true.

Example 19. Let $X =]0, 1[$ and define the function M on $X^2 \times \mathbb{R}^+$ by

$$M(x, y, t) = \begin{cases} 1, & x = y, \\ xyt, & x \neq y,\ t \leqslant 1, \\ xy, & x \neq y,\ t > 1. \end{cases}$$

It is easy to verify that $(M, \cdot)$ is a fuzzy metric on X. Now, let $x \in X$ and $t > 0$. If we take $r \in]0, 1[$ such that $1 - r > x$ then $B(x, r, t) = \{x\}$ and so M is principal. Further, τ_M is the discrete topology and then the only convergent sequences or (by Theorem 11) p-convergent sequences, are the constant sequences.

Now, X does not have Cauchy sequences, since $M(x, y, \frac{1}{2}) \leqslant \frac{1}{2}$ for each $x, y \in X$, and so X is complete.

Let (x_n) be a strictly increasing sequence convergent to 1 in the usual topology of $\mathbb{R}$, relative to X. We have $\lim_{m,n} M(x_n, x_m, 1) = 1$ and then (x_n) is p-Cauchy. Nevertheless (x_n) is not p-convergent since (x_n) is not constant, and thus X is not p-complete.

It is an open problem to characterize those fuzzy metric spaces where the family of p-Cauchy sequences and Cauchy sequences agree, or further, when it is satisfied that completeness is equivalent to p-completeness.

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