Advances in Applied Mathematics 49 (2012) 81–96

Contents lists available at SciVerse ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

Integral geometry for the 1-norm

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ARTICLE INFO

Article history:
Received 8 January 2012
Accepted 27 February 2012
Available online 22 March 2012

MSC:
primary 53C65
secondary 52A01, 52A21, 52A30, 51F99

Keywords:
1-norm
Taxicab metric
Valuation
Convex set
Geodesic space
Metric space
Hadwiger's theorem
Steiner's theorem
Crofton formula
Kinematic formula

ABSTRACT

Classical integral geometry takes place in Euclidean space, but one can attempt to imitate it in any other metric space. In particular, one can attempt this in $\mathbb{R}^n$ equipped with the metric derived from the p-norm. This has, in effect, been investigated intensively for $1 < p < \infty$, but not for $p = 1$. We show that integral geometry for the 1-norm bears a striking resemblance to integral geometry for the 2-norm, but is radically different from that for all other values of p. We prove a Hadwiger-type theorem for $\mathbb{R}^n$ with the 1-norm, and analogues of the classical formulas of Steiner, Crofton and Kubota. We also prove principal and higher kinematic formulas. Each of these results is closely analogous to its Euclidean counterpart, yet the proofs are quite different.

0. Introduction

Classical integral geometry provides definite answers to natural questions about convex subsets of Euclidean space. The Cauchy formula, for instance, tells us that the surface area of a convex body in $\mathbb{R}^3$ is proportional to the expected area of its projection onto a random plane. The Crofton formula states that it is also proportional to the measure of the set of affine lines that meet the body. The Steiner formula gives the volume of the set of points within a specified distance of a given convex body. The kinematic formula tells us the probability that a randomly-placed convex body X meets another body Y, given that it meets a larger body $Z \supseteq Y$.

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[1] Supported by an EPSRC Advanced Research Fellowship.

0196-8858/$ – see front matter
http://dx.doi.org/10.1016/j.aam.2012.02.002

It is so successful a theory that one naturally seeks to imitate it elsewhere. This has been done in several ways. For example, Alesker [3,4] (foreshadowed by Fu [9]) has developed integral geometry for manifolds, while Bernig and Fu [7] have developed Hermitian integral geometry. Others have extended integral geometry to finite-dimensional real Banach spaces, and more generally to projective Finsler spaces: see for instance Schneider and Wieacker [21] and Schneider [19,20]. (This includes $\mathbb{R}^n$ with the 1-norm [20], but it is a different generalization from that presented here.) An important role is played there by Holmes–Thompson valuations, comparable to intrinsic volumes in Euclidean space: see Schneider [18], Álvarez Paiva and Fernandes [5], and Bernig [6].

But one simple setting in which integral geometry seems not to have been fully developed is that of ordinary metric spaces. A natural notion of convexity is available there: a subset X of a metric space A is *geodesic* if for any two points $x, x' \in X$, say distance D apart, there exists an isometry $\gamma : [0, D] \to X$ with $\gamma(0) = x$ and $\gamma(D) = x'$. Using this, we can extend to an arbitrary metric space A the fundamental notion of continuous invariant valuation on convex sets. And just as for Euclidean space, the continuous invariant valuations form a vector space, $\mathrm{Val}(A)$, indexing all the ways of measuring the size of geodesic subsets of A.

Every metric space therefore poses a challenge: classify its continuous invariant valuations. A celebrated theorem of Hadwiger answers the challenge for Euclidean space $\mathbb{R}^n$, stating that $\mathrm{Val}(\mathbb{R}^n)$ is $(n+1)$-dimensional, with a basis $V_0, \dots, V_n$ in which V_i is homogeneous of degree i. The valuations V_i are the *intrinsic volumes* (also known, with different normalizations, as the quermassintegrals or Minkowski functionals). When $n = 2$, for instance, they are Euler characteristic, half of perimeter, and area.

More ambitiously, we can attempt to reproduce in an arbitrary metric space—or one with as little structure as possible—the classical results of integral geometry in the tradition of Crofton and Blaschke. For example, we can seek analogues of the formulas listed in the first paragraph. To do this, we will need our metric space to carry an affine structure; and among the most important such spaces are the Banach spaces ℓ_p^n, that is, $\mathbb{R}^n$ equipped with the metric induced by the p-norm $\|x\|_p = (\sum |x_i|^p)^{1/p}$ ($p \in [1, \infty)$).

What is known about the integral geometry (in the sense just described) of the metric spaces ℓ_p^n? Let $p \in [1, \infty)$. The case $p = 2$ is the classical Euclidean theory. For $p \neq 1$, the space ℓ_p^n is strictly convex, so the only geodesic subsets are the convex sets. On the other hand, for $p \neq 2$, the isometry group of ℓ_p^n is very small, generated by permutations of coordinates, reflections in coordinate hyperplanes, and translations. So if $p \neq 1, 2$ then ℓ_p^n has the same geodesic subsets as ℓ_2^n, but far fewer isometries. Hence $\mathrm{Val}(\ell_p^n)$ is much bigger than $\mathrm{Val}(\ell_2^n) \cong \mathbb{R}^{n+1}$; indeed, it is infinite-dimensional. Much is known about the structure of $\mathrm{Val}(\ell_p^n)$ for $p \neq 1, 2$; this is essentially the theory of translation-invariant valuations on convex subsets of $\mathbb{R}^n$ [14,15,1,2].

But the case $p = 1$ has until now been overlooked, and turns out to contain a surprise. As we shall see, the metric space ℓ_1^n behaves very much like ℓ_2^n, but very much unlike all the other spaces ℓ_p^n. For example, there is a Hadwiger-type theorem stating that $\mathrm{Val}(\ell_1^n) \cong \mathbb{R}^{n+1}$. Furthermore, $\mathrm{Val}(\ell_1^n)$ has a basis $V'_0, \dots, V'_n$ of valuations, the *ℓ_1-intrinsic volumes*, where V'_i is homogeneous of degree i. Hence there is a *canonical* isomorphism $\mathrm{Val}(\ell_1^n) \cong \mathrm{Val}(\ell_2^n)$—despite the fact that ℓ_1^n and ℓ_2^n have neither the same geodesic subsets nor the same isometry group.

The resemblance goes deeper still: as we demonstrate, all the standard Euclidean integral-geometric formulas have close analogues in ℓ_1^n. Nevertheless, the proofs are quite different: just as the classical proofs exploit special features of Euclidean geometry, ours exploit special features of ℓ_1 geometry.

A mystery remains: why are the results for ℓ_1^n and ℓ_2^n so similar to each other, yet so different from those for ℓ_p^n when $p \in (1, 2) \cup (2, \infty)$? There is no obvious common generalization of the cases $p = 1$ and $p = 2$. Yet a common generalization must surely exist.

The case $p = \infty$ appears not to have been investigated either. Since ℓ_∞^2 is isometric to ℓ_1^2, the vector space $\mathrm{Val}(\ell_\infty^2)$ is 3-dimensional, like $\mathrm{Val}(\ell_1^2)$ and $\mathrm{Val}(\ell_2^2)$ but unlike $\mathrm{Val}(\ell_p^2)$ for $p \in (1, 2) \cup (2, \infty)$. It is natural to conjecture that $\mathrm{Val}(\ell_\infty^n) \cong \mathbb{R}^{n+1}$ for all $n \geqslant 0$.

This paper is organized as follows. We begin by establishing the fundamental facts about geodesic subsets of ℓ_1^n, here called *ℓ_1-convex* sets. (They include the convex sets, but are much more general.)

Almost immediately we encounter a stark difference between ℓ_1^n and ℓ_2^n: the intersection of ℓ_1-convex sets need not be ℓ_1-convex. And yet, there is a more subtle sense in which the two situations are precisely analogous (Remark 1.10). Guided by this analogy, we prove ℓ_1 versions of all the elementary laws governing intersections, projections and Minkowski sums of ordinary convex sets (Sections 1 and 2). We also prove a result that has no clear analogue in ℓ_2^n: if the union of two ℓ_1-convex sets is ℓ_1-convex, then so is its intersection.

Having described the algebra of the space of ℓ_1-convex sets, we turn to its topology (Sections 3 and 4). We show that it has a dense subspace consisting, roughly, of the ℓ_1-convex unions of cubes. We then generalize the theorem of McMullen [14] that a monotone translation-invariant valuation on convex sets is continuous. Our generalization implies both McMullen's theorem and its ℓ_1 counterpart.

These results provide the tools that enable us to develop the integral geometry of the metric space ℓ_1^n. The ℓ_1-intrinsic volumes are defined by a Cauchy-type formula, adapted to the smaller isometry group of ℓ_1^n. We prove analogues of the core theorems of Euclidean integral geometry: first a Hadwiger-type theorem, then analogues of the Steiner, Crofton, Kubota and kinematic formulas (Sections 5–8). While Sections 1–4 depend heavily on specific features of the geometry of ℓ_1^n, Sections 5–8 are formally close to their Euclidean counterparts. Even the constants appearing in the formulas are analogous: one simply replaces the flag coefficients [12] in the Euclidean formulas by the corresponding binomial coefficients.

As this suggests, the integral geometry of ℓ_1^n can be regarded as a cousin of the integral geometry of Euclidean space. It is more simple analytically, because of the smaller isometry group of ℓ_1^n. But since there are many more geodesic sets in ℓ_1^n than in ℓ_2^n, it is also more complex geometrically.

Conventions. $\mathbb{R}^n$ denotes real n-dimensional space as a set, topological space or vector space, but with no implied choice of metric except when $n = 1$. We allow $n = 0$. Lebesgue measure on $\mathbb{R}^n$ is written as Vol_n or Vol. The metric on a metric space is usually written as d.

1. ℓ_1-Convexity

Here we define ℓ_1-convexity and give some useful equivalent conditions. We also discuss the class of intervals in $\mathbb{R}^n$, dual in a certain sense to the class of ℓ_1-convex sets. Along the way, we review some standard material on abstract metric spaces; this can be found in texts such as Gromov [10, Chapter 1] and Papadopoulos [16].

Definition 1.1. A *path* in a metric space X is a continuous map $\gamma : [c, c'] \to X$, where c and c' are real numbers with $c \leqslant c'$; it *joins* $\gamma(c)$ and $\gamma(c')$. It is *distance-preserving* if $d(\gamma(t), \gamma(t')) = |t - t'|$ for all $t, t' \in [c, c']$.

Definition 1.2. A metric space X is *geodesic* if for all $x, x' \in X$, there exists a distance-preserving path joining x and x'.

For example, a subspace of Euclidean space is geodesic if and only if it is convex.

Definition 1.3. A subset of $\mathbb{R}^n$ is ℓ_1-*convex* if it is geodesic when given the subspace metric from ℓ_1^n.

A convex subset of $\mathbb{R}^n$ is ℓ_1-convex, but not conversely. For example, let $f : \mathbb{R} \to \mathbb{R}$ be an increasing continuous function: then the graph $\{(x, f(x)) \colon x \in \mathbb{R}\} \subseteq \ell_1^2$ is ℓ_1-convex. An ℓ_1-convex set need not even have positive reach: consider an L-shaped subset of ℓ_1^2.

Definition 1.4. Let $\gamma : [c, c'] \to X$ be a path in a metric space X. The *length* of γ is the supremum of $\sum_{r=1}^{k} d(\gamma(t_{r-1}), \gamma(t_r))$ over all partitions $c = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_k = c'$.

When speaking of functions $[c,c'] \to \mathbb{R}$, we use the terms *increasing* and *decreasing* in the non-strict sense. (For example, a constant function is both.) A real-valued function is *monotone* if it is increasing or decreasing.

Definition 1.5. A path $\gamma : [c,c'] \to \mathbb{R}^n$ is *monotone* if each of its components $\gamma_i : [c,c'] \to \mathbb{R}$ $(1 \leqslant i \leqslant n)$ is monotone.

Let X be a metric space and $x, x' \in X$. A point $y \in X$ is *between* x and x' if $d(x,x') = d(x,y) + d(y,x')$, and *strictly between* if also $x \neq y \neq x'$. When $X = \ell_1^n$, a point y is between x and x' if and only if for all $i \in \{1,\dots,n\}$, either $x_i \leqslant y_i \leqslant x'_i$ or $x'_i \leqslant y_i \leqslant x_i$.

Definition 1.6. A metric space is *Menger convex* if for all distinct points x, x' there exists a point strictly between x and x'.

We will repeatedly use the following characterization theorem for ℓ_1-convex sets.

Proposition 1.7. *Let $X \subseteq \ell_1^n$. The following are equivalent:*

(i) *X is ℓ_1-convex,*
(ii) *every pair x, x' of points of X can be joined by a path of length $d(x,x')$,*
(iii) *every pair of points of X can be joined by a monotone path in X.*

When X is closed, a further equivalent condition is that X is Menger convex.

Proof. A metric space is geodesic if and only if each pair of points x, x' can be joined by a path of length $d(x,x')$ [16, Proposition 2.2.7]. This proves the equivalence of (i) and (ii).

For (ii) $\Leftrightarrow$ (iii), a path $\gamma : [c,c'] \to \ell_1^n$ has length $d(\gamma(c),\gamma(c'))$ if and only if

$$\left|\gamma_i(t) - \gamma_i(t')\right| + \left|\gamma_i(t') - \gamma_i(t'')\right| = \left|\gamma_i(t) - \gamma_i(t'')\right| \tag{1}$$

whenever $1 \leqslant i \leqslant n$ and $c \leqslant t \leqslant t' \leqslant t'' \leqslant c'$. Eq. (1) holds if and only if $\gamma_i(t')$ is between $\gamma_i(t)$ and $\gamma_i(t'')$. It follows that γ has length $d(\gamma(c),\gamma(c'))$ if and only if it is monotone.

The final statement follows from the fact that for a metric space in which every closed bounded set is compact, Menger convexity is equivalent to being geodesic [16, Theorem 2.6.2]. $\square$

The intersection of ℓ_1-convex sets need not be ℓ_1-convex, and can in fact be highly irregular. For example, every closed subset of the line occurs, up to isometry, as the intersection of a pair of ℓ_1-convex subsets of the plane. Indeed, if $\emptyset \neq K \subseteq \mathbb{R}$ is closed then the sets

$$\left\{\frac{1}{2}\bigl(x - d(x,K), x + d(x,K)\bigr) \colon x \in \mathbb{R}\right\} \subseteq \ell_1^2, \qquad \left\{\frac{1}{2}(x,x) \colon x \in \mathbb{R}\right\} \subseteq \ell_1^2$$

are ℓ_1-convex and have intersection isometric to K. We do, however, have the following.

Definition 1.8. An *interval* in $\mathbb{R}^n$ is a subset of the form $\prod_{i=1}^n I_i$ for some (possibly empty, possibly unbounded) intervals $I_1,\dots,I_n \subseteq \mathbb{R}$.

Corollary 1.9. *The intersection of an ℓ_1-convex set and an interval in $\mathbb{R}^n$ is ℓ_1-convex.*

Proof. An interval I has the property that whenever $x, x' \in I$, every monotone path from x to x' in $\mathbb{R}^n$ lies in I. The result follows from Proposition 1.7(iii). $\square$

Remark 1.10. Corollary 1.9 might seem weak when compared with the result in Euclidean space that the intersection of *any* pair of convex sets is convex. But in fact, the ℓ_1 and Euclidean results are strictly analogous—as long as one uses the correct analogy. Let A be a metric space and, for $a, a' \in A$, write $\Gamma(a, a')$ for the set of distance-preserving paths joining a and a'. For a subspace $X \subseteq A$ to be geodesic means that whenever $x, x' \in X$,

$$\exists \gamma \in \Gamma(x, x'): \quad \text{image}(\gamma) \subseteq X.$$

There is a dual condition: $X \subseteq A$ is *cogeodesic* if whenever $x, x' \in X$,

$$\forall \gamma \in \Gamma(x, x'), \quad \text{image}(\gamma) \subseteq X.$$

A subset of ℓ_2^n is cogeodesic if and only if it is convex, if and only if it is geodesic, but a subset of ℓ_1^n is cogeodesic if and only if it is an interval. It is a logical triviality that in any metric space, the intersection of a geodesic subset and a cogeodesic subset is geodesic. Applied to ℓ_2^n, this says that the intersection of two convex sets is convex. Applied to ℓ_1^n, this is Corollary 1.9.

A subset $X \subseteq \mathbb{R}^n$ is *orthogonally convex* [8] if $X \cap L$ is convex whenever L is a straight line parallel to one of the coordinate axes. Corollary 1.9 has the following special case:

Corollary 1.11. *An ℓ_1-convex set is orthogonally convex.* □

On the other hand, an orthogonally convex set need not be ℓ_1-convex, even if it is connected. For example, choose a vector $v \in \mathbb{R}^3$ none of whose coordinates is 0, and consider the set of unit-length vectors in ℓ_2^3 orthogonal to v.

A *coordinate subspace* of $\mathbb{R}^n$ is a linear subspace P spanned by some subset of the standard basis. We write π_P for the orthogonal projection of $\mathbb{R}^n$ onto P, and $P^\perp$ for the orthogonal complement of P (with respect to the standard inner product). By Proposition 1.7(iii), we have:

Corollary 1.12. *Let P be a coordinate subspace of $\mathbb{R}^n$. Then the image under π_P of an ℓ_1-convex set is ℓ_1-convex.* □

There is a further positive result on intersections of ℓ_1-convex sets.

Lemma 1.13. *Let X and Y be closed subsets of $\mathbb{R}^n$. If X, Y and $X \cup Y$ are ℓ_1-convex, then so is $X \cap Y$.*

Proof. We use the following property of ℓ_1^n: if a point a is between points x and y, and if x and y are both strictly between points z and z', then a is strictly between z and z'.

We prove that $X \cap Y$ is Menger convex. Let $z, z' \in X \cap Y$ with $z \neq z'$. Since X and Y are Menger convex, we may choose points $x \in X$ and $y \in Y$ strictly between z and z'. Since $X \cup Y$ is ℓ_1-convex, we may choose a distance-preserving path $\gamma : [c, c'] \to X \cup Y$ joining x and y. Since X and Y are closed, we may choose $t \in [c, c']$ with $\gamma(t) \in X \cap Y$. Then $\gamma(t)$ is between x and y, hence strictly between z and z', as required. □

The interior and closure of a convex set are convex. The interior of an ℓ_1-convex set need not be ℓ_1-convex: consider $[-1, 0]^2 \cup [0, 1]^2 \subseteq \mathbb{R}^2$. On the other hand, we have the following.

Lemma 1.14. *The closure of an ℓ_1-convex set is ℓ_1-convex.*

Proof. We prove that the closure $\overline{X}$ of an ℓ_1-convex set X is Menger convex. Let $x, y \in \overline{X}$. Choose sequences (x_r) and (y_r) in X converging to x and y. Choose for each r a point $z_r \in X$ with $d(x_r, z_r) = d(z_r, y_r) = d(x_r, y_r)/2$. The sequence (z_r) is bounded, so has a subsequence convergent to some point $z \in \overline{X}$. Then $d(x, z) = d(z, y) = d(x, y)/2$. □

2. Minkowski sums

In the Euclidean context [17], there are basic laws governing the algebra of intersections and Minkowski sums: (i) if X and I are convex then so is $X \cap I$; (ii) if X and I are convex then so is $X + I$; and (iii) if X, Y are closed with $X \cup Y$ convex, and I is convex, then $(X \cap Y) + I = (X + I) \cap (Y + I)$.

In the ℓ_1 context, we already have an analogue of (i) (Corollary 1.9). Here we prove analogues of (ii) and (iii). As in Remark 1.10, the analogy entails replacing some occurrences of the term 'convex set' by 'ℓ_1-convex set', and others by 'interval'.

First we note that the class of ℓ_1-convex sets is not closed under Minkowski sums.

Example 2.1. Given $x, y \in \mathbb{R}^n$, write $[x, y]$ for the closed straight line segment between x and y. Given also $z \in \mathbb{R}^n$, write $[x, y, z] = [x, y] \cup [y, z]$. Define $X, Y \subseteq \mathbb{R}^3$ by

$$X = \bigl[(0,0,0), (2,0,0), (2,2,-1)\bigr], \qquad Y = \bigl[(0,0,0), (0,-1,2), (-1,-1,2)\bigr].$$

Then X and Y are ℓ_1-convex, but $X + Y$ is not. Indeed, $(0,0,0)$ and $(1,1,1)$ are points of $X + Y$ distance 3 apart in ℓ_1^3, but there is no point of $X + Y$ distance $3/2$ from each of them.

To prove our analogue of (ii), we use the following sufficient condition for ℓ_1-convexity of a Minkowski sum.

Lemma 2.2. *Let $X \subseteq \mathbb{R}^n$ be a closed set and $I \subseteq \mathbb{R}^n$ a compact interval. Suppose that for every $x, x' \in X$ satisfying $(x + I) \cap (x' + I) = \emptyset$, there exists a point of X strictly between x and x' in the ℓ_1 metric. Then $X + I$ is ℓ_1-convex.*

Proof. The topological hypotheses imply that $X + I$ is closed, so by Proposition 1.7, it is enough to prove that $X + I$ is Menger convex. Let y and y' be distinct points of $X + I$. Write $[\![y, y']\!]$ for the interval consisting of the points between y and y'. Since X is closed and I is compact, we may choose $x, x' \in X$ such that $y \in x + I$, $y' \in x' + I$, and $d(x, x')$ is minimal for all such pairs (x, x').

The proof is in two cases. First suppose that $(x + I) \cap (x' + I) = \emptyset$. By hypothesis, we may choose a point $z \in X$ strictly between x and x'. By minimality, $y \notin z + I$ and $y' \notin z + I$. Also, $y \in x + I$, $y' \in x' + I$, and z is between x and x', from which it follows that $[\![y, y']\!] \cap (z + I) \neq \emptyset$. Any point in this intersection is strictly between y and y'.

Now suppose that $(x + I) \cap (x' + I) \neq \emptyset$. If y or y' is in $(x + I) \cap (x' + I)$ then $(y + y')/2$ is a point of $X + I$ strictly between y and y'. If not, it is enough to prove that

$$[\![y, y']\!] \cap (x + I) \cap \bigl(x' + I\bigr) \neq \emptyset.$$

This follows from the fact that if J_1, J_2, J_3 are intervals in $\mathbb{R}^n$ whose pairwise intersections are all nonempty, then $J_1 \cap J_2 \cap J_3$ is also nonempty. □

Proposition 2.3. *The Minkowski sum of a closed ℓ_1-convex set and an interval is ℓ_1-convex.*

Proof. Let $X \subseteq \mathbb{R}^n$ be a closed ℓ_1-convex set, and let $I \subseteq \mathbb{R}^n$ be an interval. We may write I as a union of compact subintervals $I^1 \subseteq I^2 \subseteq \cdots$. By Lemma 2.2, $X + I^r$ is ℓ_1-convex for each $r \geqslant 1$. But the class of ℓ_1-convex sets is closed under nested unions, so $\bigcup_r (X + I^r) = X + I$ is ℓ_1-convex. □

Here is our analogue of law (iii). The proof is similar to the proof of the Euclidean case (Lemma 3.1.1 of [17]).

Proposition 2.4. *Let $X, Y, I \subseteq \mathbb{R}^n$. Then*

$$(X \cup Y) + I = (X + I) \cup (Y + I).$$

If X and Y are closed with $X \cup Y$ ℓ_1-convex, and I is an interval, then also

$$(X \cap Y) + I = (X + I) \cap (Y + I).$$

Proof. The first equation is trivial. In the second, the left-hand side is certainly a subset of the right-hand side. For the converse, let $z \in (X + I) \cap (Y + I)$, writing

$$z = x + a = y + b$$

($x \in X$, $y \in Y$, $a, b \in I$). Choose a monotone path $\gamma : [0, 1] \to X \cup Y$ joining x and y. Define a path $\alpha : [0, 1] \to \mathbb{R}^n$ by $\alpha(t) = z - \gamma(t)$. Since I is an interval and α is a monotone path whose endpoints are in I, the whole image of α lies in I. Since X and Y are closed, there exists $t \in [0, 1]$ such that $\gamma(t) \in X \cap Y$. Then $z = \gamma(t) + \alpha(t) \in (X \cap Y) + I$, as required. □

Example 6.1 shows that the second part of Proposition 2.4 can fail when I is merely ℓ_1-convex.

Corollary 2.5. *Let $X, Y \subseteq \mathbb{R}^n$ and let P be a coordinate subspace of $\mathbb{R}^n$. Then*

$$\pi_P(X \cup Y) = \pi_P X \cup \pi_P Y.$$

If X and Y are closed and $X \cup Y$ is ℓ_1-convex then also

$$\pi_P(X \cap Y) = \pi_P X \cap \pi_P Y.$$

Proof. This follows from Proposition 2.4, since $\pi_P Z = (Z + P^\perp) \cap P$ for all $Z \subseteq \mathbb{R}^n$. □

3. Approximation of ℓ_1-convex sets

Essential to our proof of the ℓ_1 Hadwiger theorem is the result, proved in this section, that a compact ℓ_1-convex set can be approximated arbitrarily well by a finite union of cubes *that is itself ℓ_1-convex.*

Write $C_n = [-\frac{1}{2}, \frac{1}{2}]^n$ for the unit n-cube, and $\mathbf{H} = \{m + \frac{1}{2}\colon\ m \in \mathbb{Z}\}$ for the set of half-integers. Given $X \subseteq \mathbb{R}^n$ and $\lambda \geqslant 0$, write $\lambda X = \{\lambda x\colon\ x \in X\}$. When $\lambda > 0$, write

$$X_\lambda = \bigcup \bigl\{\lambda(h + C_n)\colon\ h \in \mathbf{H}^n \text{ with } \lambda(h + C_n) \cap X \neq \emptyset\bigr\} \supseteq X.$$

Proposition 3.1. *Let $X \subseteq \mathbb{R}^n$ be an ℓ_1-convex set and $\lambda > 0$. Then X_λ is ℓ_1-convex.*

Proof. Assume without loss of generality that $\lambda = 1$. Put

$$L = \bigl\{h \in \mathbf{H}^n\colon\ (h + C_n) \cap X \neq \emptyset\bigr\},$$

so that $X_1 = L + C_n$. We show that the closed set L and the interval C_n satisfy the conditions of Lemma 2.2. The result will follow.

Let $h, h' \in L$ with $(h + C_n) \cap (h' + C_n) = \emptyset$, assuming without loss of generality that $h_1 + \frac{1}{2} < h'_1 - \frac{1}{2}$. Choose $x \in (h + C_n) \cap X$ and $x' \in (h' + C_n) \cap X$. Then

$$x_1 \leqslant h_1 + \frac{1}{2} < h'_1 - \frac{1}{2} \leqslant x'_1.$$

By ℓ_1-convexity, there exists a monotone path from x to x' in X; this contains a point z with $h_1 + \frac{1}{2} < z_1 < h'_1 - \frac{1}{2}$. Since z is between x and x', we have $z \in k + C_n$ for some $k \in \mathbf{H}^n$ between h and h'. Then $k \in L$. The constraints on z_1 force $h_1 < k_1 < h'_1$, so $h \neq k \neq h'$. $\square$

For $\lambda > 0$, let us say that a subset of $\mathbb{R}^n$ is λ-*pixellated* if it is a finite union of cubes of the form $\lambda(h + C_n)$ $(h \in \mathbf{H}^n)$. A set is *pixellated* if it is λ-pixellated for some $\lambda > 0$. The role of pixellated sets in the ℓ_1 theory is similar to that of polyhedra in the Euclidean theory. Proposition 3.1 implies:

Theorem 3.2. *The set of pixellated ℓ_1-convex subsets of $\mathbb{R}^n$ is dense in the space of compact ℓ_1-convex subsets of $\mathbb{R}^n$, with respect to the Hausdorff metric.* $\square$

In a later proof, we will use a hyperplane to divide a pixellated ℓ_1-convex set into two smaller sets. We will need the following lemma.

Lemma 3.3. *Let $\lambda > 0$ and let $X \subseteq \mathbb{R}^n$ be a λ-pixellated ℓ_1-convex set. Write*

$$X^+ = \bigcup \bigl\{\lambda(h + C_n)\colon\ h \in \mathbf{H}^n \text{ with } \lambda(h + C_n) \subseteq X \text{ and } h_1 > 0\bigr\},$$
$$X^- = \bigcup \bigl\{\lambda(h + C_n)\colon\ h \in \mathbf{H}^n \text{ with } \lambda(h + C_n) \subseteq X \text{ and } h_1 < 0\bigr\}.$$

Then X^+, X^- and $X^+ \cap X^-$ are all ℓ_1-convex.

Proof. X^+ is the closure of $X \cap ((0, \infty) \times \mathbb{R}^{n-1})$, so is ℓ_1-convex by Corollary 1.9 and Lemma 1.14. Similarly, X^- is ℓ_1-convex. Now $X^+ \cup X^- = X$, and X is ℓ_1-convex, so $X^+ \cap X^-$ is ℓ_1-convex by Lemma 1.13. $\square$

The following result will not be needed later, but is of independent interest. It generalizes the classical fact that compact convex sets are Jordan measurable.

Proposition 3.4. *Every compact ℓ_1-convex subset of $\mathbb{R}^n$ is Jordan measurable.*

Proof. Let $X \subseteq \mathbb{R}^n$ be a compact ℓ_1-convex set. For $\lambda > 0$, write

$$D(\lambda) = \bigcup \bigl\{\lambda(h + C_n)\colon\ h \in \mathbf{H}^n \text{ with } \lambda(h + C_n) \cap X \neq \emptyset \text{ and } \lambda(h + C_n) \nsubseteq X\bigr\}.$$

Write ∂X for the topological boundary of X. Then $\partial X \subseteq D(\lambda)$ for all $\lambda > 0$, and we have to show that $\mathrm{Vol}(\partial X) = 0$.

For $\lambda > 0$ and $h \in \mathbf{H}^n$, the set $D(\lambda)$ cannot contain all 3^n of the cubes $\lambda(h + \sigma + C_n)$ with $\sigma \in \{-1, 0, 1\}^n$. Indeed, if $D(\lambda)$ contains all 2^n corner cubes then X has nonempty intersection with all the corner cubes, and then it follows from ℓ_1-convexity that X contains the whole central cube $\lambda(h + C_n)$; hence $D(\lambda)$ does not contain the central cube.

We therefore have

$$\mathrm{Vol}\bigl(D(\lambda)\bigr) \leqslant \bigl((3^n - 1)/3^n\bigr)\,\mathrm{Vol}\bigl(D(3\lambda)\bigr)$$

for all $\lambda > 0$. Thus, $\inf_{\lambda > 0} \mathrm{Vol}(D(\lambda)) = 0$, giving $\mathrm{Vol}(\partial X) = 0$. $\square$

4. Valuations

Here we generalize the theorem of McMullen [14, Theorem 8] that a monotone translation-invariant valuation on convex sets is continuous. This will make numerous continuity checks very easy.

For the rest of this section, let $\mathscr{K}$ be a set of compact, orthogonally convex subsets of $\mathbb{R}^n$. We suppose that $\mathscr{K}$ contains all singletons $\{x\}$, and that $X + I \in \mathscr{K}$ whenever $X \in \mathscr{K}$ and I is a compact interval. Then $\mathscr{K}$ contains all compact intervals and is closed under translations. For example, $\mathscr{K}$ might be the set $\mathscr{K}_n$ of compact convex subsets of $\mathbb{R}^n$, or the set $\mathscr{K}_n'$ of compact ℓ_1-convex subsets of $\mathbb{R}^n$ (by Corollary 1.11 and Proposition 2.3).

A *valuation* on $\mathscr{K}$ is a function $\phi : \mathscr{K} \to \mathbb{R}$ such that $\phi(\emptyset) = 0$ and

$$\phi(X \cup Y) = \phi(X) + \phi(Y) - \phi(X \cap Y)$$

whenever $X, Y, X \cup Y, X \cap Y \in \mathscr{K}$. (The hypothesis that $X \cap Y \in \mathscr{K}$ is redundant for $\mathscr{K}_n$, and also for $\mathscr{K}_n'$, by Lemma 1.13.)

We give $\mathscr{K}$ the Hausdorff metric induced by the ℓ_1 metric on $\mathbb{R}^n$. This can be defined as follows. Writing B_n for the closed unit ball of ℓ_1^n, the *Hausdorff distance* between compact sets $X, Y \subseteq \mathbb{R}^n$ is

$$d(X, Y) = \inf\{\delta > 0 \colon\ X \subseteq Y + \delta B_n \text{ and } Y \subseteq X + \delta B_n\}.$$

By compactness, the infimum is attained.

A valuation ϕ is *continuous* if it is continuous with respect to the Hausdorff metric. The notion of continuity is unaffected by the choice of the 1-norm over other norms on $\mathbb{R}^n$, since all such are equivalent. A valuation ϕ is *increasing* if $X \subseteq Y$ implies $\phi(X) \leqslant \phi(Y)$ for $X, Y \in \mathscr{K}$, and *monotone* if ϕ or $-\phi$ is increasing. It is *translation-invariant* if $\phi(X + a) = \phi(X)$ for all $X \in \mathscr{K}$ and $a \in \mathbb{R}^n$.

For $R \geqslant 0$, write $\mathscr{K}[R] = \{X \in \mathscr{K} \colon\ X \subseteq RC_n\}$.

Lemma 4.1. *Let ϕ be a monotone translation-invariant valuation on $\mathscr{K}$. Let $R \geqslant 0$. Then*

$$\lim_{\delta \to 0} \phi(X + \delta C_n) = \phi(X)$$

uniformly in $X \in \mathscr{K}[R]$.

Proof. Given $1 \leqslant i \leqslant n$, write $\nu_i : \mathbb{R} \to \mathbb{R}^n$ for the embedding of $\mathbb{R}$ as the ith coordinate axis of $\mathbb{R}^n$. Given also $X \in \mathscr{K}$, define $f_{X,i} : [0, \infty) \to \mathbb{R}$ by

$$f_{X,i}(\delta) = \phi\big(X + \nu_i[-\delta/2, \delta/2]\big) - \phi(X).$$

For all $\delta, \delta' \geqslant 0$, by orthogonal convexity of X and the valuation property, we have

$$\phi\big(X + \nu_i\big[-\delta, \delta'\big]\big) = \phi\big(X + \nu_i[-\delta, 0]\big) + \phi\big(X + \nu_i\big[0, \delta'\big]\big) - \phi(X).$$

So by translation-invariance, $f_{X,i}(\delta + \delta') = f_{X,i}(\delta) + f_{X,i}(\delta')$. But $f_{X,i}$ is monotone, so $f_{X,i}(\delta) = f_{X,i}(1) \cdot \delta$ for all $\delta \geqslant 0$. Assume without loss of generality that ϕ is increasing. Then whenever $S \geqslant 0$ and $X \in \mathscr{K}[S]$, we have

$$0 \leqslant f_{X,i}(1) \leqslant \phi\left(X + \nu_i\left[-\frac{1}{2}, \frac{1}{2}\right]\right) \leqslant \phi\big((S+1)C_n\big),$$

so

$$0 \leqslant f_{X,i}(\delta) \leqslant \phi\big((S+1)C_n\big) \cdot \delta$$

for all $\delta \geqslant 0$.

From this estimate and the fact that $C_n = \sum_{i=1}^n \nu_i[-\frac{1}{2}, \frac{1}{2}]$, we deduce that

$$\phi(X) \leqslant \phi(X + \delta C_n) \leqslant \phi(X) + n\phi\big((R+2)C_n\big) \cdot \delta$$

for all $X \in \mathscr{K}[R]$ and $\delta \in [0, 1]$. The result follows. □

Theorem 4.2. *A monotone translation-invariant valuation on* $\mathscr{K}$ *is continuous.*

Proof. Consider, without loss of generality, an increasing translation-invariant valuation ϕ. Let $X \in \mathscr{K}$ and $\varepsilon > 0$. Choose $R \geqslant 2$ with $X \subseteq (R-2)C_n$. By Lemma 4.1, we may choose $\eta > 0$ such that $\phi(Y + \eta C_n) \leqslant \phi(Y) + \varepsilon$ for all $Y \in \mathscr{K}[R]$. Put $\delta = \min\{1, \eta/2\}$.

I claim that $|\phi(Y) - \phi(X)| \leqslant \varepsilon$ whenever $Y \in \mathscr{K}$ with $d(X, Y) \leqslant \delta$. Indeed, take such a Y. Then $Y \subseteq X + \delta B_n$; but also $B_n \subseteq 2C_n$, so $Y \subseteq X + 2\delta C_n$. This implies that $Y \subseteq RC_n$, so $Y \in \mathscr{K}[R]$, and clearly $X \in \mathscr{K}[R]$ too. It also implies that $X \subseteq Y + \eta C_n$, giving

$$\phi(X) \leqslant \phi(Y + \eta C_n) \leqslant \phi(Y) + \varepsilon.$$

Similarly, $Y \subseteq X + \eta C_n$, so $\phi(Y) \leqslant \phi(X) + \varepsilon$. This proves the claim. □

Corollary 4.3. *(See McMullen [14].) A monotone translation-invariant valuation on* $\mathscr{K}_n$ *is continuous.* □

Corollary 4.4. *A monotone translation-invariant valuation on* $\mathscr{K}'_n$ *is continuous.* □

Lebesgue measure, as a real-valued function on compact subsets of $\mathbb{R}^n$, is not continuous with respect to the Hausdorff metric. It is, however, continuous when restricted to convex sets [22, Theorem 12.7]. Corollary 4.4 generalizes this classical result to the larger class of ℓ_1-convex sets:

Corollary 4.5. *The volume function* $\mathrm{Vol} : \mathscr{K}'_n \to \mathbb{R}$ *is continuous.* □

5. An analogue of Hadwiger's theorem

We are now in a position to prove ℓ_1 analogues of the classical theorems of integral geometry. We begin with Hadwiger's theorem, adopting a strategy similar in outline to that of Klain [11].

Denote by G_n the isometry group of ℓ_1^n. It is generated by translations, coordinate permutations, and reflections in coordinate hyperplanes. A valuation ϕ on $\mathscr{K}'_n$ is *invariant* if $\phi(gX) = \phi(X)$ whenever $X \in \mathscr{K}'_n$ and $g \in G_n$. The continuous invariant valuations on $\mathscr{K}'_n$ form a vector space Val'_n over $\mathbb{R}$.

Given $0 \leqslant i \leqslant n$, write $\mathrm{Gr}'_{n,i}$ for the set of i-dimensional coordinate subspaces of $\mathbb{R}^n$; it has $\binom{n}{i}$ elements. Define $V'_{n,i} : \mathscr{K}'_n \to \mathbb{R}$, the ith ℓ_1-*intrinsic volume* on $\mathbb{R}^n$, by

$$V'_{n,i}(X) = \sum_{P \in \mathrm{Gr}'_{n,i}} \mathrm{Vol}_i(\pi_P X)$$

($X \in \mathscr{K}'_n$). In the P-summand, Vol_i denotes Lebesgue measure on $P \cong \mathbb{R}^i$.

Examples 5.1. (i) The 0th ℓ_1-intrinsic volume $V'_{n,0}$ is the Euler characteristic χ, given by $\chi(\emptyset) = 0$ and $\chi(X) = 1$ whenever $X \in \mathscr{K}'_n$ is nonempty.

(ii) The unit cube C_n has ℓ_1-intrinsic volumes $V'_{n,i}(C_n) = \binom{n}{i}$, which are the same as its Euclidean intrinsic volumes.

(iii) Write B_n for the unit ball in ℓ_1^n. Then $\mathrm{Vol}(B_n) = 2^n/n!$, giving $V'_{n,i}(B_n) = \frac{2^i}{i!}\binom{n}{i}$. Taking $n = 2$ and $i = 1$ shows that the ℓ_1- and Euclidean intrinsic volumes of a convex set are not always equal (and nor are they equal up to a constant factor, by the previous example).

A valuation ϕ on $\mathscr{K}_n'$ is *homogeneous of degree i* if $\phi(\lambda X) = \lambda^i \phi(X)$ for all $\lambda \geqslant 0$ and $X \in \mathscr{K}_n'$. In the following lemma, we write $\nu : \mathbb{R}^n \to \mathbb{R}^{n+1}$ for the embedding that inserts 0 in the last coordinate.

Lemma 5.2. *Let* $0 \leqslant i \leqslant n$. *Then*:

(i) $V'_{n,i}$ *is a continuous invariant valuation on* $\mathscr{K}_n'$,
(ii) $V'_{n,i}$ *is homogeneous of degree i*,
(iii) $V'_{n,i}(X) = V'_{n+1,i}(\nu X)$ *for all* $X \in \mathscr{K}_n'$,
(iv) $V'_{n,n} = \mathrm{Vol}_n$,
(v) $V'_{n,i}$ *is increasing.*

Proof. The only nontrivial part is (i). First fix $P \in \mathrm{Gr}'_{n,i}$. By Corollary 2.5, the function $X \mapsto \mathrm{Vol}_i(\pi_P X)$ on $\mathscr{K}_n'$ is a valuation. It is also monotone and translation-invariant, and therefore continuous by Corollary 4.4. This holds for all P, and (i) follows. □

Parts (i) and (iii) allow us to write $V'_{n,i}$ as just V'_i: for if $X \in \mathscr{K}_n'$ and $\mathbb{R}^n$ is embedded as a coordinate subspace of some larger space $\mathbb{R}^N$, then the ith ℓ_1-intrinsic volume of X is the same whether X is regarded as a subset of $\mathbb{R}^n$ or of $\mathbb{R}^N$. This justifies the word 'intrinsic'.

The *dimension* of a nonempty ℓ_1-convex set $X \subseteq \mathbb{R}^n$ is the smallest $i \in \{0, \dots, n\}$ such that $X \subseteq P + q$ for some $P \in \mathrm{Gr}'_{n,i}$ and $q \in \mathbb{R}^n$. A valuation ψ on $\mathscr{K}_n'$ is *simple* if $\psi(X) = 0$ whenever $X \in \mathscr{K}_n'$ is of dimension less than n.

Proposition 5.3. *Let ψ be a simple continuous translation-invariant valuation on $\mathscr{K}_n'$. Then $\psi = c\,\mathrm{Vol}_n$ for some $c \in \mathbb{R}$.*

Proof. Put $\zeta = \psi - \psi(C_n)\,\mathrm{Vol}_n$. By Corollary 4.5, ζ is a simple continuous translation-invariant valuation with $\zeta(C_n) = 0$. We will prove that ζ is identically zero.

By dividing cubes into smaller cubes, $\zeta(\lambda C_n) = 0$ for all rational $\lambda > 0$. By continuity, $\zeta(\lambda C_n) = 0$ for all real $\lambda > 0$.

Now I claim that $\zeta(X) = 0$ whenever X is a pixellated ℓ_1-convex set. Suppose that X is λ-pixellated, where $\lambda > 0$; thus, X is a union of some finite number m of cubes of the form $\lambda(h + C_n)$ ($h \in \mathbf{H}^n$). If $m = 0$ or $m = 1$ then $\zeta(X) = 0$ immediately.

Suppose inductively that $m \geqslant 2$. Without loss of generality, the m cubes are not all on the same side of the hyperplane $\{y \in \mathbb{R}^n\colon\ y_1 = 0\}$. Define sets X^+ and X^- as in Lemma 3.3; thus, $X = X^+ \cup X^-$, and each of the sets X^+, X^-, $X^+ \cap X^-$ is ℓ_1-convex. Then $\zeta(X^+) = \zeta(X^-) = 0$ by inductive hypothesis, and $\zeta(X^+ \cap X^-) = 0$ since ζ is simple. Hence $\zeta(X) = 0$ by the valuation property, completing the induction and proving the claim.

The result now follows from Theorem 3.2. □

Theorem 5.4. *The ℓ_1-intrinsic volumes $V'_0, \dots, V'_n$ form a basis for the vector space Val'_n of continuous invariant valuations on ℓ_1-convex subsets of $\mathbb{R}^n$. In particular,* $\dim(\mathrm{Val}'_n) = n + 1$.

Proof. By Lemma 5.2(iii), (iv), we have $V'_i(C_i) = 1$ whenever $0 \leqslant i \leqslant n$, and $V'_j(C_i) = 0$ whenever $0 \leqslant i < j \leqslant n$. It follows that $V'_0, \dots, V'_n$ are linearly independent.

We prove by induction on n that $V'_0, \dots, V'_n$ span Val'_n. This is trivial when $n = 0$. Suppose that $n \geqslant 1$, and let $\phi \in \mathrm{Val}'_n$. Choose some $Q \in \mathrm{Gr}'_{n,n-1}$, and denote by ϕ' the restriction of ϕ to ℓ_1-convex subsets of Q. By inductive hypothesis, there exist constants $c_0, \dots, c_{n-1}$ such that $\phi' = \sum_{i=0}^{n-1} c_i V'_i$. Thus, for all ℓ_1-convex sets $X \subseteq Q$,

$$\phi(X) = \sum_{i=0}^{n-1} c_i V'_i(X). \tag{2}$$

Since ϕ is invariant under translations and coordinate permutations, (2) holds for all $X \in \mathscr{K}_n'$ of dimension less than n. Put $\psi = \phi - \sum_{i=0}^{n-1} c_i V_i'$. Then ψ is a continuous translation-invariant valuation, which we have just shown to be simple. Proposition 5.3 implies that $\psi = c_n \operatorname{Vol}_n$ for some $c_n \in \mathbb{R}$. Since $\operatorname{Vol}_n = V_n'$, the result follows. □

Corollary 5.5. *Let ϕ be a continuous invariant valuation on $\mathscr{K}_n'$, homogeneous of degree $i \in \mathbb{R}$. Then $i \in \{0, \dots, n\}$ and $\phi = cV_i'$ for some $c \in \mathbb{R}$.* □

Careful analysis of the proof of the theorem enables two refinements to be made.

First, when showing that every continuous invariant valuation ϕ on $\mathscr{K}_n'$ was a linear combination of ℓ_1-intrinsic volumes, we never called on the fact that ϕ was invariant under reflections in coordinate hyperplanes. Second, we did not use the full strength of the assumption that ϕ was continuous: only that ϕ was *continuous from the outside*, that is, $\lim_{Y \to X,\, Y \supseteq X} \phi(Y) = \phi(X)$ for all $X \in \mathscr{K}_n'$. (The essential point is that in Proposition 3.1, the pixellated sets X_λ contain X.) But any linear combination of ℓ_1-intrinsic volumes is continuous and invariant under the full isometry group. Hence:

Corollary 5.6. *Let ϕ be a valuation on $\mathscr{K}_n'$, continuous from the outside and invariant under translations and coordinate permutations. Then ϕ is continuous and invariant.* □

6. An analogue of Steiner's formula

The most obvious analogue of the classical Steiner formula would be an identity of the form

$$\operatorname{Vol}(X + \lambda B_n) = \sum_{i=0}^{n} c_i V_i'(X) \lambda^{n-i}$$

for $X \in \mathscr{K}_n'$ and $\lambda \geqslant 0$. Here B_n denotes the closed unit ball in ℓ_1^n, and $c_0, \dots, c_n$ are constants. However, there can be no such formula. For if there were then $\operatorname{Vol}(\cdot + B_n)$ would be a valuation on $\mathscr{K}_n'$, and the following example demonstrates that for general n, it is not.

Example 6.1. Let $X = [0, 1] \times \{0\}$ and $Y = \{0\} \times [0, 1]$, both subsets of $\mathbb{R}^2$. Then X, Y, $X \cap Y$ and $X \cup Y$ are all ℓ_1-convex sets, and it is straightforward to calculate that

$$\operatorname{Vol}\big((X \cap Y) + B_2\big) < \operatorname{Vol}\big((X + B_2) \cap (Y + B_2)\big).$$

But $(X \cup Y) + B_2 = (X + B_2) \cup (Y + B_2)$ by Proposition 2.4, so $\operatorname{Vol}(\cdot + B_2)$ is not a valuation on $\mathscr{K}_2'$.

There is, however, a Steiner-type formula in which the role of the ball is played by the cube C_n.

Theorem 6.2. *Let $X \in \mathscr{K}_n'$ and $\lambda \geqslant 0$. Then for $0 \leqslant k \leqslant n$,*

$$V_k'(X + \lambda C_n) = \sum_{i=0}^{k} \binom{n-i}{n-k} V_i'(X) \lambda^{k-i}.$$

In particular,

$$\operatorname{Vol}(X + \lambda C_n) = \sum_{i=0}^{n} V_i'(X) \lambda^{n-i}.$$

For the left-hand side of the first equation to be defined we need $X + \lambda C_n$ to be ℓ_1-convex. This follows from Proposition 2.3.

Proof. We begin by showing that $V'_k(\cdot + C_n)$ is a continuous invariant valuation. Proposition 2.4 implies that it is a valuation, since C_n is an interval. It is invariant, since C_n is invariant under isometries fixing the origin. It is also monotone, and therefore continuous by Corollary 4.4.

By the ℓ_1 Hadwiger Theorem 5.4, there are constants c_i such that $V'_k(X + C_n) = \sum_{i=0}^n c_i V'_i(X)$ for all $X \in \mathscr{K}'_n$. It follows that for $\lambda > 0$ and $X \in \mathscr{K}'_n$,

$$V'_k(X + \lambda C_n) = \lambda^k V'_k\big(\lambda^{-1} X + C_n\big) = \sum_{i=0}^n c_i V'_i(X) \lambda^{k-i}.$$

The result follows on putting $X = C_n$, using Example 5.1(ii). $\square$

7. An analogue of Crofton's formula

In this section and the next, we derive ℓ_1 analogues of Euclidean integral-geometric formulas. The formal structure is similar to that in Klain and Rota [12].

For $0 \leqslant k \leqslant n$, let $\mathrm{Graff}'_{n,k}$ denote the set of k-dimensional affine subspaces of $\mathbb{R}^n$ parallel to some k-dimensional coordinate subspace. Each element of $\mathrm{Graff}'_{n,k}$ is uniquely representable as $P + q$ with $P \in \mathrm{Gr}'_{n,k}$ and $q \in P^\perp$.

There is a natural measure on $\mathrm{Graff}'_{n,k}$, invariant under isometries of ℓ_1^n. Indeed, $\mathrm{Graff}'_{n,k}$ is in canonical bijection with the disjoint union $\coprod_{P \in \mathrm{Gr}'_{n,k}} P^\perp$, each space $P^\perp$ carries Lebesgue measure Vol_{n-k}, and summing gives the measure on $\mathrm{Graff}'_{n,k}$.

Theorem 7.1. *Let $X \in \mathscr{K}'_n$. Then for $0 \leqslant j \leqslant k \leqslant n$,*

$$\int\limits_{\mathrm{Graff}'_{n,k}} V'_j(X \cap A)\, dA = \binom{n+j-k}{j} V'_{n+j-k}(X). \tag{3}$$

In particular, for $0 \leqslant k \leqslant n$, the set

$$\big\{A \in \mathrm{Graff}'_{n,k} \colon X \cap A \neq \emptyset\big\}$$

has measure $V'_{n-k}(X)$.

Proof. We prove just the first statement, the second being the case $j = 0$.

Write $\phi(X)$ for the left-hand side of (3). Then ϕ is a monotone invariant valuation since V'_j is, and since the measure on $\mathrm{Graff}'_{n,k}$ is invariant. It is therefore continuous, by Corollary 4.4. Moreover, ϕ is homogeneous of degree $n + j - k$, since for $\lambda > 0$ and $X \in \mathscr{K}'_n$,

$$\phi(\lambda X) = \int\limits_{B \in \mathrm{Graff}'_{n,k}} V'_j(\lambda X \cap \lambda B)\, d(\lambda B) = \lambda^j \lambda^{n-k} \phi(X).$$

So by Corollary 5.5, $\phi = c V'_{n+j-k}$ for some $c \in \mathbb{R}$.

By construction of the invariant measure,

$$\phi(C_n) = \sum_{P \in \mathrm{Gr}'_{n,k}} \int_{P^\perp} V'_j\big(C_n \cap (P+q)\big)\,dq.$$

For $P \in \mathrm{Gr}'_{n,k}$ and $q \in P^\perp$, identifying $P^\perp$ with $\mathbb{R}^{n-k}$, we have

$$V'_j\big(C_n \cap (P+q)\big) = \begin{cases} V'_j(C_k) = \binom{k}{j} & \text{if } q \in C_{n-k}, \\ 0 & \text{otherwise.} \end{cases}$$

From this it is straightforward to deduce the value of c. □

A very similar argument, left to the reader, proves the following analogue of Kubota's theorem [13,12]. It can also be deduced directly from the definition of the ℓ_1-intrinsic volumes. Corollary 1.12 guarantees that the left-hand side is defined.

Theorem 7.2. *Let $X \in \mathscr{K}'_n$. Then for $0 \leqslant j \leqslant k \leqslant n$,*

$$\sum_{P \in \mathrm{Gr}'_{n,k}} V'_j(\pi_P X) = \binom{n-j}{n-k} V'_j(X). \qquad \square$$

8. Analogues of the kinematic formulas

The classical kinematic formulas concern the intrinsic volumes of sets $gX \cap Y$, where X and Y are convex and g is a Euclidean motion. Remark 1.10 suggests that fundamentally, one of X and Y should be regarded as geodesic and the other as *co*geodesic, although in the classical context the difference is invisible. This leads us to expect ℓ_1 kinematic formulas in which X is ℓ_1-convex and Y is an interval.

To state the ℓ_1 kinematic formulas, we first need a measure on the isometry group G_n of ℓ_1^n. This is constructed as follows. G_n has a subgroup H_n, the nth hyperoctahedral group, consisting of just the isometries fixing the origin. Each element of G_n is uniquely representable as $x \mapsto h(x) + q$ with $h \in H_n$ and $q \in \mathbb{R}^n$. Being a finite group, H_n has a unique invariant probability measure. Taking the product of this measure with Lebesgue measure on $\mathbb{R}^n$ gives an invariant (Haar) measure on G_n.

We also need a result on products. First observe that if $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$ are ℓ_1-convex then $X \times Y$, viewed as a subset of $\mathbb{R}^{m+n}$, is also ℓ_1-convex.

Proposition 8.1. *Let $X \in \mathscr{K}'_m$, $Y \in \mathscr{K}'_n$, and $0 \leqslant k \leqslant m+n$. Then*

$$V'_k(X \times Y) = \sum_{i+j=k} V'_i(X) V'_j(Y)$$

where $0 \leqslant i \leqslant m$ and $0 \leqslant j \leqslant n$ in the summation.

Proof. By definition of the ℓ_1-intrinsic volumes,

$$V'_k(X \times Y) = \sum_{i+j=k} \sum_{P \in \mathrm{Gr}'_{m,i},\ Q \in \mathrm{Gr}'_{n,j}} \mathrm{Vol}_{i+j}\big(\pi_{P \times Q}(X \times Y)\big)$$

$$= \sum_{i+j=k} \sum_{P \in \mathrm{Gr}'_{m,i},\ Q \in \mathrm{Gr}'_{n,j}} \mathrm{Vol}_i(\pi_P X)\,\mathrm{Vol}_j(\pi_Q Y)$$

$$= \sum_{i+j=k} V'_i(X) V'_j(Y). \qquad \square$$

Example 8.2. Let $I = I_1 \times \cdots \times I_n$ be a nonempty compact interval. Then $V'_j(I)$ is the jth elementary symmetric polynomial in the lengths of $I_1, \ldots, I_n$ (also equal to the jth Euclidean intrinsic volume of I).

We now state the principal kinematic formula for ℓ_1^n.

Theorem 8.3. *Let $X \in \mathscr{K}'_n$ and let I be a compact interval in $\mathbb{R}^n$. Then the set*

$$\{g \in G_n\colon\ gX \cap I \neq \emptyset\} \tag{4}$$

has measure

$$\sum_{i+j=n} \binom{n}{i}^{-1} V'_i(X) V'_j(I).$$

Proof. Fix I. Write $\phi(X)$ for the measure of the set (4): then $\phi(X) = \int_{G_n} \chi(gX \cap I)\,dg$. This ϕ is a monotone invariant valuation on $\mathscr{K}'_n$, and is therefore continuous by Corollary 4.4. So by the ℓ_1 Hadwiger theorem,

$$\phi = \sum_{i=0}^{n} c_i V'_i \tag{5}$$

for some real numbers c_i. We compute c_i by evaluating $\phi(\lambda C_n)$ for $\lambda > 0$. By construction of the invariant measure on G_n,

$$\phi(\lambda C_n) = \frac{1}{|H_n|} \sum_{h \in H_n} \int_{\mathbb{R}^n} \chi\bigl(\bigl(h(\lambda C_n) + q\bigr) \cap I\bigr)\,dq.$$

But λC_n is H_n-invariant, so $\phi(\lambda C_n)$ is the Lebesgue measure of the set

$$\bigl\{q \in \mathbb{R}^n\colon\ (\lambda C_n + q) \cap I \neq \emptyset\bigr\}. \tag{6}$$

If $I = \emptyset$ then the theorem holds trivially; suppose that $I \neq \emptyset$. Write $I = \prod_{r=1}^n I_r$, and write u_r for the length of the interval I_r. Then the set (6) is a product of intervals of lengths $\lambda + u_r$. Hence

$$\phi(\lambda C_n) = \prod_{r=1}^{n} (\lambda + u_r) = \lambda^n \prod_{r=1}^{n} \bigl(1 + \lambda^{-1} u_r\bigr) = \lambda^n \sum_{j=0}^{n} V'_j\bigl(\lambda^{-1} I\bigr) = \sum_{j=0}^{n} V'_j(I) \lambda^{n-j},$$

using nonemptiness of I and Example 8.2. On the other hand, we may compute $\phi(\lambda C_n)$ using (5), and comparing coefficients gives $c_i = \binom{n}{i}^{-1} V'_{n-i}(I)$. $\square$

Higher kinematic formulas for ℓ_1^n can be deduced from Theorems 7.1 and 8.3 by an argument formally identical to that in Section 10.3 of [12]:

Theorem 8.4. *Let $0 \leqslant k \leqslant n$, let $X \in \mathscr{K}_n'$, and let I be a compact interval in $\mathbb{R}^n$. Then*

$$\int_{G_n} V_k'(gX \cap I)\,dg = \sum_{i+j=n+k} \binom{n}{i}^{-1} \binom{j}{k} V_i'(X) V_j'(I)$$

where $0 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant n$ in the summation. □

Acknowledgments

I thank Andreas Bernig, Joseph Fu, Daniel Hug, Mark Meckes, Rolf Schneider and the anonymous referee for helpful conversations and suggestions.

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