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Polygons with inscribed circles and prescribed side lengths[☆]

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ABSTRACT

We prove NP-completeness of the following problem: For n given input numbers, decide whether there exists an n-sided, plane, convex polygon that has an inscribed circle and that has the input numbers as side lengths.

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1. Introduction

Let $S = \langle s_1, s_2, \ldots, s_n \rangle$ be a sequence of n positive real numbers, and let $s_{\max} = \max_{1 \le k \le n} s_k$ denote the largest number in this sequence. A plane n-sided polygon is called an *S-polygon* if its side lengths are some permutation of the numbers $s_1, s_2, \ldots, s_n$. A necessary and sufficient condition for the existence of an S-polygon is that the length of the longest side $s_{\max}$ is smaller than the sum of the remaining sides; in other words $\sum_{k=1}^{n} s_k < 2\, s_{\max} (*)$ must be satisfied. A simple continuity argument yields that under this condition $(*)$, there in fact always exists an S-polygon that has a *circumscribed* circle (such that each vertex of the polygon lies on the circle). What about S-polygons with *inscribed* circles (where each side of the polygon is tangent to the circle)? Since there is no $\langle 1, 100, 100, 100 \rangle$-polygon with an inscribed circle, condition $(*)$ alone is certainly not sufficient to guarantee the existence of such a polygon.

In this note, we will show that (perhaps somewhat surprisingly) deciding whether for a given sequence S there exists an S-polygon with an inscribed circle is an NP-complete problem [1]. This indicates that the combinatorics of such sequences is quite messy, and that we should not hope to find a nice and simple characterization.

2. The proof

Throughout this section, calculations with indices will be done modulo n, so that index $n + k$ equals index k for all integers k. Our main tool is the following elementary lemma that characterizes the side lengths of polygons with inscribed circles.

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Lemma 1. *Let $S = \langle s_1, s_2, \ldots, s_n\rangle$ be a sequence of positive real numbers. Then the following two statements are equivalent.*

(i) *There exists an n-sided, plane, convex polygon $\mathcal{P}$ with an inscribed circle that has vertices $P_1, P_2, \ldots, P_n$ and side lengths $|P_kP_{k+1}| = s_k$ (for $k = 1, \ldots, n$).*
(ii) *There exist n positive real numbers $t_1, \ldots, t_n$ that satisfy $t_k + t_{k+1} = s_k$ for $k = 1, \ldots, n$.*

Proof. First let us assume that (i) holds. For $k = 1, \ldots, n$, consider the tangency point T_k at which the line segment P_kP_{k+1} touches the inscribed circle. Then $|P_kT_k| = |P_kT_{k-1}|$ holds for all k. By setting $t_k := |P_kT_k|$ we get the desired real values for statement (ii).

Next let us assume that statement (ii) holds. We introduce an open polygonal chain $P_1, P_2, \ldots, P_n, P_{n+1}$ with n links and with hinges at the $n-1$ intermediate points $P_2, \ldots, P_n$. The length of link P_kP_{k+1} equals s_k. On every link P_kP_{k+1} we mark a red point T_k such that $|P_kT_k| = t_k$ and such that $|T_kP_{k+1}| = t_{k+1}$. Now consider an arbitrary circle $C(r)$ with radius $r > 0$ and with the center at the origin O. We slowly wind the polygonal chain around the circle $C(r)$ in the following way: The first link P_1P_2 touches $C(r)$ in the red point T_1. The second link P_2P_3 then touches $C(r)$ in the red point T_2. And so on, and so on, until the final link P_nP_{n+1} touches circle $C(r)$ at the point T_n. The angle P_kOP_{k+1} equals $\arctan(t_k/r) + \arctan(t_{k+1}/r)$. The overall angle covered by all n links then equals $2\sum_{k=1}^{n}\arctan(t_k/r)$. As r goes to 0, this overall angle goes to infinity. As r goes to infinity, this overall angle goes to 0. Hence, there exists a radius r for which the overall angle equals 2π and for which the position of point P_{n+1} coincides with the position of point P_1. This then yields the desired polygon $\mathcal{P}$ for (i). □

Now let us return to the problem of deciding whether for a given input sequence S there exists a corresponding S-polygon with an inscribed circle. This algorithmic decision problem will be denoted as INCIRCLE.

Problem: INCIRCLE
Instance: Positive integers $s_1, s_2, \ldots, s_n$.
Question: Does there exist a plane, convex polygon with an inscribed circle, whose side lengths are a permutation of the numbers $s_1, s_2, \ldots, s_n$?

Lemma 2. *The decision problem INCIRCLE lies in the complexity class NP.*

Proof. This is an almost immediate consequence of Lemma 1. The NP-certificate for a YES-instance of INCIRCLE is the corresponding permutation of the integers s_k as side lengths of the polygon together with the corresponding numbers t_k. Since the numbers t_k are the solution of a linear equation system with coefficients 0 and 1 and right hand sides s_k, the description length of the t_k is polynomially bounded in the description length of the numbers s_k. (This follows from linear algebra folklore; see for instance Schrijver [2].) Hence, there is a concise certificate that is easily verified in polynomial time. □

The NP-hardness reduction for problem INCIRCLE is done from the following variant of the NP-complete PARTITION problem (see [1]):

Problem: PARTITION
Instance: Positive integers $a_1, \ldots, a_{2m}$ and A, such that $\sum_{k=1}^{2m} a_k = 2A$.
Question: Does there exist a set $I \subset \{1, \ldots, 2m\}$ with $|I| = m$ and $\sum_{k\in I} a_k = A$?

We construct the following instance of problem INCIRCLE: The sequence $s_1, s_2, \ldots, s_{2m}$ consists of $n = 2m$ positive integers $s_k = 3A + a_k$ for $k = 1, \ldots, 2m$. We claim that the instance of PARTITION has answer YES, if and only if the constructed instance of INCIRCLE has answer YES.

First let us assume that the instance of INCIRCLE has answer YES. Let us renumber the integers $s_1, \ldots, s_{2m}$ and the numbers $a_1, \ldots, a_{2m}$ correspondingly, so that in the polygon $\mathcal{P}$ with vertices $P_1, P_2, \ldots, P_{2m}$, the side length P_kP_{k+1} equals s_k. Consider the positive real numbers $t_1, \ldots, t_{2m}$ that exist by Lemma 1. Since these numbers satisfy $\sum_{k=1}^{2m} t_k = \frac{1}{2}\sum_{k=1}^{2m} s_k = (3m+1)A$, we conclude that

$$(3m+1)A = \sum_{k=1}^{2m} t_k = \sum_{\ell=1}^{m} s_{2\ell} = 3mA + \sum_{\ell=1}^{m} a_{2\ell}.$$

Since $\sum_{\ell=1}^{m} a_{2\ell} = A$, the PARTITION instance indeed has answer YES.

Next, let us assume that the instance of PARTITION has answer YES. Let us renumber the integers $a_1, \ldots, a_{2m}$ in such a way that $\sum_{k=1}^{m} a_{2k} = \sum_{k=1}^{m} a_{2k-1} = A$ holds. For $k = 1, \ldots, m$ we define the integers

$$t_{2k-1} = \sum_{\ell=1}^{2k-2} (-1)^{\ell} s_\ell + A = (s_2 - s_1) + (s_4 - s_3) + \cdots + (s_{2k-2} - s_{2k-3}) + A$$

$$t_{2k} = \sum_{\ell=1}^{2k-1} (-1)^{\ell+1} s_\ell - A = (s_1 - s_2) + \cdots + (s_{2k-3} - s_{2k-2}) + s_{2k-1} - A.$$

Since $-a_{2k-1} \leq s_{2k} - s_{2k-1} \leq a_{2k}$ holds for all k, all these numbers t_k are positive. Furthermore, it is easily checked that $t_k + t_{k+1} = s_k$ holds for $k = 1, \ldots, 2m-1$. Finally,

$$t_{2m} + t_1 = \sum_{\ell=1}^{2m-1} (-1)^{\ell+1} s_\ell = \sum_{\ell=1}^{m} s_{2\ell-1} - \sum_{\ell=1}^{m-1} s_{2\ell} = s_{2m}.$$

Since the statement in Lemma 1.(ii) is satisfied, the constructed instance of INCIRCLE does indeed have answer YES.

The NP-hardness reduction and the result in Lemma 2 together yield the main result of this note.

Theorem 3. *Problem INCIRCLE is NP-complete.* □

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