ASYMPTOTIC BEHAVIOUR OF A GENERALIZED BURGERS’ EQUATION

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ABSTRACT. – We consider the generalized Burgers equation:

\[ u_t = \Delta (u^m) - \frac{\partial}{\partial x_1} (u^q), \]

(GBE)

with exponents \( m > 1 \) and \( q = m + (1/N) \). We study the large-time behaviour of nonnegative weak solutions of the Cauchy problem posed in \( Q = \mathbb{R}^N \times (0, \infty) \) with integrable and nonnegative data. We construct a uni-parametric family \( \{ U_M \} \) of source-type solutions of (GBE) such that:

\[ U_M(x, t) \rightarrow M \delta(x) \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^N) \quad \text{as} \quad t \rightarrow 0, \]

and prove that they give the asymptotic behaviour of all solutions of the Cauchy problem. These special solutions have the following self-similar form: \( U(x, t) = t^{-\alpha} F(x t^{-\beta}) \), with \( \alpha = 1/(m - 1 + (2/N)) \) and \( N\beta = \alpha \). The criterion to choose the right member of the family is the following mass equality:

\[ M = \int_{\mathbb{R}^N} u_0 \, dx. \]

The construction of the family \( \{ U_M \} \) and the proof of the asymptotic convergence in this nonlinear, several dimensional setting needs a new method of asymptotic analysis. The results are then extended to equations of the form

\[ u_t = \Delta \Phi(u) - \nabla \cdot F(u), \]

where \( \Phi \) and \( F \) resemble the preceding power functions as \( u \rightarrow 0 \). In this more general case the asymptotic behaviour is described by the same family \( U_M \) mentioned above. © Elsevier, Paris

1. Introduction

This paper is devoted to study the qualitative properties and asymptotic behaviour of the nonnegative solutions to the Cauchy problem for the following diffusion-convection equation:

\[ u_t = \Delta (u^m) - \frac{\partial}{\partial x_1} (u^q), \]

(1.1)

posed in \( Q = \{(x, t) : x \in \mathbb{R}^N, \ t > 0\} \), for all \( N \geq 1 \), with exponents \( m, q \) such that:

\[ m > 1, \quad q = m + 1/N. \]

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J.M. Burgers [6,7] studied this equation in the simple case of linear diffusion $m = 1$ and one space dimension $N = 1$, so that $q = 2$ and (1.1) becomes

$$u_t + 2uu_x = u_{xx},$$

as being the simplest one to combine typical convective nonlinearity with typical heat diffusion. Eq. (1.3) is therefore referred to as Burgers equation. It was probably first introduced by Bateman (1915). Burgers equation is relevant in many senses. It appears in Hydrodynamics when we model the propagation of perturbations in a weakly dissipative and weakly nonlinear fluid. In this case $u$ has the meaning of rescaled pressure (see [23]). It is also a simple model for the statistical theory of turbulence [6,7].

The integration of Burgers equation (1.3) is simplified by the Cole–Hopf transformation which allows to reduce it to the classical heat equation and obtain in such a way the general solution. For the details of such reduction see [33], where much more information about Burgers equation can be found. In this way it is easy to prove that there exists a one-parameter family of self-similar solutions of (1.3) of the form

$$U(x, t; M) = t^{-1/2} F_M(x/t^{1/2}),$$

where $M > 0$ is a constant to be determined through the mass condition

$$\int U(x, t; M) \, dx = M.$$

Such solutions are called source-type solutions because they take a Dirac mass, $M \delta(x)$, as initial data. They are important for the general theory because each of them gives the asymptotic behaviour as $t \to \infty$ of the whole class of solutions of the Cauchy problem with initial data:

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R},$$

where $u_0$ is nonnegative, integrable and $\int u_0(x) \, dx = M$. In Barenblatt’s terminology [4] the solutions (1.4) represent the Intermediate Asymptotics of the whole class of solutions.

The same result is still true in several space dimensions, $N > 1$, putting $q = (N + 1)/N$ and keeping linear diffusion, $m = 1$, but now there is no analogous transformation into the heat equation and the proof has to use the machinery of infinite-dimensional dynamical systems. Aguirre, Escobedo and Zuazua proved in [1] that a suitable self-similar solution exists and then Escobedo and Zuazua [12] proved that this special solution gives the asymptotic behaviour for the solutions with a whole class of initial data. In fact, in the work [12] the asymptotic behaviour was settled for $q \geq (N + 1)/N$. It was proved that when $q > (N + 1)/N$ the diffusive part dominates for large times, and the asymptotic behaviour is given by the fundamental solution of the heat equation (such phenomenon is called asymptotic simplification), while for $q = (N + 1)/N$ no simplification occurs, and the large-time behaviour is given by a family of self-similar solutions of the complete equation (1.1). The range $1 < q < 1 + 1/N$ has been studied in [13] and [14]. In this case the convective part dominates and the asymptotics is given by a self-similar solution of the equation with so-called partial diffusivity (diffusion in the direction of convection disappears).

We address here the situation for nonlinear diffusion, $m > 1$, and concentrate on Burgers case (1.2), where no asymptotic simplification occurs. It corresponds to $q = (N + 1)/N$ when $m = 1$, but the method used in [12] does not extend to the completely nonlinear case we consider here because it uses heavily the functional properties of the Laplace operator as a linear operator. Hence the interest in developing genuinely nonlinear methods, adapted to treat all these problems.
Through the rest of the paper we consider Eq. (1.1) with exponents $q$ and $m$, satisfying (1.2).

Theorem 1.1. – For every $M > 0$ there exists a unique self-similar solution of Eq. (1.1) of the form

$$U(x, t) = t^{-\alpha} F(x t^{-\beta}),$$

of the source-type class, i.e., as $t \to 0$

$$U(x, t; M) \to M \delta(x) \quad \text{in} \; \mathcal{D}'(\mathbb{R}^N),$$

where $F$ is a compactly supported function and the similarity exponents are

$$\alpha = \frac{N}{N(m - 1) + 2}, \quad \beta = \frac{1}{N(m - 1) + 2}.$$

The second result concerns the behaviour of general solutions of the Cauchy problem for Eq. (1.1) for large $t$.

Theorem 1.2 (Asymptotic behaviour). – Let $u(x, t)$ be the weak solution of the Cauchy problem for Eq. (1.1) with initial data $u_0 \in L^1$, $u_0 \geq 0$. Let $\int u_0(x) \, dx = M$. Then, for every $p \in [1, \infty]$

$$t^{\alpha(p)} \| u(., t) - U(., t) \|_{L^p} \to 0 \quad \text{as} \; t \to \infty,$$

where $U(x, t)$ is the unique self-similar solution with mass $M$ constructed in Theorem 1.1 and $\alpha(p) = \frac{p}{p-1} \alpha$ with $\alpha = \frac{N}{N(m - 1) + 2}$.

The construction of the family $\{U_M\}$ and the proof of (1.10) are based on the use of the Lyapunov method of asymptotic analysis. We start as in the works [30], [24] and [11], where the four-step technique introduced by Kamin and Vazquez [21] is followed. The main difficulty of the present analysis is the identification of the limit since a direct identification needs a strong uniqueness result for the source-type solution, a task that we will avoid. In this paper, identification is performed in two steps: (i) we prove that all possible limits must be self-similar and (ii) all possible limits are equal. For both (i) and (ii) we use a suitably constructed Lyapunov functional (Section 5). This method can have application to other evolution problems, cf. [32].

Let us mention that in one space dimension the above results have been proved as part of the works [24] and [30], where the asymptotic behaviour of solutions to the Cauchy problem for equation

$$u_t = (u^m)_{xx} + (u^q)_x$$

was settled in the ranges of exponents $m, q > 1$.

Since we are in the presence of slow diffusion, $m > 1$, our problem has the property of finite propagation, i.e., solutions whose initial data have compact support remain compactly supported for any given time $t > 0$ (though the support eventually covers the whole space). This applies in particular to the source-type solutions. We can also give an asymptotic estimate of the way in which the supports of $u$ and $U_M$ resemble for large $t$. Let $\Omega(t)$ be the support of $U_M$ at time $t$ and let $\Omega_0(t)$ be the support of $u$. We recall the definition of the Hausdorff distance. Let $\Omega_1$ and $\Omega_2$ be two closed subsets of a metric space $E$. Let $D_{x,y}$ be the set of all numbers $d(x, \Omega_2)$ with $x \in \Omega_1$ and $d(y, \Omega_1)$ with $y \in \Omega_2$, $d$ being the distance in $E$. Then the Hausdorff distance between $\Omega_1$ and $\Omega_2$ is defined as follows:

$$\text{dist}(\Omega_1, \Omega_2) = \sup_{x \in \Omega_1, y \in \Omega_2} D_{x,y}.$$
THEOREM 1.3. — Assume that \( u \) is a solution as in Theorem 1.2 and \( u_0 \) is compactly supported. Then the normalized Hausdorff distance between \( \Omega_u(t) \) and \( \Omega_u(t) \) goes to zero, that is

\[
\lim_{t \to \infty} \frac{\text{dist}(\Omega_U(t), \Omega_u(t))}{t^\beta} = 0.
\]

The plan of the paper is as follows. In Sections 2 and 3 we study the question of existence and uniqueness of a class of weak solutions of the Cauchy problem for Eq. (1.1). In Section 4 we prove some fundamental estimates and properties of weak solutions which will be useful in the asymptotic analysis. In Section 5, we develop the four-step technique for the proof of Theorem 1.2. Section 6 is devoted to the proof of Theorem 1.1, which is an almost direct consequence of Theorem 1.2. Section 7 is devoted to prove Theorem 1.3, while in Section 8, we study the shape of the family of attractors, considering the limit cases \( M \to 0 \) and \( M \to \infty \). As a consequence, we obtain the existence of a source-type solution for the equation with partial diffusivity

\[
(1.14) \quad u_t = \Delta y \left( u^m \right) - \left( u^q \right)_{x_1}
\]

where \( y = (x_2, x_3, \ldots, x_n) \). Finally, in Section 9, we extend our results to more general nonlinearities both in the diffusive and in the convective terms of (1.1).

Notes. — Eq. (1.1) is studied in the recent work [11] in several space dimensions in the range of exponents \( m > 1, 1 < q < m - 1 \). The asymptotic behaviour in this case is of a different type, purely convective.

The methods of this paper apply with very minor modifications to the linear-diffusion case \( m = 1 \), where the Lyapunov method has been applied by Zuazua [35]. The main difference is that solutions are always positive everywhere and the analysis of the support is replaced by suitable decay as \( x \to \infty \).

2. Existence of solutions

In this section we give a definition of weak solution for the Cauchy problem for Eq. (1.1) with initial data \( u_0 \in L^1, u_0 \geq 0 \), and proceed with its construction. The question of uniqueness is discussed later.

DEFINITION 2.1. — Let \( u_0 \in L^1(\mathbb{R}^N), u_0 \geq 0 \). We say that \( u(x, t) \) is a weak solution of Eq. (1.1) with initial data \( u_0 \) if:

(W1) \( u \) is continuous and nonnegative in \( \mathbb{R}^N \times (0, \infty) \),
(W2) \( u \in C([0, \infty); L^1(\mathbb{R}^N)) \cap L^\infty((\tau, \infty) \times \mathbb{R}^N), \forall \tau > 0 \),
(W3) \( u^m \in L^\infty_{\text{loc}}((0, \infty); H^1(\mathbb{R}^N)) \),
(W4) for all \( 0 < t_1 \leq t_2 < \infty \), all bounded open sets \( \Omega \subset \mathbb{R}^N \) with smooth boundary and all \( \phi \in C^\infty(\overline{\Omega} \times [t_1, t_2]), \phi = 0 \) on \( \partial \Omega \times [t_1, t_2] \), we have

\[
\int_{t_1}^{t_2} \int_{\Omega} \left[ \nabla u^m \cdot \nabla \phi - u \phi_t - u^q \phi_{x_1} \right] \, dx \, dt = \int_{\Omega} u(t_1, x) \phi(t_1, x) \, dx - \int_{\Omega} u(t_2, x) \phi(t_2, x) \, dx,
\]

(W5) \( u(0) = u_0 \), i.e., \( u(t) \to u_0 \) in \( L^1(\mathbb{R}^N) \) as \( t \to 0 \).
Note. We will frequently use the notation $u(t)$ to mean $u(\cdot, t)$, a function of $x$ for given $t$. $C_0^\infty(\Omega)$ denotes the space of smooth and compactly supported functions in $\Omega$ and $B_r(x_0)$ stands for the (open) ball with center in $x_0$ and radius $r$. The support (in $x$) of a function $u(x, t)$ for a fixed $t$ will be denoted by $\Omega_{u}(t)$. The rest of the notations is rather standard: By $L^p(\Omega)$ we denote the usual Lebesgue spaces and by $H^1(\Omega)$ the Sobolev space $W^{1,2}(\Omega)$.

Next we prove the following:

**Theorem 2.1.** For every $u_0 \in L^1$, $u_0 \geq 0$, there exists a weak solution of Eq. (1.1) with initial data $u_0$ in the sense of previous definition.

Hui [18] proposes a weaker form of solution and proves existence and uniqueness for bounded integrable data. The fact that our solutions have better regularity (they are energy solutions in the usual terminology) will be used in the sequel. The restriction of boundedness on the data is not natural in our context. References to other existence results will be commented upon in Section 9 and in the Appendix.

**Proof.** We adopt a constructive method starting with regular problems to which standard theory [26] can be applied, and obtaining estimates that allow to pass to the limit. Such a construction is carried out for the Porous Medium Equation in the survey paper [31]. Since our construction is very similar, we omit the details and emphasize the differences. A slightly different approach is adopted in [25].

(1) As a first step, we construct a weak solution for the homogeneous Dirichlet problem. Let $\Omega$ be an open bounded set with smooth boundary, and let $u_0 \in C_0^\infty(\overline{\Omega})$, $u_0 \geq 0$ and $u_0$ vanishes together with the first two derivatives on the boundary. Denote $Q_T = \Omega \times (0, T)$ and $Q = Q]\infty.

Consider the approximate problems

\[
\begin{aligned}
&u_{n0} = \Delta(u_n^m) = (u_n^m)_x \quad \text{in } Q_T, \\
&u_n(x, 0) = u_{n0}(x) = u_0(x) + 1/n, \\
&u_n = 1/n \quad \text{on } \partial \Omega \times [0, \infty).
\end{aligned}
\]

These problems are essentially non-degenerate, and each one has solution $u_n \in C^{2,1}_{\max}(\overline{\Omega})$. By the Maximum Principle, $u_{n+1} \leq u_n$ and $1/n \leq u_n \leq M + 1/n$, where $M = \max_{\Omega} u_0$. We obtain a point-wise (and $L^p$ with $1 \leq p < \infty$) limit $u$, such that $0 \leq u \leq M$. Multiplying the equation by $u_n^m - (1/n)^m$ and integrating in $Q_T$ we obtain, as in [31, p. 359] a uniform (in $T$ and $n$) control on $\nabla u_n^m$ (convection does not play any role). Hence, for some subsequence (not relabeled), $\nabla u_n^m \to \nabla u^m$ weakly in $L^2(Q)$, and the so-called Energy Estimate:

\[
(m + 1) \int_0^T \int_\Omega |\nabla u_n^m|^2 \ dx \ dt + \int_\Omega u_n^{m+1}(x, T) \ dx \leq \int_\Omega u_0^{m+1}(x) \ dx,
\]

holds in the limit. The convergences above allow to pass to the limit in (W4) of Definition 2.1 (which is obviously satisfied by the classical solutions $u_n$). Precisely, for each $\phi \in C^\infty(\overline{\Omega} \times [\tau_1, \tau_2])$ such that $\phi = 0$ on $\partial \Omega \times [\tau_1, \tau_2]$ with $0 \leq \tau_1 \leq \tau_2 < \infty$ we have

\[
(2.2) \int_{\tau_1}^{\tau_2} \int_\Omega \left\{ \nabla u_n^m \cdot \nabla \phi - u_n \phi_t - u^d \phi_x \right\} \ dx \ dt = \int_\Omega u(x, \tau_1) \phi(x, \tau_1) \ dx - \int_\Omega u(x, \tau_2) \phi(x, \tau_2) \ dx.
\]

It follows from the construction that $u^m \in L^2(0, T : H^1(\Omega))$. It is clear from (2.1) that the $L^{m+1}$ norm of $u$ decays in time. Multiplying by other powers of $u$ we obtain a “generalized” Energy...
Estimate (see [31, p. 366]), from which follows the $L^p$-norm decay

\[(2.3) \quad \|u(\cdot, t)\|_p \leq \|u_0\|_p,\]

for all $p \geq 1$. Moreover, if $u_0 > 0$ in $\Omega$, then the solution is positive and classical, that is, $u \in C^\infty(\bar{Q}) \cap C(\bar{Q})$. Positivity can be settled by means of the following barrier (see [16]):

\[(2.4) \quad w(x, t) = \begin{cases} (\rho^2 - |x|^2)^{1/(m-1)}e^{-\sigma t} & \text{if } |x| < \rho, \\ 0 & \text{elsewhere}, \end{cases}\]

where $\sigma$ is a positive parameter suitably chosen. Precisely, we can choose $\sigma$ large enough in such a way that $w_{x_0} = w(x - x_0, t)$ is a classical sub-solution of (1.1) in $[|x - x_0| < \rho] \times (0, \infty)$. This is true in the range $q \geq (m + 1)/2$ (see [16] for details), hence for $q = m + 1/N$. Moreover, $w_{x_0}$ vanishes on the lateral boundary of this region. For each $x_0 \in \Omega$ we can choose $\rho$ small enough in order to have $u_0 \geq w_{x_0}(0)$ in $[|x - x_0| < \rho] \subset \Omega$. Hence, the classical Maximum Principle applied to $u_n$ yields $u_n \geq w_{x_0}$ in $[|x - x_0| < \rho] \times (0, \infty)$, and in the limit $u \geq w_{x_0}$ in this region. Then, $u_0$ is bounded away from 0 in $N = B_{x_0}(r) \times (0, T)$ for some $r > 0$. By the regularity theory of quasi-linear non-degenerate parabolic equations, we conclude that $u \in C^\infty(N)$ and the initial data are taken continuously in $B_{x_0}(r)$. From our approximation process it is also clear that $u$ vanishes continuously on $\Sigma = \partial \Omega \times (0, T)$.

Since the constructed solution is a solution with “finite energy”, by the results of F. Otto [27] the Contraction Principle in $L^1$ is valid in the following form:

\[(2.5) \quad \int_{\Omega} |u_1 - u_2|(x, t) \, dx \leq \int_{\Omega} |u_{10} - u_{20}|(x) \, dx,\]

where $u_1$ and $u_2$ denote the solutions with initial data $u_{10}$ and $u_{20}$ respectively.

(2) Now we consider the initial value problem. As a first approximation, take $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N)$ and $u_0 > 0$ everywhere. Let $\zeta_n(x)$ be a sequence of “cutoff functions”, such that $\zeta_n \in C^\infty(\mathbb{R}^N)$, $0 \leq \zeta_n \leq 1$, $\zeta_n = 1$ in $B_{n-1}(0)$, $\zeta_n = 0$ in the complement of $B_n(0)$ and its derivatives up to second order are bounded in $\mathbb{R}^N$. Consider the sequence of problems:

\[
\begin{align*}
\left\{ \begin{array}{l}
  u_n = \Delta(u_m^n) - (u_k^n)_{x_1} & \text{in } B_n(0) \times (0, T), \\
  u_n(x, 0) = u_0 \zeta_n, \\
  u_n = 0 & \text{on } \partial B_n(0) \times [0, T).
\end{array} \right.
\]

These problems can be solved as in the previous step, because $u_{n0} \in C^\infty(\bar{Q})$. Since the initial data are strictly positive, the solution is classical in $B_n(0) \times (0, T)$. Then, the Maximum Principle can be applied and we obtain an ordered sequence $u_n \leq u_{n+1}$ in $\mathbb{R}^N \times (0, T)$ (We extend by 0 each $u_n$ and keep the notation). The uniformity in $n$ of estimates (2.1) and (2.3) allow us to obtain a limit $u$ in $L^\infty((0, \infty); L^p(\mathbb{R}^N))$ for all $1 \leq p$. This limit satisfies (W3) and (W4) of Definition (2.1) (in fact, (W3) is satisfied without subscript ‘loc’). Since the sequence of approximations is locally bounded below away from 0 for large $n$, it also satisfies (W1) by regularity theory.

Estimates (2.1) and (2.3) remain valid. Contraction in $L^1$ holds in the limit, (W2) and (W5) in Definition 2.1 are easily obtained as a consequence of $L^1$-contraction and approximation.

(3) Take now $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $u_0 \geq 0$. We may approximate $u_0$ in $L^1$ by $u_{n0}$ as in Step 2. Moreover, we can do it in such a way that $\|u_{n0}\|_1 \leq \|u_0\|_1$ and $\|u_{n0}\|_\infty \leq \|u_0\|_\infty$. 

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The $L^1$-contraction allows to obtain a limit in $C([0, \infty); L^1(\mathbb{R}^N))$ such that $u(0) = u_0$. The uniformity of estimates (2.1) and (2.3) enables to pass to the limit in (W4) of Definition 2.1. (W1) is true thanks to the regularity theory of degenerate parabolic equations (see [9]).

(4) In order to extend our construction to data in $L^1(\mathbb{R}^N)$, we need an $L^\infty$-estimate for $t > 0$. We recall the so called $L^1-L^\infty$ regularizing effect for the Porous Medium equation:

\begin{equation}
\|u(t)\|_{L^\infty} \leq C \|u\|_{L^1(\mathbb{R}^N)}^\frac{1}{t-a},
\end{equation}

with $\alpha = N/(N(m-1)+2)$ and $\lambda = 2/(N(m-1)+2)$, which is valid for solutions with nonnegative integrable data. It turns out that, in our case, such an estimate is also valid, thanks to a symmetrization result due to Bénilan and Abourjaily [3], which enables to compare solutions of Eq. (1.1) and solutions (with the same initial data) of the Porous Medium equation (which is the symmetrized one). Although this result is proved for regular problems, it holds in our case by approximation. For the sake of completeness, we reproduce it here.

**Theorem 2.2** (Theorem 1 of [3, p. 4]). – Let $\Omega$ be any open set in $\mathbb{R}^N$, $Q = (0, T) \times \Omega$, let $a : Q \times \mathbb{R}^{N+1} \to \mathbb{R}^N$ be a Carathéodory function and satisfy

\[ a(t, x, k) \xi \geq \sigma(\|k\|\|\xi\|^2 - F(t, x, k) \xi), \quad \forall (t, x, k, \xi) \in Q \times \mathbb{R}^{N+1} \]

where $\sigma : \mathbb{R} \to (0, \infty)$ is Hölder continuous, $F : Q \times \mathbb{R} \to \mathbb{R}^N$ is Carathéodory and satisfies

\[
\sup_{|k| \leq R} |F(., k)| \in L^2(Q), \quad \forall R > 0 \quad \text{and} \quad \text{div}(kF(., k)) \leq 0 \quad \text{in } D'(\Omega) \quad \forall (t, k) \in (0, T) \times \mathbb{R}.
\]

Let $u \in C([0, T]; L^1(\Omega)) \cap L^2_{\text{loc}}(0, T; W^{1,2}_0(\Omega))$ with $a(., u, \nabla u) \in L^2_{\text{loc}}((0, T); L^2(\Omega))$ satisfy

\[ u_t = \text{div}(a(., u, \nabla u)) \quad \text{in } D'(Q). \]

On the other hand let $\tilde{\Omega}$ be a ball $B(0, R) \subset \mathbb{R}^N$ with $|\Omega| \leq |\tilde{\Omega}|$, $v_0 \in L^1(\tilde{\Omega}) \cap L^\infty(\tilde{\Omega})$ with $\tilde{v}_0 = v_0$ and $v$ be the solution of

\[
\begin{cases}
\tilde{v}_t = \Delta \phi(v) & \text{on } (0, T) \times \tilde{\Omega}, \\
v = 0 & \text{on } (0, T) \times \partial \tilde{\Omega}; \quad v(0, .) = v_0 & \text{on } \tilde{\Omega},
\end{cases}
\]

where $\phi(v) = \int_0^v \alpha(k) \, dk$. If $\int_{\tilde{\Omega}} (|u_0| - k)^+ \, dx \leq \int_{\tilde{\Omega}} (v_0 - k)^+ \, dx$ for any $k \geq 0$, then

\begin{equation}
\int_{\tilde{\Omega}} (|u(t)| - k)^+ \, dx \leq \int_{\tilde{\Omega}} (v(t) - k)^+ \, dx \quad \forall (t, k) \in (0, T) \times \mathbb{R}^+.
\end{equation}

Clearly, all the hypotheses of Theorem 2.2 hold in our case, and the symmetrized equation is the Porous Medium equation. Applying this result in $Q = \mathbb{R}^N \times (0, T)$, putting $k = C \|u\|_{L^1}^\frac{1}{t-a}$ and taking the same initial data for (1.1) and the Porous Medium equation, we obtain the validity of (2.6) for the solutions of (1.1).

We also need the following important result:
Theorem 2.3 (Conservation of mass). Let \( u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \). Then, for all \( t > 0 \) we have

\[
\int_{\mathbb{R}^N} u(x, t) \, dx = \int_{\mathbb{R}^N} u_0(x) \, dx.
\]

(That is, the \( L^1 \)-norm of the solution is an invariant of the evolution.)

Proof. Suppose \( u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N) \). As a test function in the definition of weak solution, take \( \phi(x, t) = \zeta_n(x) \), where \( \zeta_n(x) \) is as in Step 2 of our construction. We obtain

\[
\int_{\mathbb{R}^N} u(x, t) \zeta_n(x) \, dx = \int_{\mathbb{R}^N} u_0 \zeta_n(x) \, dx = \int_{\mathbb{R}^N} \left\{ u^{m}(x, t) \Delta \zeta_n(x) - u(x, t) \zeta_{n1} \right\} \, dx \, dt.
\]

Notice that \( \Delta \zeta_n \) and \( \zeta_{n, x_1} \) converge point-wise to 0 in \( \mathbb{R}^N \), and \( \zeta_n \) converges to 1. By the dominated convergence theorem, we can pass to the limit in all the integrals. We obtain

\[
\int_{\mathbb{R}^N} u(x, \tau) \, dx = \int_{\mathbb{R}^N} u_0 \, dx = 0,
\]

which is the desired result. For general data \( u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \), we proceed by approximation.

Take now \( u_0 \in L^1(\mathbb{R}^N) \), \( u_0 \geq 0 \). Consider the sequence of approximations \( u_{n0} \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \), such that \( u_{n0} \to u_0 \) in \( L^1 \) and \( \| u_{n0} \|_{L^1} \leq \| u_0 \|_{L^1} \). The \( L^1 \)-contraction produces a limit in \( C([0, \infty); L^1(\mathbb{R}^N)) \). By (2.6), \( \{ u_n(t) \} \) is uniformly bounded in \( \mathbb{R}^N \times [\tau_1, \tau_2] \), with \( 0 < \tau_1 \leq \tau_2 \). Thanks to this fact and the conservation of mass, the Energy Estimate (2.1) gives uniform estimates for \( \nabla u_n \) in the same regions. We can pass to the limit in the weak formulation if \( \tau_1 > 0 \). All the conditions (W1)–(W5) in Definition 2.1 are seen to hold. The \( L^1 \)-contraction and the conservation of mass remain valid by approximation.

This ends the construction of the solution. Theorem 2.1 is proved.

3. Uniqueness

The uniqueness of solutions of diffusive-convective equations of the form

\[
(3.1) \quad u_t = \Delta \Phi(u) - \text{div}(F(u))
\]

has been discussed by a number of authors, though the results do not exactly match the class of solutions we are dealing with. Here is the result that we need at this stage:

Theorem 3.1. There exists at most one solution to (1.1) with initial data \( u_0 \in L^1(\mathbb{R}^N) \), \( u_0 \geq 0 \) in the sense of Definition 2.1. Moreover, for two solutions \( u_1, u_2 \) with data \( u_{01} \) and \( u_{02} \) respectively, we have:

\[
(3.2) \quad \int_{\mathbb{R}^N} |u_1 - u_2|(x, t) \, dx \leq \int_{\mathbb{R}^N} |u_{10} - u_{20}|(x) \, dx.
\]
We will devote a final Appendix to explain the proof of this result in the context of a more general class of equations of the form (3.1) and we will briefly discuss the corresponding literature.

4. Properties of weak solutions

Before proceeding with the asymptotic study we need to establish a number of properties of the weak solution and its support. First of all, we state and prove an estimate for the time derivative of the solution.

**Theorem 4.1.** – For every weak solution \( u \) the following estimate holds:

\[
\left\| \frac{\partial u}{\partial t} \right\|_{L^2(\mathbb{R}^n \times (t, +\infty))} \leq C_4 \left( \| u \|_1, m, N \right) t^{-\frac{m(m+1)}{2}}, \quad t > 0.
\]

**Proof.** – Let \( u \) denote the solution of the approximate problem for (1.1) (see Step 2 in Section 2) and \( Q_T = \Omega \times (0, T) \) the corresponding bounded space-time domain. Multiplying Eq. (1.1) by \( \frac{\partial u^m}{\partial t} \) and integrating in \( \Omega \) by parts we obtain

\[
\int_{\Omega} \left( \frac{\partial u^{m+1/2}}{\partial t} \right)^2 dx + \frac{(m + 1)^2}{8m} \int_{\Omega} \left| \nabla u^m(x, t) \right|^2 dx = \frac{(m + 1)^2}{4m} \int_{\Omega} \left| (u^m)'(u^q)'_{x_1} \right| dx.
\]

Applying Young’s inequality in the last integral we get:

\[
\frac{(m + 1)^2}{4m} \int_{\Omega} \left| (u^m)'(u^q)'_{x_1} \right| dx = C(m, N) \int_{\Omega} u^{q+m-2} |u_{x_1}||u_t| dx
\]

\[
\leq \frac{1}{2} \int_{\Omega} \left( \frac{\partial u^{m+1/2}}{\partial t} \right)^2 dx + C_1(m, N) \int_{\Omega} u^{m-1} |(u^q)'_{x_1}|^2 dx.
\]

Hence

\[
\int_{\Omega} \left( \frac{\partial u^{m+1/2}}{\partial t} \right)^2 dx + \frac{(m + 1)^2}{8m} \int_{\Omega} \left| \nabla u^m(x, t) \right|^2 dx \leq C_1(m, N) \int_{\Omega} u^{m-1} |(u^q)'_{x_1}|^2 dx.
\]

The last term can be estimated as follows:

\[
\int_{\Omega} u^{m-1} |(u^q)'_{x_1}|^2 dx \leq C_2(m, N) \| u(t) \|_{\infty}^{2q-(m+1)} \int_{\Omega} |\nabla (u^m)|^2 dx.
\]

We get

\[
\frac{1}{2} \int_{\Omega} \left( \frac{\partial u^{m+1/2}}{\partial t} \right)^2 dx + \frac{(m + 1)^2}{8m} \int_{\Omega} \left| \nabla u^m(x, t) \right|^2 dx \leq C_3(m, N) \| u(t) \|_{\infty}^{2q-(m+1)} \int_{\Omega} |\nabla (u^m)|^2 dx.
\]

\[
\left\| \frac{\partial u}{\partial t} \right\|_{L^2(\mathbb{R}^n \times (t, +\infty))} \leq C_4 \left( \| u \|_1, m, N \right) t^{-\frac{m(m+1)}{2}}, \quad t > 0.
\]

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In particular, we have

$$
\frac{d}{dt} \int_{\Omega} |\nabla u^m|^2 \, dx \leq C_4(m, N) \|u(t)\|_{L_\infty}^{2q-(m+1)} \int_{\Omega} |\nabla (u^m)|^2 \, dx.
$$

Put, for simplicity

$$
\gamma(t) = \int_{\Omega} |\nabla u^m(x, t)|^2 \, dx, \quad t \geq 0.
$$

Integrating (4.3) in $t$ from $t_1$ to $t_2$ with $0 < t_1 < t_2$, and taking into account (2.6) and (2.1) (observe that, since our solution is obtained as a monotone limit, (2.6) is valid for the approximations), we obtain

$$
\gamma(t_2) \leq \gamma(t_1) + C_5(m, N, \|u\|_1) t_1^{-\alpha(2q-1)}.
$$

Now we integrate in $t_1$ from $t_2/2$ to $t_2$ and use (2.1) once again. This gives:

$$
\gamma(t_2) \leq C_6(m, N, \|u\|_1) t_2^{-\alpha m-1}.
$$

Next we integrate (4.2) in $t$ from $\tau > 0$ to $T$. We obtain

$$
\frac{1}{2} \int_{\Omega_\tau^T} \left| \left( u^{(m+1)/2} \right)_t \right|^2 \, dx \, dt + \frac{(m+1)^2}{8m} \int_{\Omega} \left| \nabla u^m(x, \tau) \right|^2 \, dx
$$

$$
\leq \frac{(m+1)^2}{8m} \int_{\Omega} \left| \nabla u^m(x, \tau) \right|^2 \, dx + C_3(m, N, \|u\|_1) t^{-\alpha(2q-(m+1))} \int_{\Omega_\tau^T} \left| \nabla (u^m) \right|^2 \, dx.
$$

From the last inequality, (2.1) and (4.5) we infer

$$
\int_{\Omega_\tau^T} \left| \left( u^{(m+1)/2} \right)_t \right|^2 \, dx \, dt \leq C(m, N, \|u\|_1) t^{-\alpha m-1}.
$$

The identity

$$
\left( u^m \right)_t = \left( 2m/(m+1) \right) u^{(m-1)/2} \left( u^{(m+1)/2} \right)_t,
$$

combined with the last inequality and (2.6) gives the desired result for the approximate problems. Since the solution of the Cauchy problem can be obtained as a limit of such approximate solutions and (4.1) holds uniformly for them, we get (4.1) passing to the $L^2$-weak limit. The derivative should be understood in the sense of distributions. This estimate, together with (2.1), implies $u^m \in H^1(\mathbb{R}^N \times (t, +\infty))$ for all $t > 0$.

The following properties of the support can be found in [16] in one space dimension.

**Theorem 4.2 (Retention property).** – Let $u$ be a weak solution of problem (1.1) with $u_0 \in C^0_0(\mathbb{R}^N)$. Suppose that $u(x_0, t_0) > 0$ at some point $(x_0, t_0)$, $t_0 \geq 0$. Then $u(x_0, t) > 0$ for all $t \geq t_0$. □

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We omit the proof, since it follows the one given by Gilding [16, p. 211], valid in the range $q > (m + 1)/2$. As a consequence, the positivity set is expanding in time, i.e., for $t_2 > t_1 > 0$:

$$u(x, t_1) > 0 \subset u(x, t_2) > 0.$$  

\[ (4.8) \]

**Corollary 4.1.** If the support of the solution is connected at $t_1 \geq 0$, then it stays connected for all $t > t_1$.

**Proof.** Let $t_2 > t_1$ be such that the positivity set $\{u(x, t_2) > 0\}$ has a connected component $\Omega$, disjoint from the one that contains the set $\{u(x, t_1) > 0\}$. Then the Maximum Principle, applied to the cylinder $\mathbb{R}^n \times [t_1, t_2]$ gives a contradiction. ($u = 0$ in the lateral boundary of this cylinder by the retention property).

**Theorem 4.3 (Penetration property).** Let $u$ be a weak solution of problem (1.2) with initial data in $C^1_0(\mathbb{R}^n)$. Then $\text{supp } u(t)$ eventually reaches any point $x \in \mathbb{R}^n$.

The proof is based on comparison with the following explicit sub-solution with expanding support:

$$z(x, t) = \sigma(t + \tau)^{-\gamma} \left[ \rho - x^2(t + \tau)^{-\beta} \right]^{1/\sigma},$$

where the positive parameters $\gamma, \beta, \rho, \sigma$ and $\tau$ are chosen appropriately. See Kalashnikov [19] for the one-dimensional case.

We perform next a more delicate task, estimating the growth of the support.

**Theorem 4.4.** Let $u$ be a solution of Eq. (1.1) with initial data in $C^\infty_0(\mathbb{R}^n)$. Then, its support at time $t > 0$ is contained in the neighborhood of radius $Ct$ of the initial support, where $C$ is some constant, depending only on $m$, $N$ and $\|u\|_{L^1}$, and $\beta = 1/(N(m - 1) + 2)$, as above.

**Proof.** This is proved by combining three arguments:

(a) a result of Díaz and Véron [10], based on the local energy method introduced by Antontsev (see [2]) in order to estimate the growth of the support in intervals of the form $[t_1, t_2]$, $0 < t_1 < t_2$.

This estimate depends only on $\|u\|$ and structural constants,

(b) the similarity properties of the equation, which allow to estimate the growth of the support in any interval of the form $[\tau, t]$, with $\tau > 0$, independently of $\tau$,

(c) a comparison argument, in order to control the initial (in $[0, \tau]$ with arbitrarily small $\tau$) growth of the support.

Let us see these results in detail.

**Lemma 4.1.** There exists $\varepsilon > 0$ such that the growth of the support in the interval $[1, 1 + \varepsilon]$ is bounded by 1. The value of $\varepsilon$ depends only on $m$, $N$ and $\|u\|_{L^1}$.

**Proof.** We use theorem 2, p. 151 of the work [10]. The hypotheses of that result are satisfied in our case with $q = 1$, $M_1 = 1$, $M_2 = 1$, $\beta = 1$, $\alpha = (1 - m)/2m < 0$. We also need to slightly displace the origin of time, e.g., taking $\hat{u}(t) = u(t + \tau)$, $\tau > 0$. The boundedness of $u$, (2.6), guarantees condition (21) of p. 151. The theorem asserts that there exists a function $F(t, \rho) = C(\rho) t^\lambda$ with $C, \lambda \geq 0$ and a time $T^*$ (depending only on structural constants) such that whenever $u_0$ vanishes in a ball $B_{\rho_0}(x_0)$ then $u(t)$ vanishes a.e. in a smaller ball $B_{\rho_1}(x_0)$, $\rho_1^\gamma = \rho_0^\gamma - F(t, \rho_0)$, for all $t \in (0, T^*)$ such that

$$\rho_0^\gamma > F(t, \rho_0).$$
More precisely, $F(t, \rho_0)$ is defined by:

\begin{equation}
(4.10) \quad F(t, \rho_0) = C^* t^\lambda \min_{\frac{m}{2m} \leq \tau \leq 1} \left\{ \frac{E'(t, \rho_0)}{2m \tau - m - 1} \max\{1, \rho_0^{\nu-1}\} \max\{b^\mu(t, \rho_0), b^\eta(t, \rho_0)\} \right\}
\end{equation}

where $C^* > 0$ depends only on the structure constants (here $m$, $N$ and $\|u\|_1$) and

\begin{align*}
\sigma &= \frac{1}{m}, \\
\lambda &= \frac{m + 1}{N(m - 1) + 2(m + 1)}, \\
\mu &= \frac{2(1 - \tau)}{N(1 - \sigma) + 2(\sigma + 1)}, \\
\nu &= \frac{2(\sigma + 1) + N(1 - \sigma)}{\sigma + 1}, \\
\eta &= \frac{1 - \sigma}{(\sigma + 1) + \gamma},
\end{align*}

while

\begin{equation}
(4.11) \quad E(t, \rho) = \int_0^t \int_{B_\rho(x_0)} |\nabla u^m|^2 \, dx \, ds,
\end{equation}

\begin{equation}
(4.12) \quad b(t, \rho) = \sup_{0 < \tau < t} \int_{B_\rho(x_0)} |u(\tau, x)|^{m+1} \, dx.
\end{equation}

If $C^* = 0$, $\rho_1 = \rho_0$ and the result follows. Suppose $C^* > 0$. We apply this result above to the solution $v(x, t) = u(x, t + 1)$. Take $\rho_0 = 1$ for simplicity. Using estimates (2.1) in $\mathbb{R}^N \times [1, t]$, (2.6) and (2.8) we have:

\begin{equation}
(4.13) \quad E(t, 1) = \int_1^{t+1} \int_{B_1(x_0)} |\nabla u^m|^2 \, dx \, ds \leq C_1(\|u\|_1, m) \quad \forall t > 0,
\end{equation}

\begin{equation}
(4.14) \quad b(t, 1) = \sup_{1 < \tau < t+1} \int_{B_1(x_0)} |u(\tau, x)|^{m+1} \, dx \leq C_2(\|u\|_1, m) \quad \forall t > 0.
\end{equation}

Therefore, putting $\tau = 1$:

\begin{equation}
(4.15) \quad F(\varepsilon, 1) \leq C_3 \varepsilon^{\lambda} \left\{ \frac{E'(1)(\varepsilon, 1)}{m - 1} \max\{1, b^\eta(\varepsilon, 1)\} \right\} \leq C_3 \varepsilon^{\lambda} \quad \text{for all } \varepsilon > 0,
\end{equation}

where $C = C(m, \|u_0\|_1)$.

Take now $\varepsilon < T^*$ such that $1 > C_3 \varepsilon^{\lambda}$ (this implies $1 > F(\varepsilon, 1)$). We deduce from the quoted result that if $u(1) = 0$ in $B_1(x_0)$, then

\begin{equation}
(4.16) \quad u(t) = 0 \quad \text{in } (1, 1 + \varepsilon) \times B_{\rho_1}(x_0),
\end{equation}

where $\rho_1^\nu = 1 - F(\varepsilon, 1) > 1 - C_3 \varepsilon^{\lambda} > 0$. If $x_0$ is not contained in the 1-neighborhood of $\Omega_u(1)$ (recall that $\Omega_u(t)$ denotes the support of $u$ at the time $t$), then $u = 0$ in $B_1(x_0)$. Consequently, by (4.16), $u(1 + \varepsilon/2) = 0$ in $B_{\rho_1}(x_0)$. In particular, $u(1 + \varepsilon/2) = 0$ in $x_0$. Therefore, the support of the solution at time $1 + \varepsilon/2$ is contained in a 1-neighborhood of the support at time $t = 1$. Lemma 4.1 is proved.  \[ \Box \]
LEMMA 4.2. – The support of any solution at time \( t_0 > 0 \) is contained in a neighborhood of radius \( C \| x \|^{1+\epsilon} \) of the support at any time \( \tau \), with \( 0 < \tau < t_0 \).

Proof. – The idea is to use the similarity properties of the equation to transform the interval \([1, 1+\epsilon]\) into the intervals \([(1+\epsilon)^{-1}, 1], [(1+\epsilon)^{-2}, (1+\epsilon)^{-1}]\), and so on, converging to \( t = 0 \). Consider the rescaled solution
\[
\lambda \eta (x, t) = \lambda \eta (\lambda \beta x, \lambda t).
\]
\( \lambda \eta \) has the same mass as \( \eta \). Therefore, by Lemma 4.1, the growth of its support in the interval \([1, 1+\epsilon]\) is less than 1. Take \( \lambda = (1+\epsilon)^{-1} \). Then the growth of the support of \( \eta (\lambda \beta x, t) \) in \([(1+\epsilon)^{-1}, 1] \) is less than 1, and this in turn implies that the growth of the support of \( \eta \) in \([1, 1+\epsilon]^{-1}, 1\) is less than \( (1+\epsilon)^{-\beta} \). Repeating the same argument, we obtain that the growth of the support of \( \eta \) in the interval \([(1+\epsilon)^{-n}, (1+\epsilon)^{-n+1}] \) is less than \( (1+\epsilon)^{-n\beta} \). Taking into account that \( (1+\epsilon)^{-n} \to 0 \) and that
\[
\sum (1+\epsilon)^{-n\beta} < \infty,
\]
it turns out that the growth of the support in any interval of the form \((\tau, 1)\) is bounded by some constant.

The next step is to transform the interval \((0, 1)\) into \((0, t_0)\). Consider \( \lambda \eta \) with \( \lambda = t_0 \). Arguing as before, in any interval \((\tau, t_0)\) the growth of the support of \( \eta (\lambda \beta x, t) \) is less than \( C \), which in turn implies that the growth of the support of \( \eta \) in the same interval is less than \( C \| x \|^{1+\epsilon} \).

In the following lemma we study the initial growth of the support:

LEMMA 4.3. – Let \((x_0, t_0) \in \mathbb{R}^N \times [0, \infty) \) with \( u(x_0, t_0) = 0 \) and \( \delta > 0 \) such that:
\[
 u(x, t_0) \leq C \| x - x_0 \|^{1+\epsilon} \quad \text{in} \quad \{ \| x - x_0 \| \leq \delta \}.
\]
Then, there exists \( t_1 > t_0 \) such that:
\[
 u(x_0, t) = 0 \quad \text{in} \quad [t_0, t_1].
\]

For the one-dimensional case, the proof is in Gilding [16]. His proof applies without changes to our case.

Combining Lemmata 4.1, 4.2 and 4.3, we prove Theorem 4.4 as follows: Fix \( \delta > 0 \). By Lemma 4.3, there exists \( \tau > 0 \) such that
\[
\Omega_u (\tau) \subset \left( \Omega_u (0) \right)_{\delta^\epsilon}.
\]
(\( \delta \) denotes the \( \delta \)-neighborhood of the set \( S \)). Now, applying Lemma 4.2, for any \( t_0 > 0 \):
\[
\Omega_u (t) \subset \left( \Omega_u (}\right)_{\| x \|^{1+\epsilon}}.
\]
Combining both results, we obtain:
\[
\Omega_u (t_0) \subset \left( \Omega_u (0) \right)_{\| x \|^{1+\epsilon}}.
\]
Now let \( \delta \to 0 \) and we get the desired result, that is,
\[
\Omega_u (t_0) \subset \left( \Omega_u (0) \right)_{\| x \|^{1+\epsilon}}.
\]
5. Asymptotic behaviour

This section is devoted to the proof of our main result, Theorem 1.2.

Proof of Theorem 1.2. – As we said in the Introduction, we adopt the four-step method, introduced by Kamin and Vazquez [21].

Step 1. Rescaling. In view of the invariance properties of our equation, we consider the following family of rescaled solutions:

\[ u_\lambda(x, t) = \lambda^\alpha u(\lambda^{\beta} x, \lambda t) \]  

with \(\alpha = N/(m - 1) + 2\) and \(N\beta = \alpha\). As it is easily seen by a change of variables, proving our main result is equivalent to proving that

\[ u_\lambda(., t_0) \rightarrow U(., t_0) \quad \text{when} \quad \lambda \rightarrow \infty, \]  

with convergence in \(L^p(\mathbb{R}^N)\) at any fixed \(t_0 > 0\).

Step 2. Uniform estimates and compactness. In this section we gather some estimates which will hold uniformly for all the family \(\{u_\lambda\}\). These estimates allow us to prove its compactness in some functional space.

1. The \(L^1\)-estimate (Theorem 2.3)

\[ \|u(t)\|_{L^1(\mathbb{R}^N)} = C_1, \quad t \geq 0. \]  

(E1)

2. The \(L^\infty\)-estimate (2.6)

\[ \|u(t)\|_{L^\infty(\mathbb{R}^N)} \leq C_2(\|u\|_{1, m, N} t^{-\alpha}), \quad t > 0. \]  

(E2)

3. The Energy estimate, applied in \(\mathbb{R}^N \times (t, +\infty)\) with \(t > 0\), combined with estimates (E1) and (E2) gives

\[ \|\nabla u^m\|_{L^2(\mathbb{R}^N \times (t, +\infty))} \leq C_3(\|u\|_{1, m, N} t^{-\frac{m}{2}}), \quad t > 0. \]  

(E3)

4. The estimate of \((u^m)^l\) in \(L^2\), proved in Section 4:

\[ \|u^m\|_{L^2(\mathbb{R}^N \times (t, +\infty))} \leq C_4(\|u\|_{1, m, N} t^{\frac{2m - 1}{2}}), \quad t > 0. \]  

(E4)

It is an easy matter to check that the above estimates hold uniformly for the family \(\{u_\lambda\}\). (E1) and (E2) give uniform bounds in all \(L^p\), \(p \geq 1\). This fact, together with (E3) and (E4), imply boundedness of \((u^m)^l\) in \(H^1(\mathbb{R}^N \times (t, +\infty))\) for all \(t > 0\). It follows from Rellich–Kondrachov theorem that \((u^m)^l\) is precompact in \(L^2_{\text{loc}}(Q)\) and therefore \(\{u_\lambda\}\) is precompact in \(L^1_{\text{loc}}(Q)\). Denote by \(U\) the limit of some convergent subsequence \(\{u_{\lambda_n}\}\), \(\lambda_n \rightarrow \infty\).

Step 3. Passage to the limit. Now we proceed to the analysis of the limit orbits \(U(t)\). First of all, we observe that our estimates allow to pass to the limit in (W4) of Definition 2.1. In particular, \(U\) is a distributional solution of (1.1) in \(Q\). The regularity theory of [9], applied to a
convergent subsequence \( \{u_{\lambda_n}\} \), says that it is bounded in some Hölder space \( C^\varepsilon(\Omega \times [t_1, t_2]) \), \( 0 < \varepsilon < 1 \), for each bounded \( \Omega \) and \( 0 < t_1 < t_2 \). By Ascoli–Arzelà’s theorem, some finer subsequence converges to a certain continuous function \( U \) uniformly on compact subsets of \( \Omega \).

Next we state and prove some additional properties of these limit orbits, which will be useful in the next step, and which have some interest by themselves.

**Theorem 5.1.** – Let \( u \) be a weak solution of (1.1) with \( u_0 \in C_0^\infty(\mathbb{R}^N) \). Let \( U \) be the limit of some convergent subsequence of scalings \( u_{\lambda_n} \). Then the support of \( U \) is:

1. **compact**.
2. **connected**.
3. **non-contracting**.

Moreover,

4. \( \text{mass}(U) = \text{mass}(u) \).
5. \( U(0) = M\delta \).

**Proof.** – (1). Compactness of the support of \( U \) is an easy consequence of Theorem 4.4. First of all, we notice that the growth of the support from \( t = 0 \) to \( t = t_0 \) does not depend on \( \lambda \).

We have, for all \( t > 0 \)

\[
\Omega_\delta(t) \subseteq \left( \Omega_{\lambda_n}(0) \right)_{R(t)} \quad \text{with} \quad R(t) = Ct^\beta, \quad \beta = 1/(N(m - 1) + 2).
\]

The uniform boundedness of the supports implies that some subsequence \( u_{\lambda_n} \) converges uniformly to \( U \) in sets of the form \( \mathbb{R}^N \times [t_1, t_2] \) with \( 0 < t_1 < t_2 \), in particular \( u_{\lambda_n}(t) \) converges uniformly to \( U(t) \) in \( \mathbb{R}^N \) for each fixed \( t > 0 \). Now take some \( \lambda^* > 0 \). We have

\[
\Omega_u(t) \subseteq \bigcup_{\lambda_n > \lambda^*} \Omega_{\lambda_n}(t) \subseteq \bigcup_{\lambda_n > \lambda^*} \left( \Omega_{\lambda_n}(0) \right)_{R(t)} = \left( \Omega_{\lambda_n}(0) \right)_{R(t)}.
\]

Passing now to the limit \( \lambda^* \to \infty \) we obtain

\[
\Omega_U(t) \subseteq B_{R(t)}.
\]

Compactness is proved. Moreover, we have obtained an estimate on the rate of growth of the support.

(2) **Connectedness.** Suppose for \( t_0 > 0 \) there is a connected component \( G \) of the set \( \{U(t_0) > 0\} \) which does not contain the origin. Take \( \tau < t_0 \) such that

\[
\Omega_U(\tau) \cap G = \emptyset.
\]

(Such \( \tau \) exists, because of the estimate of the support). Now consider the function \( \hat{U}(t) = U(t + \tau) \), which is again a weak solution of Eq. (1.1), in the cylinder \( \overline{G} \times [0, t_0 - \tau] \). In its lateral boundary \( \hat{U} = 0 \) by the retention property, and in \( \overline{G} \times \{0\} \), \( \hat{U} = 0 \) by the assumption above. This contradicts the maximum principle. Therefore, every connected component of \( \{U(t_0) > 0\} \) contains the origin. This implies that there is only one component and \( \Omega_U = \{U > 0\} \) is connected.

(3) **Expansion.** We must show that, given \( t_2 \geq t_1 > 0 \), we will have

\[
\Omega_U(t_1) \subseteq \Omega_U(t_2).
\]

This is a direct consequence of the retention property applied to \( \hat{U}(t) = U(t + \tau) \).
(4)–(5). The fact that \( \text{mass}(U) = M \) is a consequence of the convergence of \( u_\lambda \) towards \( U \) in \( L^1(\mathbb{R}^N) \) for each \( t > 0 \) (the compactness of the supports allows to change convergence in \( L^1_{\text{loc}}(\mathbb{R}^N) \) for fixed \( t > 0 \) into convergence in \( L^1(\mathbb{R}^N) \)). (5) follows from (4) and the estimate of the support \( (R(t) \to 0 \text{ when } t \to 0) \). Theorem 5.1 is proved. □

Once we know that the limit orbit is a source-type solution, a standard way of identifying it relies on proving that in our class of solutions there is a unique solution of the evolution problem with initial data a Dirac delta (a unique source-type solution). This approach could be used in the purely diffusive case (porous medium equation) thanks to Pierre’s uniqueness result for measures as initial data, [28]. It was possible for the \( p \)-Laplacian equation because of the uniqueness result of [21]. We will not attempt such a general result here and we will use a different way to arrive at the identification of \( U \) as the unique self-similar solution. We even prove in the process that such a solution exists. The following property is crucial in order to identify the limit orbits.

**Theorem 5.2.** For every initial data in \( C_0^{\infty}(\mathbb{R}^N) \), and every convergent subsequence \( u_{\lambda,n} \), the limit \( U \) is self-similar.

**Proof.** For the proof, we adopt the Lyapunov approach, that is, we construct a suitable functional, decreasing along orbits. Such a construction is carried out for the Porous Medium equation in the Notes [32], and for Eq. (1.1) with linear diffusion, \( m = 1 \), by Zuazua [35]. The main difference of our case with respect to the Porous Medium case is that for the latter the asymptotic limit is known \textit{a priori} and has an explicit self-similar form (Barenblatt solutions), while here we need to prove the self-similarity of the limit orbits. It turns out that for this purpose we can use a Lyapunov functional which is very similar to the one used in [32], and the proof of its properties is, with minor modifications, the same. In Step 4 (Theorem 5.3) we will make use of this approach again in order to identify the limit orbits.

For every weak solution \( u(t) \) and fixed \( h > 0 \) we define the functional \( J_u(t) = J_{u,h}(t) \) by

\[
J_u(t) = \int_{\mathbb{R}^N} |u(x,t) - u_h(x,t)| \, dx, \quad t > 0,
\]

where \( u_h \) denotes the rescaled solution, that is,

\[
u_h(x,t) = h^{\beta} u(h^\alpha x, ht),\]

with \( \beta = 1/(N(m - 1) + 2) \) and \( \alpha = N\beta \). This functional measures the lack of self-similarity of \( u \). Here are some properties of \( J_u \).

**Lemma 5.1.** \( J_u(t) \) is monotonically non-increasing in \( t \) for every \( u \). Consequently, it has a limit when \( t \to \infty \).

This lemma is a direct consequence of the \( L^1 \)-contraction, which is valid in our case (see Theorem 3.2). We denote \( J_u(\infty) = \lim_{t \to \infty} J_u(t) \).

**Lemma 5.2.** \( J_U(t) \) is constant.

**Proof.** We study the action of \( J \) on rescaled orbits. We have:

\[
J_{u_\lambda}(t) = \int_{\mathbb{R}^N} |u_\lambda(x,t) - (u_\lambda)_h(x,t)| \, dx = \int_{\mathbb{R}^N} |u_\lambda(x,t) - (u_h)_\lambda(x,t)| \, dx = J_u(\lambda t).
\]
Then:

\[
\lim_{\lambda \to \infty} J_{u_\lambda}(t) = \lim_{\lambda \to \infty} J_{u}(\lambda t) = J_u(\infty) \quad \forall t > 0.
\]

Now, \( J_u(t) \) depends continuously on \( u \) for the type of convergence that we have (recall that \( u_\lambda(t) \to U(t) \) in \( L^1(\mathbb{R}^N) \) for each fixed \( t > 0 \)). Therefore,

\[
J_U(t) = J_u(\infty). \quad \square
\]

**Lemma 5.3.** – For any orbit \( u(t) \) there exists a time \( t_0 \geq 0 \) such that \( J_u(t) \) is strictly decreasing in any time interval \( (t_1, t_2) \) with \( t_0 < t_1 < t_2 \) unless \( u = u_h \) for all \( h > 0 \) in \( (t_1, t_2) \).

**Proof.** – Solutions of our problem have the property of penetration, that is, an initially connected support grows and tends to occupy the whole space. If the initial support is not connected, each component remains connected (Corollary to the retention property) and evolves in the same way. Let \( t_0 \) be such that the supports of \( u(t_0) \) and \( u_h(t_0) \) are connected and are not disjoint. Let \( t_1 > t_0 \). Consider the solution \( w \) to our equation with data at \( t = t_1 \) given by

\[
w(x, t_1) = \max \left\{ u(x, t_1), v(x, t_1) \right\}
\]

where \( v(t) = u_h(t) \). By the maximum principle:

\[
w(t) \geq \max \left\{ u(t), v(t) \right\}.
\]

We have the identities:

\[
w(x, t_1) - u(x, t_1) = \left[ v(x, t_1) - u(x, t_1) \right]_+,
\]

\[
w(x, t_1) - v(x, t_1) = \left[ u(x, t_1) - v(x, t_1) \right]_+,
\]

which imply:

\[
J(t_1) = \int (w(x, t_1) - u(x, t_1)) \, dx + \int (w(x, t_1) - v(x, t_1)) \, dx
\]

and, because of (5.9), we have for all \( t \geq t_1 \)

\[
J(t) \leq \int (w(x, t) - u(x, t)) \, dx + \int (w(x, t) - v(x, t)) \, dx \quad \forall t \geq t_1.
\]

Constancy of \( J \) in \( [t_1, t_2] \) implies constancy of both integrals in the last inequality, since both of them are non-increasing in time. This in turn implies:

\[
w(t) = \max \left\{ u(t), v(t) \right\} \quad \text{in} \ [t_1, t_2].
\]

We can use the strong maximum principle to conclude that since the supports of \( u \) and \( v = u_h \) are connected and not disjoint, and their masses coincide,

\[
w(t) = u(t) = v(t) \quad \text{in} \ [t_1, t_2]. \quad \square
\]

We can now complete the proof of Theorem 5.2. Lemmata 5.2 and 5.3 imply that \( J_U(t) = 0 \) for large \( t \). Since \( J_U(t) \) is constant by Lemma 5.2, we have \( J_U(t) = 0 \). This in turn implies \( U = U_h \), for all \( h > 0 \). Therefore, \( U \) is self-similar and Theorem 5.2 is proved. \( \square \)
Step 4. Identification of the limit. In this step we prove that the constructed limit solution $U$ is unique, in the sense that it does not depend on the convergent subsequence $\{u_{n}\}$ nor on the initial data in the class $C_0^\infty(\mathbb{R}^N)$.

Take some fixed initial data in $C_0^\infty(\mathbb{R}^N)$ with connected support, and mass $M$. Let $U^*_M$ denote the limit of a certain convergent subsequence $\{u^*_n\}$. We prove that $U^*_M$ is the unique possible limit.

**Theorem 5.3.** For any $u_0 \in C_0^\infty(\mathbb{R}^N)$ with mass $M$, $U^*_M$ is the limit of any convergent subsequence $\{u_{n}\}$.

**Proof.** We use again the Lyapunov approach. This time we put, as in [32]:

$$J_u(t) = \int |u(x,t) - U^*_M(x,t)| \, dx. \tag{5.18}$$

Lemmata 5.1 and 5.2 of Step 3 are true for this $J^*$ (use, in Lemma 5.2, the fact, proved above, that $U^*_M$ is self-similar). Lemma 5.3 is also true, by virtue of the proved properties of the support of $U^*_M$. Therefore, $\lim u_{\lambda} = U^*_M$. \qed

We can now give the proof of Theorem 1.2 for initial data in the class $C_0^\infty(\mathbb{R}^N)$. According to Theorem 5.3, for $t=1$ we have:

$$u_\lambda(x,1) \rightarrow U^*_M(x,1) \quad \text{in } L^1(\mathbb{R}^N). \tag{5.19}$$

Translating this result to the original orbit, we obtain:

$$\|u(t) - U^*_M(t)\|_{L^1} \rightarrow 0 \quad \text{when } t \rightarrow \infty, \tag{5.20}$$

which is the result of Theorem 1.2 for $p=1$.

**Uniform convergence.** Once we identified the limit of $\{u_{\lambda}\}$ in $L^1$, we will have, for the type of data we are considering (see Step 3)

$$u_{\lambda}(1) \rightarrow U^*_M(1) \quad \text{as } \lambda \rightarrow \infty, \text{ uniformly in } \mathbb{R}^N. \tag{5.21}$$

Translating this result to the original orbit, we obtain

$$t^\alpha \|u(t) - U^*_M(t)\|_{L^\infty} \rightarrow 0, \tag{5.22}$$

with $\alpha = N/(N(m-1) + 2)$. This result, combined with the proved $L^1$-convergence, gives (by simple interpolation) the following convergence result in all $L^p$ with temporal rate of decay, depending on $p$:

$$t^{\alpha(p)} \|u(t) - U^*_M(t)\|_{L^p} \rightarrow 0 \tag{5.23}$$

with $\alpha(p) = \frac{p}{p-1} \alpha$. Theorem 1.2 is proved for initial data in $C_0^\infty(\mathbb{R}^N)$.

**Extension to general data.** The next step is to extend these results to data in $L^1(\mathbb{R}^N)$. We begin with $L^1$-convergence. Thanks to the $L^1$-contraction, we can use a density argument. Precisely, given $u_0 \in L^1(\mathbb{R}^N)$, $u_0 \geq 0$ and $\varepsilon > 0$, we choose $u'_0 \in C_0^\infty(\mathbb{R}^N)$, $u'_0 \geq 0$, such that:

$$\|u'_0 - u_0\|_1 \leq \varepsilon. \tag{5.24}$$
Denote by \( u'(t) \) the solution with initial data \( u'_0, M' = \int_{\mathbb{R}^N} u'_0 \, dx \). By the triangle inequality

\[
\|u(t) - U^*_M(t)\|_1 \leq \|u(t) - u'(t)\|_1 + \|u'(t) - U^*_M(t)\|_1 + \|U^*_M(t) - U^*_M(t)\|_1.
\]

By construction, \( |M - M'| \leq \|u_0 - u'_0\|_1 \leq \varepsilon \). Hence \( \|U^*_M - U^*_M\|_1 \leq \varepsilon \). By the \( L^1 \)-contraction

\[
\|u(t) - u'(t)\|_1 \leq \|u(0) - u'(0)\|_1 \leq \varepsilon.
\]

Therefore,

\[
\|u(t) - U^*_M(t)\|_1 \leq 2\varepsilon + \delta(t),
\]

where \( \delta(t) \to 0 \) when \( t \to \infty \) according to previous result. Passing to the limit \( t \to \infty \),

\[
\lim_{t \to \infty} \|u(t) - U^*_M(t)\|_1 \leq 2\varepsilon.
\]

Since \( \varepsilon \geq 0 \) is arbitrary, we conclude the proof for \( p = 1 \).

**Uniform convergence with general data.** Let \( U_M \) denote the Barenblatt solution of the Porous Medium equation. Let \( \{u_\lambda\} \) denote the rescaled family of solutions (the scaling parameters are the same as in our case). Let \( R \) be such that the support of \( U_M(1) \) is contained in \( B_R(0) \). Clearly, as a result of the uniform convergence of \( u_\lambda \) towards \( U_M \) in compact subsets of \( \mathcal{Q} \) and the \( L^1 \)-convergence of \( u_\lambda(1) \) to \( U_M(1) \), we know that, given \( \varepsilon > 0 \)

\[
\int_{\mathbb{R}^N - B(0,R)} u_\lambda(1) \, dx < \varepsilon,
\]

if \( \lambda \) is large enough. In the work [32] (Chapter 9, Theorem 3.2) the following result is proved:

**Proposition 5.2.** – There is a function \( C(\varepsilon) \) with \( C(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \) such that

\[
0 < \varepsilon \leq u_\lambda(x, 1) \leq C(\varepsilon) \quad \text{for } |x| \geq R.
\]

This estimate is a consequence of the following result:

**Lemma 5.4.** – For every solution \( v \) of the initial value problem for the Porous Medium equation in \( \mathcal{Q} \) with initial data such that

\[
0 < \varepsilon \leq v_0(x) \leq K \quad \text{and} \quad \int_{\mathbb{R}^N} (v_0(x) - \varepsilon) \, dx \leq \varepsilon,
\]

we have the \( L^\infty \)-estimate

\[
v(x, t) \leq C(\varepsilon, K) \quad \forall t \geq 1/2,
\]

where \( C \) is a continuous function such that \( C(0, K) = 0 \).

We would like to have a result analogous to Lemma 5.4. It can be obtained by using again the symmetrization result of [3]. Indeed, consider the solution to Eq. (1.1) with the same initial data as in Lemma 5.4. Strict positivity of the data implies that in fact, we are dealing with a non-degenerate problem (modify the power function \( r \to r^m \) near the origin). The symmetrized
problem will be a non-degenerate problem which is equivalent to the Porous Medium equation with initial data as above. Then, putting \( k = C(\varepsilon, K) \) in Theorem 2.2, the validity of Lemma 5.4 follows in our case.

The fact that Lemma 5.4 implies Proposition 5.2 can be proved exactly as in [32]. Proposition 5.2 obviously gives the desired result, that is

\[
(5.33) \quad u_\lambda(1) \to U_M^*(1) \quad \text{when} \quad \lambda \to \infty \quad \text{uniformly in} \quad x.
\]

As for data in \( C^\infty_0(\mathbb{R}^N) \), a simple interpolation gives the convergence result in all \( L^p \)-spaces.

The main Theorem 1.2 is proved.

6. Uniqueness of the source-type solution

The existence of the family \( \{U_M, \ M > 0\} \) was established as a part of the proof of Theorem 1.2. Let us worry about the question of uniqueness of such solutions and thus complete the proof of Theorem 1.1. Suppose that a nonnegative function \( \tilde{u} \in L^1(\mathbb{R}^N) \) satisfies:

\[
\begin{align*}
(S1) & \quad \tilde{u} \in C((0, +\infty); L^1(\mathbb{R}^N)) \cap C(Q) \cap L^\infty((\tau, +\infty) \times \mathbb{R}^N) \quad \text{for} \quad \tau > 0. \\
(S2) & \quad \tilde{u} \text{ is a weak solution of (1.1) in the sense that it satisfies (W1), (W3) and (W4) of Definition 2.1.} \\
(S3) & \quad \tilde{u} \text{ is self-similar, that is, of the form (1.7).} \\
(S4) & \quad \int_{\mathbb{R}^N} \tilde{u}(x,t)\phi(x) \, dx \to M\phi(0) \quad \text{when} \quad t \to 0 \quad \text{for some} \quad M > 0 \quad \text{and all} \quad \phi \in \mathcal{D}(\mathbb{R}^N).
\end{align*}
\]

Our goal is to prove that \( \tilde{u} = U_M \), where \( U_M \) is the limit orbit with mass \( M \) constructed in Section 5.

**Proof of Theorem 1.1.** – First of all, we observe that the values of the exponents \( \alpha \) and \( \beta \) in (1.7) are uniquely determined. One relation between \( \alpha \) and \( \beta \) is obtained by substituting (1.7) in (1.1) and eliminating \( t \). Another relation is due to the fact that the mass \( \int_{\mathbb{R}^N} \tilde{u} \, dx \) must be constant (as a consequence of (1.7)) and equal to \( M \) by (S4). In this way we obtain the following values for \( \alpha \) and \( \beta \):

\[
(6.1) \quad \alpha = N/(N(m-1)+2), \quad N\beta = \alpha.
\]

Fix \( \tau > 0 \). Since, by hypothesis, \( \tilde{u}(\tau) \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \), we can consider the Cauchy problem for (1.1) with initial data \( \tilde{u}(\tau) \). Let \( \tilde{w}(x,t) \) be the weak solution of this problem. By uniqueness, \( \tilde{w}(x,t) = \tilde{U}(t + \tau) \). Then, by the results of Section 5,

\[
(6.2) \quad \tilde{w}_\lambda(1) = \tilde{U}_\lambda(1 + \tau/\lambda) \to U_M(1) \quad \text{in} \quad L^1(\mathbb{R}^N).
\]

By the triangle inequality, and taking into account the self-similarity of \( \tilde{U} \),

\[
(6.3) \quad \|\tilde{U}(1) - U_M(1)\|_{L^1} \leq \|\tilde{U}(1) - \tilde{U}(1 + \tau/\lambda)\|_{L^1} + \|\tilde{U}(1 + \tau/\lambda) - U_M(1)\|_{L^1} = \|\tilde{U}(1) - \tilde{U}(1 + \tau/\lambda)\|_{L^1} + \|\tilde{U}_\lambda(1 + \tau/\lambda) - U_M(1)\|_{L^1}.
\]

Now choose \( \lambda \) (according to (6.1) and condition (S1) above) large enough such that

\[
(6.4) \quad \|\tilde{U}_\lambda(1 + \tau/\lambda) - U_M(1)\|_{L^1} \leq \varepsilon \quad \text{and} \quad \|\tilde{U}(1) - \tilde{U}(1 + \tau/\lambda)\|_{L^1} \leq \varepsilon,
\]

for arbitrary \( \varepsilon > 0 \). (6.3) and (6.4) imply that

\[
(6.5) \quad \tilde{U}(1) = U_M(1).
\]

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Since both $\bar{U}$ and $U_M$ are self-similar with the same exponents $\alpha$ and $\beta$, the last equality implies $\bar{U} = U_M$ in $Q$, which was the assertion of Theorem 1.1. □

7. The shape and evolution of the support

In this section we investigate the evolution of the support of general compactly supported solutions, relating it to the shape of the support of the source-type solution with the same mass. Preliminary information on the form of the support is given in Sections 4 and 5.

We describe first the main properties of the support of the source-type solution $U = U_M(x, t)$, $M > 0$. Since the source-type solution is self-similar the support at any time $t > 0$, $\Omega_U(t)$, is related in a simple way to the support at time $t = 1$ by homothety:

(7.1) $\Omega_U(t) = \Omega_U(1) t^\beta$,

with the similarity exponent $\beta$ given in (1.9). We also know that $\Omega_U(1)$ is a bounded set, hence there exists a constant $R(m, N, M) > 0$ such that:

(7.2) $\Omega_U(1) \subseteq B_R(0)$.

There is also a similar estimate from below. This is a consequence of the following facts: (i) the solution $U$ at time $t = 1$ is a continuous function with mass $M > 0$, hence there will be some ball $B_r(x_0)$ where $U > 0$, (ii) the existence of sub-solutions (4.9) which penetrate into the whole space means that for some later time $t_1 > 1$ a neighborhood of the origin will belong to $\Omega_U(t_1)$, (iii) the self-similarity (7.1) implies then that $\Omega_U(1)$ must also contain one such neighborhood.

We conclude in this way that there exists another constant $R'(m, N, M) > 0$ such that:

(7.3) $B_{R'}(0) \subseteq \Omega_U(1)$.

By self-similarity the same estimates are true for the set $t^{-\beta} \Omega_U(t)$ for every $t > 0$. In the sequel we take the optimal values of $R$ and $R'$, that is:

(7.4) $R_1 = \inf \{ R > 0 \text{ such that (7.2) holds} \}$,

(7.5) $R_0 = \sup \{ R' > 0 \text{ such that (7.3) holds} \}$.

We observe that both radii depend on the mass $M$ and of course on $m$ and $N$. When there is no convection the source-type solution is radially symmetric and then $R_1 = R_0$. In the presence of convection we have $R_0 < R_1$.

Another interesting property of the support is its star-shapedness around the origin: for every $x \in \Omega_U(t)$ the whole segment joining $x$ with 0 lies in $\Omega_U(t)$. This property follows immediately from self-similarity (7.1) and retention, Theorem 4.2.

Now we are in a position to prove Theorem 1.3. It contains a delicate growth argument.

Proof of Theorem 1.3. – A lower bound for the extension of the support $\Omega$ of $u$ compared with $\Omega_U$ is an easy consequence of the asymptotic convergence of $u$ towards $U$. Theorem 1.2, and the continuity of $U$. Therefore, we need only estimate the maximum distance from the points of $\partial \Omega_U(t)$ to $\Omega_U(t)$. Let us call this maximum excess distance $d_u(t)$. It is a nonnegative quantity, and we need to prove that, once renormalized, it tends to zero:

(7.5) $\tilde{d}_u(t) = \frac{d_u(t)}{t^\beta} \to 0$ as $t \to \infty$. 

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Observe that if $\tilde{u}_{t1}$ is the rescaled version of $u$ that takes time $t_1$ to 1, then $d_u(st_1) = t_1^\beta d_{\tilde{u}}(s)$ for every $s > 0$. It is convenient to use the normalized distance so that

$$(7.6) \quad \tilde{d}_u(st_1) = \tilde{d}_{\tilde{u}}(s), \quad s > 0.$$ 

In the following we will use such rescalings to make our basic calculations at $t = 1$. For notational simplicity, we drop the subindex $u$ and write $d(s)$ and $d(s)$ instead of $d_u(s)$ and $\tilde{d}_u(s)$ respectively.

**Basic distance calculation.** We take a point $x_0$ which lies at a distance $d.h / C > d.h / C$ from the support of $U$. From the possible growth of the support of $U$, cf. (7.1), this support and the cylinder will be disjoint for a time $0 \ll t$ if $t$ satisfies

$$(7.7) \quad R_1(1 + \tau)^\beta - R_1 \leq d(1 + \delta/2).$$ 

Since the leading-order term in the left-hand side expression is $R_1\beta \tau$ as $\tau \to 0$, there exists $\tau_1 > 0$ such that $R_1(1 + \tau)^\beta - R_1 \leq 2R_1 \beta \tau$ for $0 < \tau < \tau_1$. Then, for $\tau_0(\delta) = \min(\tau_1, (d(1 + \delta/2)/2R_1\beta))$, (7.7) holds. According to Díaz–Véron’s estimate, Lemma 4.1, the function $u$ still vanishes at $x = x_0$ for $t \in (1, 1 + \tau)$ if $F(\tau, \delta/2) \leq (\delta/2)^\theta$. Since $F(\tau, \delta/2)$ can be estimated in the form $F(\tau, \delta) < C\tau^\lambda$, we can choose $\tau < \tau_0(\delta)$ small enough, in order to have

$$(7.8) \quad d(1 + \tau) \leq d(1 + \delta/2).$$

We also remark the following estimate from below for $d(1 + \tau)$:

$$(7.9) \quad d(1 + \tau) \geq d(1) - 2R_1\beta \tau,$$

valid for $\tau < \tau_1$, which is a consequence of two facts: (i) the expansion of the support of $u$ and (ii) the fact that for $\tau < \tau_1$, the support of $U$ grows less than $2R_1\beta \tau$. A consequence of (7.8) and (7.9) is

**Lemma 7.1.** For a given solution $u$, the normalized distance $\tilde{d}$ is a continuous function from $(0, \infty)$ to $[0, \infty)$.

**Proof.** Let $t_1 > 0$. Considering the rescaled solution $\tilde{u}_{t1}$, it will be enough to prove the continuity of $\tilde{d}(s)$ at $s = 1$. (7.8) and (7.9) show that $d$ is continuous from the right at $s = 1$. Indeed, for $d$ small enough, if we put $C\tau^\lambda = (\delta/2)^\theta$, both conditions [10] and (7.7) will be satisfied, since $\nu/\lambda > 1$. Therefore, for small enough $\tau$,

$$(7.10) \quad d(1) - 2R_1\beta \tau \leq d(1 + \tau) \leq d(1) + C\tau^{\lambda/\nu}$$

and continuity from the right follows. In order to prove the continuity from the left, we observe that for arbitrary $t > 0$, (7.10) reads:

$$(7.11) \quad d(t) - At^\beta h \leq d(t + h) \leq d(t) + Bt^\beta h^{\lambda/\nu},$$

for $0 < h < h_0(t)$, where $h_0$ can be chosen uniformly for all $t$ in sets of the form $[0 < t_0 < t < t_1]$, and $A$, $B$ are positive constants. Therefore, $d$ (and, consequently, $\tilde{d}$) is continuous. Lemma 7.1 is proved. □

Continuing the proof of Theorem 1.3, we have the following partial result:
Lemma 7.2. For any solution $u$, 

\begin{equation}
\lim_{t \to \infty} \inf \tilde{d}(t) = 0.
\end{equation}

Proof. The main idea for the proof is the observation that the function $F(\tau, \delta)$ can be estimated in the form $C \varepsilon \tau^\delta$, being $\varepsilon$ the mass of the solution contained in the ball $B_{\delta/2}(x_0)$ and $C$ a structural constant. By the asymptotic convergence in $L^1$ norm, given any solution $u$ and a number $\varepsilon > 0$ there exists a time $t(\varepsilon) > 0$ such that the mass of $u(t)$ contained in the complement of $\Omega(t)$ is less than $\varepsilon$ for all $t > t(\varepsilon)$. By rescaling we may always take one such time, say $t_1 > t(\varepsilon)$, and make it $t = 1$ and then the previous assertion is true for all $t \geq 1$.

Taking this fact into account, it is not difficult to prove the lemma by contradiction. Suppose that such a limit is $a > 0$. Then for any $\varepsilon > 0$ very small there exists a time $t_0$ large enough such that $d(t) > a - \varepsilon$ and the condition of outer mass less than $\varepsilon$ holds for $t \geq t_0$. Moreover, for a sequence $t_n \to \infty$ we have $d(t_n) \to a$. We rescale one such $t_n$ to make it 1, hence we may assume that $a + \varepsilon \geq \tilde{d}(1) > a - \varepsilon > 0$. If we take $t_0 = (a - \varepsilon)/4R_1\beta$ then (7.7) will be true for all $d > 0$. Putting $\delta = 2(C \varepsilon \tau^\delta)^{1/\nu}$, the condition of [10] will hold, therefore:

\begin{equation}
d(1 + \tau) \leq d(1) + C \varepsilon^{1/\nu} \tau^{\lambda/\nu}
\end{equation}

for all $0 < \tau < t_0$, with arbitrarily small $\varepsilon > 0$. In terms of the renormalized distance, (7.22) reads:

\begin{equation}
\tilde{d}(1 + \tau) \leq \frac{\tilde{d}(1)}{(1 + \tau)^\beta} + \frac{C \varepsilon^{1/\nu} \tau^{\lambda/\nu}}{(1 + \tau)^\beta}.
\end{equation}

For small values of $\tau$, say $0 < \tau \leq \tau'$ we will have

\begin{equation}
\tilde{d}(1 + \tau) \leq \tilde{d}(1)(1 - \beta \tau/2) + C \varepsilon^{1/\nu} \tau^{\lambda/\nu}.
\end{equation}

Moreover, by rescaling we conclude that the same happens for other initial times:

\begin{equation}
\tilde{d}(\tau_1(1 + \tau)) \leq \tilde{d}(\tau_1)(1 - \beta \tau/2) + C \varepsilon^{1/\nu} \tau^{\lambda/\nu}.
\end{equation}

If we choose now $\varepsilon$ so small that

\begin{equation}
2\varepsilon + C \varepsilon^{1/\nu} \tau^{\lambda/\nu} < a\beta \tau'/2,
\end{equation}

then we will have $\tilde{d}(1 + \tau') < a - \varepsilon$ against the hypothesis. The claim is proved. \hfill \Box

The last step is to prove that also $\limsup_{t \to \infty} \tilde{d}(t) = 0$. Otherwise, there exists an $a > 0$ such that for a sequence $t_n \to \infty$ we have $\tilde{d}(t_n) \to a$. By virtue of Lemma 7.2, there exists a set of disjoint intervals $[[t_{0n}, t_{1n}]]$ with $t_{0n} \to \infty$ such that $\tilde{d}(t_{0n}) = a/3$, $\tilde{d}(t_{1n}) = 2a/3$ and for $t_{0n} < t < t_{1n}$, we have $a/3 < \tilde{d}(t) < 2a/3$. There are two possibilities:

1. $t_n = t_{1n}/t_{0n}$ contains a subsequence (not relabeled) with limit 1.
2. $t_n > 1 + r$ for some fixed $r > 0$ and $n > N$.

If (1) holds, we take $n$ large enough, so that $t_n = 1 + \bar{t}$ with $\bar{t} < a/6R_1\beta$ and $C \varepsilon \bar{t}^\delta < (\delta/2)^\nu$. Then, by (7.8) with $\delta/2 \leq a/3$ we would have $\tilde{d}(t_{1n}) < 2a/3$, false.

In the second case, there is a $\bar{t} < a/6R_1\beta$ such that $t_{1n}/(1 + \bar{t}) > t_{0n}$ for $n > N$. First, we choose $\varepsilon$ (and the corresponding $t_{0n}$) such that (7.17) holds for $\bar{t}$. Consider, for such $n$, the point $t_n = t_{1n}/(1 + \bar{t})$. Then, by the same argument as in the proof of Lemma (7.2), we will have $\tilde{d}(t_{1n}) = \tilde{d}(t_n(1 + \bar{t})) < \tilde{d}(t_n) - \varepsilon < 2a/3$, again false. Theorem 1.3 is proved. \hfill \Box

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Corollary 7.1. Let \( r_0(t) \) and \( r_1(t) \) denote the minimal (respectively, the maximal) distance from \( \partial \Omega \) to the origin. Then, as \( t \to \infty \) we have

\[
\lim_{t \to \infty} \frac{r_1(t)}{t^p} = R_1(M), \quad \lim_{t \to \infty} \frac{r_0(t)}{t^p} = R_0(M).
\]

8. More on the shape of the self-similar solutions

In this section we discuss the change of shape of the family of self-similar solutions \( \{U_M\} \) as \( M \) varies from 0 to infinity. A natural question in this context is the following: is the parameter \( M \) scalable, i.e., is it possible to produce the whole family \( U_M \) out of a single universal function, e.g., \( U_1 \)? We will see that the answer is negative. Moreover, it turns out that passing to the limit in the parameter \( M \), we obtain the standard Barenblatt solution of the purely diffusive equation

\[
u_t = \Delta u^m
\] when \( M \to 0 \), while in the limit \( M \to \infty \) we find a self-similar solution of the equation with partial diffusivity

\[
u_t = \Delta_y (u^m) - (u^q)_{x_1}
\] where \( y = (x_2, x_3, \ldots, x_n) \). In this way, Burgers equation exhibits in itself the same phenomenon of asymptotic simplification that occurs when we consider Eq. (1.1) with exponents \( q > m + 1/N \) (supercritical or diffusive case) and \( 1 < q < m + 1/N \) (sub-critical or convective case), thus establishing a continuous connection between both cases.

First we consider the case \( M \to 0 \). Let \( U_M \) denote the self-similar source type solution with mass \( M \), as above. Consider the following rescaled functions:

\[
\tilde{U}_M = M^{-2\beta} U_M \left( M^{(m-1)\beta} x, t \right),
\]

where \( \beta = 1/(N(m - 1) + 2) \), as above. It is clear that \( \int_{\mathbb{R}^N} \tilde{U}_M \, dx = 1 \) for every \( M > 0 \). Moreover, \( \tilde{U}_M \) satisfies (in the weak sense of Section 6) the following equation, a perturbation of (1.1):

\[
u_t = \Delta (u^m) - M^{1/N} (u^q)_{x_1}.
\] At this point, it is formally clear that, when \( M \to 0 \), the \( \tilde{U}_M \)'s must converge to the self-similar (and source-type) solution of the Porous Medium equation with mass equal to 1. The rigorous justification is based on the estimates proved in previous sections. Indeed, consider the family \( \{\tilde{U}_M, M > 0\} \). As we noticed before, \( \|\tilde{U}_M\|_{L^1} = 1 \). An easy calculation shows that the \( L^\infty \)-estimate (E2) holds uniformly in \( M \) (for this purpose, it is relevant the exact dependence on \( \|u\|_{L^1} \) in the estimate (E2), see (2.6)). Estimates (E3) and (E4) also hold uniformly. Then, arguing as in Section 5, we obtain a limit \( \tilde{U} \) in \( L^1_{\text{loc}}(Q) \) for some subsequence \( \tilde{U}_{M_n} \) with \( M_n \to 0 \). The mentioned estimates also allow to pass to the limit in (8.4). We thus obtain a self-similar and source-type solution of the Porous Medium equation. By uniqueness (see [21]) it is the Barenblatt solution with mass 1.

Next, we consider the case \( M \to \infty \). This time, the appropriate rescaled functions are the following:

\[
\tilde{U}_M = M^{-(1+1/N)} U_M \left( M^{-1/N} x, y, M^{-\frac{N(m-1)}{N+1}} t \right).
\]
As before, we notice that \( \int_{\mathbb{R}^N} U_M \, dx = 1 \). As in the previous case, all our estimates hold uniformly in \( M \), and in the limit we obtain a self-similar and source-type solution with mass 1 of the equation with partial diffusivity (8.2).

The changing shape of the self-similar solutions with \( M \) is portrayed in Figs. 1 and 2 for \( m = 1 \) and \( m = 2 \) \((N = 1)\).

9. Extension to general nonlinearities

Our study of asymptotic behaviour has been presented on the equation with power-like nonlinearities. But the result can be extended to a more general class of equations which includes (1.1) as a particular case. Thus, we may consider the equation

\[
(9.1) \quad u_t = \Delta \Phi(u) - \text{div}(F(u))
\]

with conditions on the functions \( \Phi \) and \( F \) for which a theory of existence and uniqueness can be performed and moreover \( \Phi(u) \sim u^m \), \( F(u)/u^q \sim (c_0, \ldots, 0) \) for \( u \sim 0 \). More precisely, we assume that

(H1) \( \Phi \in C^1([0, +\infty)) \) is an increasing function with \( \Phi(0) = 0 \) and satisfies

\[
(9.2) \quad d_1 r^{m-1} \leq \Phi'(r) \leq d_2 r^{m-1}, \quad r \geq 0,
\]
for some positive constants $d_1$ and $d_2$. Besides, we require that

$$\lim_{r \to 0} \frac{\Phi'(r)}{m^m - 1} = a \quad \text{with } a > 0. \quad (9.3)$$

(H2) The vector field $\mathbf{F} = (f_1, \ldots, f_N) \in C^1([0, +\infty); \mathbb{R}^N)$ satisfies

$$\lim_{r \to 0} \frac{\mathbf{F}'(r)}{qr^q - 1} = \mathbf{b} \quad \text{with } \mathbf{b} \neq 0. \quad (9.4)$$

Note that since vector $\mathbf{F}$ appears in (9.1) only through its derivatives, we can always suppose that $\mathbf{F}(0) = \mathbf{0}$. By rotating the axes and rescaling, we can attain $a = 1$ and $\mathbf{b} = (1, \ldots, 0)$. We will assume this in the sequel.

The conditions on $\mathbf{F}$ above entail the existence of positive constants $c_1$ and $\varepsilon$ such that, for $0 < r \leq \varepsilon$, we have

$$\frac{|f_i'(r)|}{r^q - 1} \leq c_1 \quad \text{for all } 1 \leq i \leq N. \quad (9.5)$$

Condition (H2) on the vector field $\mathbf{F}$ means that, in some sense, the convection term acts mainly in the direction of the vector $\mathbf{b}$ (assumed to be equal to $(1, \ldots, 0)$) when $u$ is near zero, while in the other directions diffusion dominates.

We assert that under these conditions the Cauchy problem is well-posed for integrable and nonnegative initial data and that the asymptotic behaviour of Theorem 1.2 remains valid,
the $U_M$ being the same self-similar solutions of (1.1) constructed above. We thus encounter a phenomenon of asymptotic simplification towards a simpler equation. That is, the only relevant information about $\Phi$ and $F$ for large times is their behaviour near 0. It is completely natural, since as $t \to \infty$, $u$ goes to 0.

A theory of existence and uniqueness for Eq. (9.1) has been developed by different authors under different assumptions on the nonlinearities and the class of solutions, cf. [15,27,34,8]. Similar (but stricter) power-like conditions are used in [11] and also in [14], where diffusion is linear, $m = 1$.

We may re-do the constructive existence proof and derivation of the main properties of the solutions needed in the proof of the asymptotic behaviour by introducing minor modifications in the definitions and proofs of preceding sections. Uniqueness is discussed in the Appendix. For example, in the definition of solution (Definition 2.1) we need only change condition (W4) into:

$$ (W4') \quad \text{for all} \ 0 < r_1 \leq r_2 < \infty, \ \text{all} \ \Omega \subset \mathbb{R}^N \ \text{bounded with smooth boundary and all} \ \phi \in L^\infty(\Omega \times [r_1, r_2]), \ \phi = 0 \ \text{on} \ \partial \Omega \times [r_1, r_2], \ \text{we have} $$

$$ (9.6) \quad \int_{r_1}^{r_2} \int_{\Omega} \left\{ \nabla \Phi(u) \cdot \nabla \phi - u \phi_t - F \cdot \nabla \phi \right\} \, dx \, dt = \int_{\Omega} u \phi \, dx \bigg|_{r_1}^{r_2}. $$

Existence of this class of solutions is obtained like in Section 2 since the corresponding references to [26,9,27] and [3] are still valid. In this way Theorem 2.1 applies. Let us worry about the fundamental estimates (E1)–(E4). The mass conservation (E1) can be proved in the same way as in Theorem 2.3. In fact, (E1) only depends on the divergence form of the right hand side in (9.1). Estimate (E2) is proved in [20] for general filtration equations of the type

$$ (9.7) \quad u_t = \Delta \Phi(u). $$

Note. – In this general case, the constant $C$ in (E2) depends also on the constant $d_1$ in (9.2). Arguing as in Section 2, the validity of (E2) extends to Eq. (9.1).

Concerning the Energy estimate (E3), it can be obtained by the standard technique of multiplying Eq. (9.1) by $\Phi(u)$ and integrating by parts. We obtain

$$ (9.8) \quad \int_0^\infty \int_{\mathbb{R}^N} \left| \nabla \Phi(u) \right|^2 \, dx \, dt \leq \frac{d_2}{m} \|u_0\|_{L^1} \|u(t)\|^m_{L^\infty}, $$

being $d_2$ the constant from (9.2). Taking into account the $L^\infty$-estimate (E2) we obtain the validity of (E3) for Eq. (9.1), with a constant depending on $d_1, d_2, m, N, \|u\|_1$ and $t$.

The $L^2$-estimate of the time derivative (E4) partially uses the restrictions imposed on the convection vector $F$. We reproduce the main lines of the proof (compare with Theorem 4.1). We multiply (9.1) by $\Phi(u)_t$ and integrate in $\Omega$ by parts. We get the following:

$$ (9.9) \quad \int_\Omega \left| \Psi(u) \right|^2 \, dx + \frac{d}{d\tau} \int_\Omega \left| \nabla \Phi(u) \right|^2 \, dx = \sum_{i=1}^N \int_\Omega \Phi'(u) f_i'(u) u_t u_{x_i} \, dx, $$

where $\Psi(t) = \int_0^t \sqrt{\Phi'(s)} \, ds$. Applying Young’s inequality to the right-hand side of (9.9), and taking into account (9.2), (9.5), (E2) and the continuity of $f_i'$ we arrive at the following inequality,
analogous to (4.3):

\[ \frac{d}{dt} \int \nabla \phi^2 \, dx \leq C(m, N, \|u\|_1, d_1, c) \|u(t)\|_\infty^{2q-m+1} \int \nabla \phi^2 \, dx, \]

where \( c \) is a positive constant such that \( f'_i(u)/u^{q-1} < c \) for the values of \( u(s) \), \( s \geq t > 0 \) (such a constant exists by virtue of (9.5) and the continuity of \( f'_i \)). Next we derive an estimate for the quantity

\[ \gamma'(t) = \int \nabla \phi(u(t))^2 \, dx, \]

analogous to (4.5), the only difference being that the constant also depends on \( d_1, d_2 \) and \( c \). Next we integrate in \( t \) from \( t_1 > 0 \) to \( \infty \), use (9.9) and the estimate on \( \gamma' \) thus obtaining the estimate

\[ \int_{t_1}^{\infty} \int \nabla \phi(u(t))^2 \, dx \, dt \leq C(N, m, \|u\|_1, d_1, d_2, c, t_1). \]

The left inequality in (9.2) can be used in order to estimate from below the integral in the left hand side of (9.12), and conclude as in Theorem 4.1. The only difference of this general case with respect to the case of pure powers is that the constant in (E4) depends also on \( d_1, d_2 \) and \( c \).

This is the asymptotic result for solutions of Eq. (9.1).

**Theorem 9.1.** – Let \( \Phi \) and \( F \) satisfy conditions (9.2) and (9.4), and let \( u_0 \in L^1(\mathbb{R}^N) \), \( u_0 \geq 0 \), \( \int u_0 \, dx = M > 0 \). Then the weak solution of the Cauchy problem for Eq. (9.1) satisfies the thesis of Theorem 1.2, with the same \( U_M \).

**Proof.** – We introduce the same family of rescalings (5.1) of Section 5. \( u_\lambda \) satisfies the following equation

\[ u_{\lambda t} = \Delta \Phi_\lambda(u_\lambda) - \text{div} F_\lambda(u_\lambda), \]

where

\[ \Phi_\lambda(r) = \lambda^{\alpha m} \Phi(\lambda^{-\alpha} r), \quad F_\lambda(r) = \lambda^{\alpha-\beta+1} F(\lambda^{-\alpha} r). \]

It is easy to check that (9.2) holds for all \( \Phi_\lambda \) with the same constants \( d_1 \) and \( d_2 \) and that (9.5) is valid for all \( F_\lambda \) with \( \lambda \geq 1 \) with the same \( \varepsilon \) and \( c_1 \). Therefore, since all the \( u_\lambda \) have the same mass, all the estimates (E1)–(E4) are valid uniformly in \( \lambda \). Arguing as in Section 5, Step 2, we obtain a convergent sequence in \( L^1_{\text{loc}}((t_0, +\infty) \times \mathbb{R}^N) \), which we denote again \( \{u_\lambda\} \). Let \( U \) be the limit of this sequence. It is an easy matter to check that

\[ \Phi_\lambda(r) \rightarrow r^m \quad \text{and} \quad F_\lambda(r) \rightarrow r^q \]

as \( \lambda \rightarrow \infty \), and moreover, this convergence is uniform on compact subsets of \([0, \infty)\). Taking into account these convergences, we can pass to the limit in the weak formulation, and we conclude that \( U \) satisfies (1.1) in the weak sense.

Another important property of the solutions of (9.1) is the fact that solutions with initially compact support have compact support for all times \( t > 0 \). This fact can be proved exactly as in Section 4, by using the energy method of Antontsev, since the necessary structural assumptions
are satisfied, by virtue of (9.2) and (9.4). This property enables to get convergence in $L^1$ (not only in $L^1_{\text{loc}}$) for solutions with $u_0 \in C_0^\infty$.

The problem of the identification of the limit can be solved by means of the method of asymptotic analysis of [17, pp. 443–446]. In the sequel, we adopt the notation of this work. First of all, we perform in (9.1) the continuous rescaling, corresponding to (5.1). It is given by

$$\theta(\xi, \tau) = t^\alpha u(\xi t^\beta, t),$$

where $\tau = \ln t$, and $\alpha$ and $\beta$ are as in (5.1). $\theta(\xi, \tau)$ satisfies

$$\theta_t = \Delta_\xi \Phi_\tau(\theta) - \text{div}_\xi (F_\tau(\theta)) + \alpha \theta + \beta \xi \nabla_\xi \theta,$$

with

$$F_\tau(r) = e^{\alpha \tau} \Phi(re^{\beta \tau}), \quad F_\tau(r) = e^{(\alpha - \beta + 1)\tau} F(re^{-\tau})$$

(compare with (9.13) and (9.14)). We denote the (non-autonomous) operator in the right hand side of (9.17) by $B(\theta, \tau)$. We also consider the following autonomous operator:

$$A(\theta) = \Delta_\xi \theta^{m} - (\theta^q)_{\xi_1} + \alpha \theta + \beta \xi \nabla_\xi \theta,$$

which is the operator that appears in the equation for $\theta$ given by (9.16) when $u$ is a solution of (1.1). As a class $S$ of solutions of (9.1), we take the class of solutions with $u_0 \in C_0^\infty(\mathbb{R}^N)$ and a fixed mass $M = \int u_0 \, dx > 0$. The corresponding solutions of (9.17) are in $C([0, \infty); L^1(\mathbb{R}^N))$, have the same mass (which is conserved in time) and its supports are contained in some fixed ball for all $t > 0$. As the ambient space we take the following closed subset of $L^1$:

$$X = \{u \in L^1(\mathbb{R}^N), \ u \geq 0 \text{ with fixed mass } M > 0\}.$$

In order to reduce the asymptotics of the perturbed problem (9.17) to the asymptotics of the non-perturbed one

$$\theta_t = A(\theta),$$

three basic conditions are imposed in the paper [17]. The first of them concerns the compactness of the orbits in the class $S$ of the perturbed equation

$$\theta_t = B(\theta, \tau)$$

in $X$. It holds in our case, as a consequence of estimates (E1)–(E4) (more exactly, their translation in terms of $\theta$). (E4) implies equicontinuity of the set $[\theta^q(\tau)]_{\tau > 0} : [\tau_1, \tau_2] \to L^1(\mathbb{R}^N)$ with $0 \leq \tau_1 < \tau_2$, hence compactness in $L^1_{\text{loc}}([0, +\infty); L^1(\mathbb{R}^N))$.

The second condition, i.e. the fact that the limits of all the convergent subsequences $[\theta(\tau + s_i)]$ in $L^1_{\text{loc}}([0, +\infty); L^1(\mathbb{R}^N))$ with $s_i \to \infty$ and $\theta \in S$ are solutions of (9.20), has been proved above.

Finally, the third condition of uniform Lyapunov stability of the global $\omega$-limit set of (9.20) (which in our case reduces to a point: $\Omega = \{F_M\}$, where $F_M$ denotes the profile of $U_M$), is a consequence of the $L^1$-contraction property enjoyed by the solutions of (1.1) shown in Section 3 (this property is inherited by (9.20)). Therefore, the main result of [17, Theorem 3], is valid in our case. That is, the $\omega$-limit set of a solution of (9.21) in the class $S$ is contained in $\Omega$. Therefore, $\theta(\xi, t) \to F_M$ as $t \to \infty$, which is the desired result.
For general initial data (with non-compact support) a final density argument is needed. Since it follows the lines of the argument used in Section 5, we omit the details. □

Appendix on uniqueness

We discuss in some detail the uniqueness of the solutions to Eq. (9.1) which includes as a particular case (1.1). Our first result uses Kalashnikov’s duality proof for the porous media equation, which has been adapted by Hui [18] to the power case (1.1) and works for more general nonlinearities. See also [5,29]. It is to be noted that no regularity is assumed on $ut$.

**Theorem A1.** Suppose that $\Phi$ satisfies assumption (H1) of Section 8 with $m \geq 1$ and $F$ satisfies (H2) with $q > (m + 1)/2$. Then for every $u_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ there exists at most one function $u \in L^\infty(0, T; L^1(\mathbb{R}^N)) \cap L^\infty(Q_T)$ satisfying Eq. (9.1) in the sense:

$$\text{(A.1)} \quad \int_0^T \int_\Omega \left\{ \nabla \Phi(u) \cdot \nabla \phi - F \cdot \nabla \phi - u \phi_t \right\} \, dx \, dt = \int_\Omega u_0(x) \phi(x, 0) \, dx - \int_\Omega u(x, t) \phi(x, t) \, dx.$$  

**Proof.** Suppose $u_1$ and $u_2$ are two solutions with initial data $u_{10}$ and $u_{20}$ respectively. Take a test $\eta \in C^\infty_0(B_R(0) \times [0, T])$, $R > 0$, such that $\eta = 0$ on $\partial B_R(0) \times [0, t]$. Subtracting (A.1) for $u_1$ and $u_2$ and integrating by parts, we obtain

$$\text{(A.2)} \quad \int_{B_R(0)} (u_1 - u_2)(x, T) \eta(x, T) \, dx = \int_{B_R(0)} (u_{10} - u_{20})(x) \eta(x, 0) \, dx + \int_0^T \int_{B_R(0)} (u_1 - u_2)(x, t) \Delta \eta \, dx \, dt,$$

where

$$\text{(A.3)} \quad A = \begin{cases} \frac{\Phi(u_1) - \Phi(u_2)}{u_1 - u_2} & \text{for } u_1 \neq u_2, \\ \Phi'(u_1) & \text{for } u_1 = u_2, \end{cases}$$

and

$$\text{(A.4)} \quad B = \begin{cases} \frac{F(u_1) - F(u_2)}{u_1 - u_2} & \text{for } u_1 \neq u_2, \\ F'(u_1) & \text{for } u_1 = u_2. \end{cases}$$

Let $C_1$ be such that

$$\|u_1\|_{L^\infty}, \|u_2\|_{L^\infty} \leq C_1.$$  

Then, applying the Cauchy mean value theorem to $A$ and $B$, and taking into account the conditions imposed on $\Phi$ and $F$, there exist constants $k_1$ and $k_2$ (depending on $d_1$ and $c_1$) such that

$$\text{(A.5)} \quad \frac{(B^j)^2}{2A} \leq k_1 C_1^{2q-m-1}, \quad \frac{k_2}{B^j} \leq k_2 C_1^{q-m}, \quad \forall j,$$

(here $B^j$ represents the $j$-th coordinate of vector $B$.) Next, we construct suitable smooth approximations of $A$ and $B$ in $B_R \times (0, T))$. As in [18] we take sequences of smooth functions.
\[ \begin{align*}
0 < c_i & \leq A_{i,R} \leq d_2 C_1^{m-1}, \\
0 & \leq B_{i,R}^j \leq c_1 C_1^{q-1} + 1, \\
B_{i,R}^j & \leq \frac{k_1 C_1^{2q-m-1} + 1}{A_{i,R}} = C_2, \\
B_{i,R}^j & \leq \frac{k_2 C_1^{q-m} + 1}{A_{i,R}} = C_3.
\end{align*} \]

and, moreover, the following convergences take place:

\[ \frac{(A_{i,R} - A)}{A_{i,R}^{1/2}} \to 0 \quad \text{and} \quad B_{i,R}^j - B^j \to 0 \]

in \( L^2(B_R(0) \times (0, T)) \) as \( i \to \infty \) for all \( R > 0 \). Next we consider the dual problem:

\[ \begin{align*}
\eta_t + A_{i,R} \Delta \eta + B_{i,R} \eta - \lambda \eta & = 0 \quad \text{for} \ (x, s) \in B_R(0) \times (0, T), \\
\eta(x, s) & = 0 \quad \text{for} \ (x, s) \in \partial B_R(0) \times (0, T), \\
\eta(x, T) & = \theta(x) \quad \text{for} \ x \in B_R(0),
\end{align*} \]

where \( \theta \in C_0^\infty(B_R(0)), 0 \leq \theta \leq 1 \). Problem (A.9) has a unique smooth solution \( \eta_{i,R} \). By the maximum principle, \( 0 \leq \eta_{i,R} \leq 1 \). We need estimates of \( \Delta \eta_{i,R} \) and \( \nabla \eta_{i,R} \) in \( L^2(B_R(0) \times (0, T)) \). Such estimates are obtained in [2]. The basic estimate is the following:

\[ \begin{align*}
\int_0^T \int_{B_R(0)} A_{i,R}(\Delta \eta_{i,R})^2 \, dx \, ds + 2(\lambda - C_2) \int_0^T \int_{B_R(0)} \| \nabla \eta_{i,R} \|^2 \, dx \, ds \leq \int \int_{B_R(0)} \| \nabla \theta \|^2 \, dx.
\end{align*} \]

As in [18] (see also [29]), we introduce two suitable super-solutions of (A.9). The first of them has the form

\[ g(x, s) = e^{h(x)} \left( \frac{1 + R_0^2}{1 + |x|^2} \right)^\beta, \]

where \( h(x, s) = C'(T - s) \) and \( \beta > 0 \). Choosing \( C' \) large enough, it is easy to check that \( g \) is a super-solution of the dual equation and, moreover, it is greater than \( \eta_{i,R} \) in the parabolic boundary. Hence, by the maximum principle [26], \( g \geq \eta_{i,R} \) in \( B_R(0) \times (0, T) \).

The second super-solution has the form

\[ g^*(x, s) = a e^{h(x)} \Gamma(|x|), \]

where

\[ \Gamma(r) = (R - r) - C_3(R - r)^2. \]

Again, it is easy to show that \( g^* \) is a super-solution of (A.9) in the set \( B_\alpha = \{ R - \alpha < r < R, 0 < s < T \} \) if we take \( \alpha = 1/2(C_3 + n - 1) \). Moreover, if we put

\[ a = \left( 1 + R_0^2 \right) \beta / \left( \Gamma(R - \alpha) \left( 1 + (R - \alpha)^2 \right)^\beta \right), \]

we get the condition

\[ g^* = g \quad \text{in} \ \partial B_{R-\alpha} \times (0, T). \]
Therefore, since in the exterior boundary of $B_R$ obviously $\eta_{1,R} = 0 = g^*$, by the maximum principle we have $g^* \geq \eta_{1,R}$ in $B_R$. Then, the following estimate on the normal derivative of $\eta_{1,R}$ holds:

\[(A.16) \quad \|\partial \eta_{1,R}/\partial n\|_{L^\infty((\partial B_R(0)) \times (0,T)} \leq \|g^*/\partial v\|_{L^\infty((\partial B_R(0)) \times (0,T)}) \leq CR^{-2\beta}.
\]

Now, putting $\eta = \eta_{1,R}$ in (A.2), using (A.16) and (A.10) we proceed as in [18, p. 1691], thus obtaining, for all $\theta \in C_0^\infty(B_R(0))$, $0 \leq \theta \leq 1$, $R > 2$:

\[
\int_{B_R(0)} (u_1 - u_2) (x,s) \theta(x) \, dx \leq \int_{R^n} (u_{10} - u_{20})_+ (x) \, dx + 2C \left( \frac{A_i,R - A_i}{A_{i,R}} \right) \| \nabla \theta \|_{L^2} \\
+ (2C_1/(2(\lambda - C_2)))^{1/2} \|B_{i,R} - B\|_{L^2} \| \nabla \theta \|_{L^2} \\
+ \lambda \int_0^T \int_{R^n} (u_1 - u_2)_+ \, dx \, ds + CR^{n-1-2\beta} T.
\]

(A.17)

Choosing now $\beta = n/2$ and letting first $i \rightarrow \infty$ and then $R \rightarrow \infty$, $\lambda \rightarrow C_2$, we get the following:

\[
(A.18) \quad \int_{R^n} (u_1 - u_2)(x,s) \theta(x) \, dx \leq \int_{R^n} (u_{01} - u_{02})_+ \, dx + C \int_0^T \int_{R^n} (u_1 - u_2)_+ \, dx \, ds.
\]

Next we put $\theta = \chi_{[u_1 \leq u_2]} \ast \rho_\varepsilon$, where $\rho_\varepsilon$ denotes a sequence of mollifiers. Letting now $\varepsilon \rightarrow 0$ and $R \rightarrow \infty$ we get

\[
(A.19) \quad \int_{R^n} (u_1 - u_2)_+ (x,s) \, dx \leq \int_{R^n} (u_{01} - u_{02})_+ \, dx + C \int_0^T \int_{R^n} (u_1 - u_2)_+ \, dx \, ds \quad \forall s \in (0, T).
\]

After a standard application of Gronwall’s inequality, and a similar argument for the negative part of $u_1 - u_2$, we obtain the desired result:

\[
(A.20) \quad \int_{R^n} |u_1 - u_2|(x,s) \, dx \leq e^{C\beta} \int_{R^n} |u_{01} - u_{02}| \, dx.
\]

We want to extend this result to the class of unbounded solutions considered in our paper.

**THEOREM A2.** – There exists at most one solution to Eq. (9.1) in the sense of Definition 9.1. Moreover, the map $u_0 \mapsto u(\cdot, t)$ is a contraction in $L^1(R^N)$.

**Proof.** – Take $u_0 \in L^1$ and let $u_1(t)$ and $u_2(t)$ be two solutions. Fixed $\varepsilon > 0$ by part (W2) of the definition of solution we can choose $\tau$ small enough, such that

\[
(A.21) \quad \|u_i(t) - u_0\|_{L^1} < \varepsilon \quad \text{for } i = 1, 2.
\]

Hence, $\|u_1(t) - u_2(t)\|_{L^1} < 2\varepsilon$ for $\tau$ small enough. Taking now $u_1(\tau)$ and $u_2(\tau)$ as initial data, by Theorem A1 we will have:

\[
(A.22) \quad \|u_1(t) - u_2(t)\|_{L^1} < 2e^{C(\tau-t)} \varepsilon \quad \forall t > \tau.
\]
Since \( \varepsilon \) is arbitrary, the previous inequality entails

\[
(A.23) \quad u_1(t) = u_2(t) \text{ a.e.}
\]

Therefore, all solutions coincide with the ones we have constructed before, for which we can prove that

\[
(A.24) \quad \int_{\mathbb{R}^N} |u_1 - u_2|(x, t)\,dx \leq \int_{\mathbb{R}^N} |u_{10} - u_{20}|(x)\,dx.
\]

This result is proved by Otto [27] for bounded domains and we have obtained it in the limit. \( \square \)

**Comment.** Alternative proofs of uniqueness use Kruzhkov's ideas [22] and apply to much more general diffusive-convective equations. We refer to Carrillo’s unpublished work [8] for a very general result which covers also entropy solutions for purely convective equations (in bounded domains). See also [27], which is used in several instances in this paper. The present duality method works only for diffusion-dominated equations but has the advantage of being comparatively simpler. On the other hand, a number of uniqueness proofs are based on the extra assumption that \( u \) is a function of bounded variation, cf. [34,15].

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