Uniqueness of positive solutions for a class of elliptic systems

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Received 6 August 2005
Available online 4 November 2005
Submitted by G. Bluman

Abstract

In this article, we consider uniqueness of positive radial solutions to the elliptic system \( \Delta u + a(|x|) f(u, v) = 0 \), \( \Delta v + b(|x|) g(u, v) = 0 \), subject to the Dirichlet boundary condition on the open unit ball in \( \mathbb{R}^N \) (\( N \geq 2 \)). Our uniqueness results applies to, for instance, \( f(u, v) = u^p v^q \), \( g(u, v) = u^p v^q \), \( p, q > 0 \), \( p + q < 1 \) or more general cases.

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Keywords: Elliptic system; Positive solutions; Uniqueness

1. Introduction

Existence and nonexistence of positive radial solutions for the elliptic system

\[
\begin{align*}
\Delta u + a(|x|) f(u, v) &= 0, \\
\Delta v + b(|x|) g(u, v) &= 0,
\end{align*}
\]

with different boundary conditions has received much attention recently [1–4]. Meanwhile, the uniqueness of positive radial solutions of the well-known Lane–Emden system (the special case of system (1.1)),

\[
\Delta u = -v^q, \quad \Delta v = -u^p,
\]

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Supported by the NSF of Gansu Province (No. 3ZS051-A25-016), NWNU-KJCXGC, the Foundation of excellent Young Teacher of the Chinese Education Ministry, Spring-Sun Program (No. Z2004-1-62033).

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has been studied clearly, see [5] and the references therein. However, there are very few results on the uniqueness of positive radial solutions of system (1.1). Provided that  
\[ a(|x|) = \lambda, \quad b(|x|) = \mu, \quad f(u, v) = f(v), \quad g(u, v) = g(u), \]  
Dalmasso [6] investigated the existence and uniqueness of positive radial solutions of the boundary value problem

\[ \begin{align*}
\Delta u &= -\lambda f(v) \quad \text{in } B, \\
\Delta v &= -\mu g(u) \quad \text{in } B, \\
u &= v = 0 \quad \text{on } \partial B,
\end{align*} \]

(1.2)

where \( B \) is the open unit ball in \( \mathbb{R}^N (N \geq 2) \), \( f, g : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \) are continuous functions, and \( a, b : \mathbb{R}^+ \to (0, \infty) \) are continuous functions, \( R^+ = [0, +\infty) \). He obtained some results in the sublinear case when \( f, g \) are nondecreasing and there exist positive numbers \( p, q \) with \( pq < 1 \) such that

\[ \frac{f(x)}{x^q} \quad \text{and} \quad \frac{g(x)}{x^p} \quad \text{are nonincreasing on } \mathbb{R}^+. \]

D.D. Hai [7] considered system (1.2) and extended (H) to hold only for large \( x \). Similar results in the case of a single equation were obtained in [8,9]. On superlinear problems associated with (1.2), we refer to [10,11] for literature.

Motivated by [1–9] and the references therein, we consider the uniqueness of positive solutions of the general problem

\[ \begin{align*}
\Delta u + a(|x|) f(u, v) &= 0 \quad \text{in } B, \\
\Delta v + b(|x|) g(u, v) &= 0 \quad \text{in } B, \\
u &= v = 0 \quad \text{on } \partial B,
\end{align*} \]

(1.3)

where \( B \) is the open unit ball in \( \mathbb{R}^N (N \geq 2) \), \( f, g : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \) are continuous and \( a, b : \mathbb{R}^+ \to (0, \infty) \) are continuous functions.

In this paper we will show that, under appropriate conditions, system (1.3) admits at most one positive radial solution. Our main result is new.

According to a result of Troy [12], positive solutions of (1.3) are radially symmetric and decreasing. It leads us to write system (1.3) in the form

\[ \begin{align*}
(r^{N-1} u')' &= -a(r) r^{N-1} f(u(r), v(r)), \quad r \in (0, 1), \\
(r^{N-1} v')' &= -b(r) r^{N-1} g(u(r), v(r)), \quad r \in (0, 1), \\
u'(0) &= v'(0) = u(1) = v(1) = 0.
\end{align*} \]

(1.4)

From now on, we investigate uniqueness of positive solutions for system (1.4).

For \((x, y) \in \mathbb{R}^2\), denote \(|(x, y)| = \sqrt{x^2 + y^2}\). We make the following assumptions:

\begin{enumerate}
\item[(H1)] \( f, g : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+ \) are continuous, nondecreasing in each variable for any value of the other variable, and \((C^1)^2\) (i.e., \( f, g \) having continuous first partial derivatives) on \((0, \infty) \times (0, \infty)\) and

\[ \lim_{|(x, y)| \to 0} \sup (xf_x + yf_y) < \infty, \quad \lim_{|(x, y)| \to 0} \sup (xg_x + yg_y) < \infty. \]

\item[(H2)] There exist nonnegative numbers \( p, q, A, B \) with \( A, B > 0, \ p + q < 1 \) such that

\[ \lim_{\min\{|x|, |y|\} \to \infty} \frac{f(x, y)}{x^q y^p} = A, \quad \lim_{\min\{|x|, |y|\} \to \infty} \frac{g(x, y)}{x^p y^q} = B, \]

and for \( p_1 > p, q_1 > q, \ p_1 + q_1 < 1, \)
are nonincreasing with respect to every variable for large x and y.

(H3) There exist positive constants $K_1, K_2$ such that
$$f(x, y) \geq K_1 x^q y^p, \quad g(x, y) \geq K_2 x^p y^q.$$  

(H4) $a, b : \mathbb{R}^+ \to (0, \infty)$ are continuous functions. Let $a_0 = \min_{x \in [0, 1]} a(x) > 0, b_0 = \min_{x \in [0, 1]} b(x) > 0$ and $a_1 = \max_{x \in [0, 1]} a(x) > 0, b_1 = \max_{x \in [0, 1]} b(x) > 0$. There exist positive numbers $L_1, L_2$ independent of $a(\cdot), b(\cdot)$ such that
$$\frac{a_1}{a_0} \leq L_1, \quad \frac{b_1}{b_0} \leq L_2.$$

Our main result is:

**Theorem 1.** Assume (H1)–(H4) hold. Then there exists a positive number $\eta$ such that system (1.4) has at most one positive solution for
$$\min\left(a_1 - q b_0, a_0 b_1 - q b_0, \frac{1}{a_0 (1 - q) b_0}ight) \geq \eta.$$

**Remark 2.** Theorem 1 is a new result for system (1.4), and correspondingly, we will obtain the uniqueness of positive radial solutions for system (1.3).

The rest of this paper is organized as follows. In Section 2, we introduce some preliminary lemmas. Theorem 1 is proved in Section 3. Finally, in Section 4 we give some related remarks and examples to which our theorems apply.

### 2. Preliminary results

**Lemma 3.** Let $h$ be continuous on $\mathbb{R}^+ \times \mathbb{R}^+$ and $(C^1)^2$ on $(0, \infty) \times (0, \infty)$ such that
$$\lim_{|(x,y)| \to 0} \sup_{(x,y)} (xh_x + yh_y) < \infty.$$  

Let $M, \epsilon, r$ be positive numbers with $\epsilon < 1$. Then there exists a positive number $C$ such that
$$|h(\gamma x, \gamma y) - \gamma^r h(x, y)| \leq C (1 - \gamma)$$
for $\epsilon \leq \gamma < 1$ and $0 \leq |(x, y)| \leq M$.

**Proof.** Let $0 \leq |(x, y)| \leq M$. Define
$$H(\gamma) = h(\gamma x, \gamma y) - \gamma^r h(x, y) \quad (\epsilon \leq \gamma < 1).$$

By the mean value theorem, there exists $c \in (\gamma, 1)$, such that
$$|H(\gamma)| = |H(\gamma) - H(1)|$$
$$= (1 - \gamma) \left|xh_1'(cx, cy) + yh_2'(cx, cy) - rc^{r-1}h(x, y)\right|$$
$$= (1 - \gamma) \frac{1}{c} \left[cxh_1'(cx, cy) + cyh_2'(cx, cy)\right] - rc^{r-1}h(x, y)$$
$$\leq C (1 - \gamma),$$
where
\[ C = \sup_{\varepsilon} \{ |xh'_x(x, y) + yh'_y(x, y)|: 0 < |(x, y)| \leq M \} \]
\[ + r \max(\varepsilon^{-1}, 1) \sup \{|h(x, y)|: 0 \leq |(x, y)| \leq M \}. \]

The following lemma establishes upper and lower estimates for possible positive solutions of system (1.4).

**Lemma 4.** Let \((u, v)\) be a positive solution of system (1.4). Then there exist positive constants \(M_i, i = 1, 2, 3, 4\) and \(\eta\) independent of \(u, v\), such that
\[ M_1(a_0^{1-q} b_0^p)^{\frac{1}{1-q^2-p^2}} (1 - r) \leq u(r) \leq M_2(a_0^{1-q} b_0^p)^{\frac{1}{1-q^2-p^2}} (1 - r), \]
\[ M_3(a_0^p b_0^{1-q})^{\frac{1}{1-q^2-p^2}} (1 - r) \leq v(r) \leq M_4(a_0^p b_0^{1-q})^{\frac{1}{1-q^2-p^2}} (1 - r), \]
for \(\min(a_0^{1-q} b_0^p, a_0^p b_0^{1-q}) \geq \eta\) and \(0 < r < 1\).

**Proof.** Let \((u, v)\) be a positive solution of the system (1.4). By integrating two equations in (1.4) we obtain respectively,
\[ u(r) = \int_r^1 \frac{1}{s^{N-1}} \left( \int_0^s a(\tau) \tau^{N-1} f(u, v) d\tau \right) ds, \]
(2.1)
\[ v(r) = \int_r^1 \frac{1}{s^{N-1}} \left( \int_0^s b(\tau) \tau^{N-1} g(u, v) d\tau \right) ds. \]
(2.2)

From now on, we shall denote by \(C_i, i = 1, 2, \ldots, \) positive constants independent of \(u, v, a_0, b_0\).

Since \(u, v\) are decreasing and \(f(x, y)\) is nondecreasing with respect to every variable \(x, y,\)
\[ u \left( \frac{1}{2} \right) \geq \int_{1/2}^1 \frac{1}{s^{N-1}} \left( \int_0^{1/2} a(\tau) \tau^{N-1} f(u, v) d\tau \right) ds \geq \frac{1}{2} a_0 f \left( u \left( \frac{1}{2} \right), v \left( \frac{1}{2} \right) \right) \int_0^{1/2} \tau^{N-1} d\tau \]
\[ = \frac{a_0}{N 2^{N+1}} f \left( u \left( \frac{1}{2} \right), v \left( \frac{1}{2} \right) \right). \]
(2.3)

Similarly,
\[ v \left( \frac{1}{2} \right) \geq \frac{b_0}{N 2^{N+1}} g \left( u \left( \frac{1}{2} \right), v \left( \frac{1}{2} \right) \right). \]
(2.4)

By (H3), we have
\[ f(x, y) \geq K_1 x^q y^p, \]
(2.5)
\[ g(x, y) \geq K_2 x^p y^q. \]
(2.6)

Both (2.5) and (2.6) together with (2.3), (2.4) give us
\[ u \left( \frac{1}{2} \right) \geq \frac{a_0}{N 2^{N+1}} f \left( u \left( \frac{1}{2} \right), v \left( \frac{1}{2} \right) \right) \geq \frac{a_0 K_1}{N 2^{N+1}} u^q \left( \frac{1}{2} \right) v^p \left( \frac{1}{2} \right) \]
(2.7)
and
\[
v\left(\frac{1}{2}\right) \geq \frac{b_0}{N2^{N+1}} f\left(\frac{1}{2}, \frac{1}{2}\right) \geq \frac{b_0 K_2}{N2^{N+1}} v^q\left(\frac{1}{2}\right).
\] (2.8)

Solving the inequalities (2.7), (2.8), we obtain
\[
u^{1-q}\left(\frac{1}{2}\right) \geq \frac{a_0 K_1}{N2^{N+1}} v^p\left(\frac{1}{2}\right) \geq \frac{a_0 K_1}{N2^{N+1}} \left(\frac{b_0 K_2}{N2^{N+1}}\right)^\frac{p}{q} u^{\frac{2}{q}}\left(\frac{1}{2}\right).
\] (2.9)

It follows that
\[
\left[u\left(\frac{1}{2}\right)\right]^{-\frac{1}{1-q}} - \frac{1}{1-q} = \left[u\left(\frac{1}{2}\right)\right]^{\frac{1}{(1-q)^2-p^2}} \geq \frac{a_0 K_1}{N2^{N+1}} \left(\frac{b_0 K_2}{N2^{N+1}}\right)^\frac{p}{q}.
\] (2.10)

Note that \(p + q < 1\), that is, \((1 - q)^2 - p^2 > 0\), we have
\[
u\left(\frac{1}{2}\right) \geq \left[\frac{a_0 K_1}{N2^{N+1}} \left(\frac{b_0 K_2}{N2^{N+1}}\right)^\frac{p}{q}\right]^{-\frac{1}{(1-q)^2-p^2}} = C_1 \left[a_0^{1-q} b_0^p\right]^{-\frac{1}{(1-q)^2-p^2}}.
\] (2.11)

In a similar manner, we get
\[
v\left(\frac{1}{2}\right) \geq C_2 \left[a_0^p b_0^{1-q}\right]^{-\frac{1}{(1-q)^2-p^2}}.
\] (2.12)

Considering the case \(r \geq \frac{1}{2}\), it follows from (2.11), (2.12)
\[
u'\left(\frac{1}{2}\right) = \int_0^r a(s)s^{N-1} f\left(u(s), v(s)\right) ds \geq a_0 \int_0^{1/2} s^{N-1} f\left(u(s), v(s)\right) ds
\]

\[
\geq \frac{a_0}{N2^N} f\left(u\left(\frac{1}{2}\right), v\left(\frac{1}{2}\right)\right) \geq \frac{a_0 K_1}{N2^N} u^q\left(\frac{1}{2}\right) v^p\left(\frac{1}{2}\right)
\]

\[
\geq \frac{K_1 C_2^p}{N2^N} a_0 \left[a_0^{1-q} b_0^p\right]^{-\frac{q}{(1-q)^2-p^2}} \left[a_0^p b_0^{1-q}\right]^{-\frac{1}{(1-q)^2-p^2}}
\]

\[
= C_3 \left[a_0^{1-q} b_0^p\right]^{-\frac{1}{(1-q)^2-p^2}},
\] (2.13)

and after integrating,
\[
u\left(\frac{1}{2}\right) \geq C_3 \left[a_0^{1-q} b_0^p\right]^{-\frac{1}{(1-q)^2-p^2}} (1 - r)
\] for every \(r \geq \frac{1}{2}\).

Similarly,
\[
v\left(\frac{1}{2}\right) \geq C_4 \left[a_0^p b_0^{1-q}\right]^{-\frac{1}{(1-q)^2-p^2}} (1 - r)
\] for every \(r \geq \frac{1}{2}\).

Since \(u, v\) are decreasing, this implies that there exist positive constants \(M_1, M_3\) independent of \(u, v\) such that the left-side inequalities for \(u, v\) in Lemma 4 hold.

From (2.1), (2.2), we have
\[
|u|_0 \leq a_1 f\left(|u|_0, |v|_0\right), \quad |v|_0 \leq b_1 g\left(|u|_0, |v|_0\right),
\] (2.16)
where $| \cdot |_0$ denotes the sup-norm.

By (2.14), (2.15), for large $a_0^{1-q} b_0^p$ and $a_0 b_0^{1-q}$,

$$|u_0| \geq C_5 \left[ \frac{a_0^{1-q} b_0^p}{a_0} \right]^{\frac{1}{(1-q)^2-p^2}} \gg 1 \quad (\text{i.e., } |u_0| \text{ is large}),$$

$$|v_0| \geq C_6 \left[ \frac{a_0 b_0^{1-q}}{a_0} \right]^{\frac{1}{(1-q)^2-p^2}} \gg 1 \quad (\text{i.e., } |v_0| \text{ is large}).$$

Therefore, by the conditions

$$\lim_{\min \{ |x|, |y| \} \to \infty} \frac{f(x, y)}{x^q y^p} = A > 0, \quad \text{and} \quad \lim_{\min \{ |x|, |y| \} \to \infty} \frac{g(x, y)}{x^p y^q} = B > 0$$

in (H2), we have from (2.16) that

$$|u_0| \leq a_1 f \left( |u_0|, |v_0| \right) \leq a_1 C_7 |u_0|^q |v_0|^p,$$

$$|v_0| \leq b_1 g \left( |u_0|, |v_0| \right) \leq b_1 C_8 |u_0|^p |v_0|^q.$$  \hfill (2.17)

It follows that

$$|u_0| \leq \left[ a_1 C_7 |v_0|^p \right]^{\frac{1}{1-q}}, \quad |v_0|^p \leq \left[ b_1 C_8 |u_0|^p \right]^{\frac{1}{1-q}}.$$  \hfill (2.18)

Consequently, we obtain

$$|u_0| \leq \left[ \frac{a_1}{a_0} a_0 C_7 \right]^{\frac{1}{1-q}} \left[ \frac{b_1}{b_0} b_0 C_8 |u_0|^p \right]^{\frac{p}{(1-q)^2-p^2}} \leq (L_1 C_7)^{\frac{1}{1-q}} (L_2 C_8)^{\frac{p}{(1-q)^2-p^2}} a_0 b_0^{2(1-q)^2} \left| u_0 \right|^{\frac{p}{(1-q)^2}} |u_0|^p$$

or

$$|u_0| \leq C_9 a_0^{\frac{1}{1-q}} b_0^{\frac{p}{(1-q)^2}} \left| u_0 \right|^{\frac{p}{(1-q)^2-p^2}},$$

or

$$|v_0| \leq C_9 a_0^{\frac{1}{1-q}} b_0^{\frac{p}{(1-q)^2-p^2}}.$$  \hfill (2.19)

Similarly,

$$|v_0| \leq C_{11} \left[ a_0^p b_0^{1-q} \right]^{\frac{1}{(1-q)^2-p^2}}.$$  \hfill (2.20)

Recall back to the equation with respect to $u'$ in (2.13) and the condition (H2), we get

$$-u'(r) \leq a_1 f \left( |u_0|, |v_0| \right) \leq C_7 \frac{a_1}{a_0} a_0 |u_0|^q |v_0|^p$$

$$\leq C_7 L_1 C_7^{\frac{q}{p}} C_8^{\frac{p}{(1-q)^2-p^2}} a_0 b_0^{2(1-q)^2} \left[ a_0^p b_0^{1-q} \right]^{\frac{1}{(1-q)^2-p^2}}$$

or

$$u'(r) \leq C_{12} \left[ a_0^{1-q} b_0^p \right]^{\frac{1}{(1-q)^2-p^2}} (1-r), \quad 0 < r < 1.$$  \hfill (2.21)

In a similar manner, we have the upper estimate for $v(r)$. This completes the proof of Lemma 4. \hfill \Box
3. Proof of Theorem 1

Let \((u, v)\) and \((u_1, v_1)\) be positive solutions of system (1.4) and let \(\min(a_0^{1-q}b_0^p, a_0^{p}b_0^{1-q})\) be large enough so that Lemma 4 applies. By Lemma 4, one has

\[
\frac{M_1}{M_2} u_1 \leq u \leq \frac{M_2}{M_1} u_1 \quad \text{and} \quad \frac{M_3}{M_4} v_1 \leq v \leq \frac{M_4}{M_3} v_1,
\]

\(t \in (0, 1)\).

Let \(\alpha = \sup \{c > 0: u \geq cu_1 \text{ in } (0, 1)\}\) and \(\beta = \sup \{d > 0: v \geq dv_1 \text{ in } (0, 1)\}\). Then obviously \(\alpha_0 \leq \alpha < \infty, \beta_0 \leq \beta < \infty\) and \(u \geq \alpha u_1, v \geq \beta v_1 \text{ in } (0, 1)\), where \(\alpha_0 = \frac{M_1}{M_2} > 0, \beta_0 = \frac{M_3}{M_4} > 0\).

We claim that \(\alpha \geq 1, \beta \geq 1\).\(^{(3.1)}\)

Without loss of generality, we may assume that \(\alpha \leq \beta\), then we need only to prove that \(\alpha \geq 1\). Suppose to the contrary that \(\alpha < 1\). Since

\[
(r^{N-1}u)' = -a(r)r^{N-1}f \left( \int \frac{1}{s^{N-1}} \left( \int a(\tau)\tau^{N-1}f(u, v)\,d\tau \right) ds \right),
\]

and

\[
(r^{N-1}u_1)' = -a(r)r^{N-1}f \left( \int \frac{1}{s^{N-1}} \left( \int a(\tau)\tau^{N-1}f(u_1, v_1)\,d\tau \right) ds \right),
\]

and \(v \geq \beta v_1 \geq \alpha v_1\), it is clear that

\[
\left[r^{N-1}(u - \alpha u_1)\right]' \leq -a(r)r^{N-1} \left[ f \left( \int \frac{1}{s^{N-1}} \left( \int a(\tau)\tau^{N-1}f(\alpha u_1, \alpha v_1)\,d\tau \right) ds \right) \right.
\]

\[
\int \frac{1}{s^{N-1}} \left( \int b(\tau)\tau^{N-1}g(\alpha u_1, \alpha v_1)\,d\tau \right) ds \bigg) - \alpha f \left( \int \frac{1}{s^{N-1}} \left( \int a(\tau)\tau^{N-1}f(u_1, v_1)\,d\tau \right) ds \right) \bigg]
\]

\[
\int \frac{1}{s^{N-1}} \left( \int b(\tau)\tau^{N-1}g(u_1, v_1)\,d\tau \right) ds \bigg).\] (3.2)
Let \( q_1 > q_2 > q, \ p_1 > p_2 > p \) and \( p_1 + q_1 < 1 \). We claim that

\[
\int_0^s a(\tau)\tau^{N-1} f(a u_1, a v_1) \, d\tau \geq \alpha^{p_1+q_1} \int_0^s a(\tau)\tau^{N-1} f(u_1, v_1) \, d\tau, \quad s \geq 0, \quad (3.3)
\]

\[
\int_0^s b(\tau)\tau^{N-1} g(a u_1, a v_1) \, d\tau \geq \alpha^{p_1+q_1} \int_0^s b(\tau)\tau^{N-1} g(u_1, v_1) \, d\tau, \quad s \geq 0. \quad (3.4)
\]

In fact, since \( \alpha \geq \alpha_0 > 0 \), and \( f(x, y) \) is nonincreasing with respect to every variable for large \( x \) and \( y \), we obtain

\[
\frac{f(\alpha x, \alpha y)}{(\alpha x)^{q_2}(\alpha y)^{p_2}} \geq \frac{f(x, y)}{x^{q_2}y^{p_2}}.
\]

It gives us

\[
f(\alpha x, \alpha y) \geq \alpha^{q_2+p_2} f(x, y) \quad \text{for large } x \text{ and } y.
\]

Let \( \frac{1}{2} \leq T < 1 \). By Lemma 4

\[
u_1(s) \geq M_1(1-T)(a_0^{1-q} b_0^{p})^{\frac{1}{1-(q_2+p_2)}} \gg 1, \quad s \leq T,
\]

and

\[
v_1(s) \geq M_3(1-T)(a_0^{p} b_0^{1-q})^{\frac{1}{1-(q_2+p_2)}} \gg 1, \quad s \leq T,
\]

therefore

\[
\int_0^s a(\tau)\tau^{N-1} \left[ f(\alpha u_1, \alpha v_1) - \alpha^{q_1+p_1} f(u_1, v_1) \right] \, d\tau \geq 0, \quad s \leq T.
\]

For the case \( s > T \),

\[
\int_0^s a(\tau)\tau^{N-1} \left[ f(\alpha u_1, \alpha v_1) - \alpha^{q_1+p_1} f(u_1, v_1) \right] \, d\tau
\]

\[
= \int_0^T a(\tau)\tau^{N-1} \left[ f(\alpha u_1, \alpha v_1) - \alpha^{q_1+p_1} f(u_1, v_1) \right] \, d\tau
\]

\[
+ \int_T^s a(\tau)\tau^{N-1} \left[ f(\alpha u_1, \alpha v_1) - \alpha^{q_1+p_1} f(u_1, v_1) \right] \, d\tau
\]

\[
\geq (\alpha^{q_2+p_2} - \alpha^{q_1+p_1}) \int_0^T a(\tau)\tau^{N-1} f(u_1, v_1) \, d\tau - C(1-\alpha)(1-T)L_1 a_0,
\]
where we have used Lemma 3 with \( f(x, y) = h(x, y) \). By \((2.5), (2.11)\) and \((2.12)\),
\[
\int_0^T a(\tau)\tau^{N-1} f(u_1, v_1) d\tau \geq \int_0^{1/2} a(\tau)\tau^{N-1} f(u_1, v_1) d\tau \geq \frac{a_0}{N2^N} f\left(\frac{1}{2}, \frac{1}{2}\right)
\]
\[
\geq \frac{a_0K_3}{N2^N} > 0,
\]
since there exists a positive number \( k_1 > 0 \) such that
\[
\alpha \gamma_2^q + p_2 - \alpha \gamma_1^q + p_1 \geq k_1(1 - \alpha) \quad \text{for } 0 < \alpha_0 \leq \alpha < 1,
\]
it follows that
\[
\int_0^s a(\tau)\tau^{N-1} \left[ f(\alpha u_1, \alpha v_1) - \alpha \gamma_1^q + p_1 f(u_1, v_1) \right] d\tau > 0, \quad s > T,
\]
if \( T \) is sufficiently close to 1. This proves the claim \((4.3)\). We can show \((4.4)\) similarly.

Inserting \((3.3), (3.4)\) into \((3.2)\) and integrating gives us
\[
z^{N-1}(u' - \alpha u')(z) \leq - \int_0^z B(\alpha, r) dr,
\]
where
\[
B(\alpha, r) = a(r)r^{N-1} \left[ \alpha \gamma_1^q + p_1 \int_r^1 \frac{1}{s^{N-1}} \left( \int_0^s a(\tau)\tau^{N-1} f(u_1, v_1) d\tau \right) ds \right.
\]
\[
- \alpha f\left( \int_r^1 \frac{1}{s^{N-1}} \left( \int_0^s b(\tau)\tau^{N-1} g(u_1, v_1) d\tau \right) ds \right)
\]
\[
\left. + \frac{1}{s^{N-1}} \left( \int_0^s b(\tau)\tau^{N-1} g(u_1, v_1) d\tau \right) \right].
\]

Using \((2.5)\) and Lemma 4, we obtain for \( r \leq T \),
\[
\int_r^1 \frac{1}{s^{N-1}} \left( \int_0^s a(\tau)\tau^{N-1} f(u_1, v_1) d\tau \right) ds
\]
\[
\geq a_0 \int_0^1 \frac{1}{s^{N-1}} \left( \int_0^T a(\tau)\tau^{N-1} f(u_1, v_1) d\tau \right) ds
\]
\[
\geq a_0 f(u_1(T), v_1(T)) \frac{T^N}{N} (1 - T) \geq a_0 \frac{T^N(1 - T)}{N} u_1^q(T) v_1^p(T)
\]
where \( c_1(T) = T^N (1 - T)^{p+q+1} M_1^p M_3^q K_1 / N \). In a similar way, we also have

\[
\int_0^1 \frac{ds}{s^{N-1}} \int_0^s b(\tau) \tau^{N-1} g(u_1, v_1) d\tau \geq c_2(T) (a_0 b_0^{1-q})^{(1-q)^2 - p^2} \gg 1.
\]

Since \( \alpha \geq \alpha_0 > 0 \), and \( f(x, y) / x^{\alpha} y^{\beta} \) is nonincreasing with respect to every variable for large \( x \) and \( y \), we obtain

\[
f(\alpha_{p+q} + q_1) \int_0^1 \frac{ds}{s^{N-1}} \int_0^s a(\tau) \tau^{N-1} f(u_1, v_1) d\tau, \quad \alpha_{p+q} \int_0^1 \frac{ds}{s^{N-1}} \int_0^s b(\tau) \tau^{N-1} g(u_1, v_1) d\tau.
\]

Note that there exists a positive number \( k_2 > 0 \) such that

\[
\alpha_{(p+q)}^2 - \alpha \geq k_2 (1 - \alpha) \quad \text{for } 0 < \alpha_0 \leq \alpha < 1,
\]

and therefore for \( r \leq T \),

\[
B(\alpha, r) \geq a(r) r^{N-1} (\alpha_{(p+q)}^2 - \alpha) f \left( \int_0^1 \frac{ds}{s^{N-1}} \int_0^s a(\tau) \tau^{N-1} f(u_1, v_1) d\tau, \right.
\]

\[
\geq r^{N-1} k_2 (1 - \alpha) K_1 c_1^q (T) c_2^p (T) a_0 (a_0 b_0^{1-q})^{(1-q)^2 - p^2} \gg 1 - \alpha
\]

\[
= c_3(T) r^{N-1} (a_0 b_0^{1-q})^{(1-q)^2 - p^2} (1 - \alpha) > 0.
\]

This proves that

\[ z^{N-1} (u' - \alpha u_1') (z) < 0, \quad 0 < z \leq T. \]

For \( z > T \), we have

\[
\int_0^{1/2} B(\alpha, r) dr \geq \int_0^1 B(\alpha, r) dr + \int_0^T B(\alpha, r) dr
\]

\[
\geq c_3(1/2) N^{2N} a_0 (a_0 b_0^{1-q})^{(1-q)^2 - p^2} (a_0 b_0^{1-q})^{(1-q)^2 - p^2} (1 - \alpha)
\]
\[-C(1-T)(1-\alpha)L_1 a_0 \]
\[= (1-\alpha)a_0 \left[ \frac{c_1(1/2)}{N/2^N} \left( a_0^{1-q} b_0^p \right)^{\frac{q}{1-q} - p} \left( a_0^p b_0^{1-q} \right)^{\frac{p}{1-q} - p^2} - C(1-T)L_1 \right] \]
\[> 0 \]

for large \( a_0^{1-q} b_0^p \) and \( a_0^p b_0^{1-q} \) and \( T \) sufficiently close to 1. Hence

\[(u'-\alpha u'_1)(z) < 0, \quad \text{for } 0 < z \leq 1.\]

It follows that there exists \( \tilde{\alpha} > \alpha \) such that \( u \geq \tilde{\alpha} u_1 \) in \((0, 1)\), a contradiction! Thus \( \alpha \geq 1 \) and moreover \( \beta \geq 1 \). Hence, \( u \geq u_1, v \geq v_1 \). Similarly, we can prove that \( u \leq u_1, v \leq v_1 \). Consequently, \( u = u_1, v = v_1 \) in \((0, 1)\), completing the proof of Theorem 1.

4. Related remarks and examples

If \( f(x, y) = m_1(x)w_1(y) \) and \( g(x, y) = m_2(x)w_2(y) \), we give below the precise statements and the results.

\[\text{(H1)' } m_i, w_i : R^+ \to R^+ \text{ are continuous, nondecreasing, and } C^1 \text{ on } (0, \infty) \text{ and }\]
\[
\lim_{(x, y) \to 0} \sup_{y > 0} (xm_i'(x)w_i(y) + ym_i(x)w_i'(y)) < \infty \quad (i = 1, 2).\]

\[\text{(H2)' } \text{There exist nonnegative numbers } p, q, A_i, B_i \quad (i = 1, 2) \quad \text{with } A_i, B_i > 0, p + q < 1 \text{ such that }\]
\[\lim_{x \to 0^+} \frac{m_i(x)}{x^q} > 0, \quad \lim_{y \to 0^+} \frac{m_i(x)}{y^p} > 0, \]
\[\lim_{x \to \infty} \frac{m_i(x)}{x^q} = A_i, \quad \lim_{y \to \infty} \frac{m_i(x)}{y^p} = B_i \quad (i = 1, 2), \]

and for \( p_1 > p, q_1 > q, p_1 + q_1 < 1, \)
\[
\frac{m_i(x)}{x^{q_1}} \quad \text{and} \quad \frac{m_i(x)}{y^{p_1}} \quad (i = 1, 2) \]

are nonincreasing for large \( x \) and \( y \), respectively.

**Theorem 5.** Assume (H1)', (H2)', and (H4) hold. Then there exists a positive number \( \eta \) such that system (1.4) has at most one positive solution for \( \min(a_0^{1-q} b_0^p, a_0^p b_0^{1-q}) \geq \eta \).

**Remark 6.** In Theorem 5, (H3) is not requested, since (H2)' implies (H2) and (H3).

We conclude this section with some examples.

**Example 7.** Assume \( f(x, y) = x^q y^p, \quad g(x, y) = x^p y^q \) where \( p, q > 0 \) with \( p + q < 1 \), and \( a(x) = \lambda > 0, b(x) = \mu > 0 \). Selecting \( L_1 = L_2 = 1 \). Clearly, for \( p_1 > p, q_1 > q, p_1 + q_1 < 1, \)
\[
\frac{f(x,y)}{x^{q_1}y^{p_1}} = x^{q-q_1} y^{p-p_1} \quad \text{and} \quad \frac{g(x,y)}{x^{p_1}y^{q_1}} = x^{p-p_1} y^{q-q_1} \]
are nonincreasing with respect to every variable \( x \) and \( y \) on \( R^+ \times R^+ \), and Theorem 5 implies the uniqueness of positive radial solutions of problem (1.4) with \( \min(a_0^{1-q} b_0^p, a_0^p b_0^{1-q}) = \min(\lambda^{1-q} \mu^p, \lambda^p \mu^{1-q}) \) large enough.
Example 8. Assume \( f(x, y) = m_1(x)w_1(y) \), \( g(x, y) = x^p y^q \), and

\[
m_1(x) = \begin{cases} e^x, & \text{if } x < D, \\ e^D \left(1 + \frac{x^q - D^q}{q D^q - 1}\right), & \text{if } x \leq D, \\
\end{cases}
\]

where \( p, q > 0 \), \( p + q < 1 \) and \( D = 2/p \). Let \( a(x) = \lambda > 0 \), \( b(x) = \mu > 0 \). Select \( L_1 = L_2 = 1 \). Note that there is no positive numbers \( q_1, p_1 \) with \( p_1 + q_1 < 1 \) such that \( m_1(x) x^{q_1} \) is nonincreasing on \( R^+ \), however, \((H1)', (H2)', and (H4) are hold. Hence, Theorem 5 gives the uniqueness of positive radial solutions of problem (1.4) with

\[
\min(a_1^{1-q} b_0^p, a_0^{p_1} b_0^{1-q}) = \min(\lambda^{1-q} \mu^p, \lambda^p \mu^{1-q})
\]

large enough.

Example 9. Assume \( f(x, y) = [\arctan(x + y) + 10]x^q y^p \), \( g(x, y) = x^p y^q \) where \( p, q > 0 \) with \( p + q < 1 \), and \( a(x) = \sin x + l_1 \), \( b(x) = \cos x + l_2 \) with \( l_1 > 1 \), \( l_2 > 1 \). Select \( L_1 = L_2 = 2 \). Similarly, there is no positive numbers \( q_1, p_1 \) with \( p_1 + q_1 < 1 \) such that \( f(x, y) x^{q_1} y^{p_1} \) is nonincreasing with respect to every variable on \( R^+ \times R^+ \), however, \((H1)–(H4) are satisfied to give the uniqueness of positive solutions to system (1.4) with

\[
\min(a_1^{1-q} b_0^p, a_0^{p_1} b_0^{1-q}) = \min(l_1^{1-q} l_2^p, l_1^p l_2^{1-q})
\]

large enough.

Acknowledgments

We thank the referees for carefully reading this paper and suggesting many valuable comments.

References