Tasoev’s continued fractions and Rogers–Ramanujan continued fractions

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Abstract

We show some new variations on Tasoev’s continued fractions $[0; a^k, \ldots, a^k]_{k=1}^\infty$, where the periodic parts include the exponentials in $k$ instead of the polynomials in $k$. We also mention some relations with other kinds of continued fractions, in particular, with Rogers–Ramanujan continued fractions.

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1. Introduction

As usual, $\alpha = [a_0; a_1, a_2, \ldots]$ denotes the simple (or regular) continued fraction expansion of $\alpha$, where

$$\alpha = a_0 + \theta_0, \quad a_0 = [\theta],$$

$$1/\theta_{n-1} = a_n + \theta_n, \quad a_n = [1/\theta_{n-1}] \quad (n \geq 1).$$

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Hurwitzian continued fraction expansions have the form
\[ \left[ c_0; c_1, \ldots, c_n, \frac{Q_1(k)}{Q_2(k), \ldots, Q_p(k)} \right]_{k=1}^{\infty} \]
if \( c_0 \) is an integer, \( c_1, \ldots, c_n \) are positive integers, \( Q_1, \ldots, Q_p \) are polynomials with rational coefficients which take positive integral values for \( k = 1, 2, \ldots \) and at least one of the polynomials is not constant (see [1, 8, 9], e.g.).

\[
\begin{align*}
e &= [2; 1, 2k, 1]_{k=1}^{\infty}, & e^{1/s} &= [1; (2k-1)s-1, 1, 1]_{k=1}^{\infty} \quad (s \geq 2), \\
\sqrt{\nu} \tan \frac{1}{\sqrt{\nu u}} &= [0; (4k-3)u, (4k-1)v]_{k=1}^{\infty} \quad \text{and} \\
\sqrt{\nu} \tan \frac{1}{\sqrt{\nu u}} &= [0; u-1, 1, (4k-1)v-2, 1, (4k+1)u-2]_{k=1}^{\infty}
\end{align*}
\]
are well-known examples. But, no example has been known such that \( \deg Q_j(k) > 1 \) in \( k \).

Tasoev [11, 12] proposed the new continued fraction \([0; a^k, \ldots, a^k]_{k=1}^{\infty}\), which is like Hurwitzian but \( Q_j(k) \) is exponential in \( k \) instead of polynomials. He showed the closed form when \( m = 1 \), but it had an error. The author obtained the correct closed form for any positive integer \( m \) in [5]. In this paper some new results belonging to this new class of continued fractions shall be shown.

This continued fractions by Tasoev may be new but it is related to some known continued fractions. In this paper, some relations with Rogers–Ramanujan continued fractions are mentioned.

2. Tasoev’s continued fractions \((m = 1)\)

Let \( a \) be any positive integer with \( a \geq 2 \). The author [5] obtained the corrected form which Tasoev proposed with a mistake. Namely,
\[
[0; a, a^2, a^3, a^4, \ldots] = \frac{\sum_{s=0}^{\infty} a^{-(s+1)^2} \prod_{i=1}^{s} (a^{2i} - 1)^{-1}}{\sum_{s=0}^{\infty} a^{-s^2} \prod_{i=1}^{s} (a^{2i} - 1)^{-1}}.
\]
Throughout the paper, in the case where \( s = 0 \), an empty product denotes 1. This result is slightly extended as
\[
[0; ua, ua^2, ua^3, \ldots, ua^{n+1}, \ldots] = \frac{\sum_{s=0}^{\infty} u^{-2s} a^{-(s+1)^2} \prod_{i=1}^{s} (a^{2i} - 1)^{-1}}{\sum_{s=0}^{\infty} u^{-2s} a^{-s^2} \prod_{i=1}^{s} (a^{2i} - 1)^{-1}},
\]
where \( u \) is rational so that \( ua \) is a positive integer [6]. In [6] we considered the power series \( f_n(z) = \sum_{s=0}^{\infty} r_{n,s} z^s \), satisfying the relation \( f_n(z) = ua^{n+1} f_{n+1}(z) + \)
\[ z^{n+2}(z) \ (n = 0, 1, \ldots). \] By \( f_n(z)/f_{n+1}(z) = u^{n+1} + z/(f_{n+1}(z)/f_{n+2}(z)) \) we could have \( f_1(1)/f_0(1) = [0; ua, ua^2, ua^3, \ldots] \). By induction we showed \( r_{n,s} = u^{-n-2s} a^{-n(n+1)/2-2sn-s^2} \prod_{i=1}^{s}(a^2i - 1)^{-1}. \)

Therefore, if we replace \( a \) by \( r \), and \( ur \) by \( a \), we have the continued fraction whose partial quotients consist of geometric progressions.

**Fact 1.**

\[ [0; a, ar, ar^2, \ldots, ar^n, \ldots] = \frac{\sum_{s=0}^{\infty} a^{-2s-1} r^{-s^2} \prod_{i=1}^{s}(r^{2i} - 1)^{-1}}{\sum_{s=0}^{\infty} a^{-2s} r^{-s^2+2s} \prod_{i=1}^{s}(r^{2i} - 1)^{-1}}. \]

It is interesting to compare with the continued fraction whose partial quotients consist of arithmetic progressions

\[ [0; a, a+d, a+2d, \ldots, a+nd, \ldots] = \frac{I_{a/d}(\frac{2}{a})}{I_{(a/d)-1}(\frac{2}{a})}, \]

where \( I_{\lambda}(z) \) are the modified Bessel functions of the first kind, defined by [2,7].

\[ I_{\lambda}(z) = \sum_{v=0}^{\infty} \frac{(z/2)^{\lambda+2v}}{v! \Gamma(\lambda + v + 1)} \]

The first new result is a variation of this type, stating the following.

**Theorem 1.**

\[ [0; ua^{2k-1} - 1, 1, va^{2k} - 1]_{k=1}^{\infty} = \frac{\sum_{s=0}^{\infty} u^{-s-1} v^{-s} a^{-(s+1)^2} \prod_{i=1}^{s}(a^{2i} - (-1)^i)^{-1}}{\sum_{s=0}^{\infty} (-1)^s u^{-s} v^{-s} a^{-s^2} \prod_{i=1}^{s}(a^{2i} - (-1)^i)^{-1}}. \]

**Lemma 1.**

\[ \frac{1}{a_1'} - \frac{1}{a_2'} + \frac{1}{a_3'} - \frac{1}{a_4'} + \ldots = [0; a_1' - 1, 1, a_2' - 1, a_3' - 1, 1, a_4' - 1, \ldots]. \]

**Proof.** By the equivalence transformation in [4, 2.3.23, p. 35] and the formula \([\ldots, a, -b, \gamma] = [\ldots, a-1, 1, b-1, -\gamma] \) in [10, Section 6], the left-hand side equals

\[ [0; a_1', -a_2', -a_3', a_4', a_5', -a_6', -a_7', \ldots] = [0; a_1' - 1, 1, a_2' - 1, a_3' - a_4', -a_5', a_6', a_7', \ldots] \]
\[ = [0; a'_1 - 1, 1, a'_2 - 1, a'_3 - 1, 1, a'_4 - 1, 1, a'_5, -a'_6, -a'_7, \ldots] = \ldots = [0; a'_{2k-1} - 1, 1, a'_{2k} - 1]_{k=1}^\infty. \square \]

**Proof of Theorem 1.** Consider a power series of \( z \)

\[ f_n(z) = r_{n,0} + r_{n,1}z + r_{n,2}z^2 + \cdots \quad (n = 0, 1, 2, \ldots), \quad (1) \]

satisfying the recurrence relation

\[ f_n(z) = w_n a^{n+1} f_{n+1}(z) + (-1)^{n+1} z f_{n+2}(z), \quad (2) \]

where

\[ w_n = \begin{cases} u & \text{if } n \text{ is even,} \\ v & \text{if } n \text{ is odd.} \end{cases} \]

Then from

\[ \frac{f_0(z)}{f_1(z)} = ua - \frac{z}{f_1(z)} = ua - \frac{z}{va^2 + \frac{z}{ua^3 - \frac{z}{va^4 + \ldots}}} \]

and Lemma 1 we have

\[ \frac{f_1(1)}{f_0(1)} = [0; ua - 1, 1, va^2 - 1, ua^3 - 1, 1, va^4 - 1 \ldots]. \]

Now, by comparing the constant term and the coefficient of \( z^s \) \( (s = 1, 2, \ldots) \) in (2) we have

\[ r_{n,0} = w_n a^{n+1} r_{n+1,0}, \quad (3) \]

\[ r_{n,s} = w_n a^{n+1} r_{n+1,s} + (-1)^{n+1} r_{n+2,s-1} \quad (s = 1, 2, \ldots). \quad (4) \]

From (3) we have

\[ r_{0,0} = \begin{cases} \frac{n}{u^2} \frac{n}{v^2} a^{\frac{n(n+1)}{2}} r_{n,0} & \text{if } n \text{ is even,} \\ \frac{n+1}{u} \frac{n-1}{v} a^{\frac{n(n+1)}{2}} r_{n,0} & \text{if } n \text{ is odd.} \end{cases} \]
We can put \( r_{0,0} = 1 \) without loss of generality. Otherwise, \( r_{0,0} \) will be cancelled in the form of the fraction \( f_1(1)/f_0(1) \). Hence,

\[
r_{n,0} = \begin{cases} 
  u - \frac{n}{2} v - \frac{n}{2} a - \frac{n(n+1)}{2} & \text{if } n \text{ is even}, \\
  u - \frac{n+1}{2} v - \frac{n-1}{2} a - \frac{n(n+1)}{2} & \text{if } n \text{ is odd}.
\end{cases}
\] (5)

Let \( s \geq 1 \). From (4), if \( n \) is even then

\[
r_{0,s} = u^2 v^2 a^{\frac{n(n+1)}{2}} r_{n,s} + \sum_{k=1}^{n} (u^k v^{k-1} a^k r_{2k+1,s-1} - u^{k-1} v^{k-1} a^{k-1} r_{2k,s-1}),
\] (6)

if \( n \) is odd then

\[
r_{0,s} = u \frac{n+1}{2} v \frac{n-1}{2} a \frac{n(n+1)}{2} r_{n,s} - \sum_{k=1}^{n-1} (u^{k-1} v^{k-1} a^{k-1} r_{2k,s-1}) + \sum_{k=1}^{n-1} u^k v^{k-1} a^k r_{2k+1,s-1}.
\] (7)

Put \( s = 1 \). If \( n \) is even, by (5) and (6) we have

\[
r_{0,1} = u^2 v^2 a^{\frac{n(n+1)}{2}} r_{n,1} + \sum_{k=1}^{n} (u^k v^{k-1} a^k - u^{k-1} v^{k-1} a^{k-1} - u^{k-1} v^{k-1} a^{k-1})
\]

\[
= u^2 v^2 a^{\frac{n(n+1)}{2}} r_{n,1} + u^{-1} v^{-1} \sum_{k=1}^{n} (a^{-4k-1} - a^{-4k+1})
\]

\[
= u^2 v^2 a^{\frac{n(n+1)}{2}} r_{n,1} + u^{-1} v^{-1} a^{-1} (a^{-2n-1} - (a^2 + 1)^{-1}).
\]

Separating the terms including \( n \) from the constant terms without \( n \), we have

\[
r_{n,1} = -u^2 v^2 a^{\frac{n(n+1)}{2}} r_{n,1} + u^{-1} v^{-1} a^{-1} (a^{-2n-1} - (a^2 + 1)^{-1}).
\]
If $n$ is odd, by (5) and (7) we have

$$r_{0,1} = u \frac{n+1}{2} v \frac{n-1}{2} a \frac{n(n+1)}{2} r_{n,1}$$

$$-\sum_{k=1}^{n+1} u^k v^{k-1} a (k-1)(2k-1) u^{-k} v^{-k} a^{-k(2k+1)}$$

$$+\sum_{k=1}^{n-1} u^k v^{k-1} a (k-1)(2k-1) u^{-k} v^{-k} a^{-k(2k+1)}$$

$$= u \frac{n+1}{2} v \frac{n-1}{2} a \frac{n(n+1)}{2} r_{n,1} + u^{-1} v^{-1} \left( -\sum_{k=1}^{n+1} a^{-4k+1} + \sum_{k=1}^{n-1} a^{-4k-1} \right)$$

$$= u \frac{n+1}{2} v \frac{n-1}{2} a \frac{n(n+1)}{2} r_{n,1} - u^{-1} v^{-1} a^{-1} a^{-2n+1} (a^2 + 1)^{-1}.$$ 

Together with $r_{0,1} = u a r_{1,1} - r_{2,0} = u a r_{1,1} - u^{-1} v^{-1} a^{-3}$ by (4) and (5), we obtain

$$u \frac{n+1}{2} v \frac{n-1}{2} a \frac{n(n+1)}{2} r_{n,1} = u^{-1} v^{-1} a^{-1} a^{-2n} (a^2 + 1)^{-1}.$$ 

Therefore,

$$r_{n,1} = u \frac{n+3}{2} v \frac{n+1}{2} a \frac{n(n+1)}{2} -2n^{-1} (a^2 + 1)^{-1}.$$ 

In general, we can prove that

$$r_{n,s} = \begin{cases} 
(-1)^s u \frac{n}{2} v \frac{n-s}{2} a \frac{n(n+1)}{2} -2s(a^2 - (-1)^s)^{-1} & \text{if } n \text{ is even,} \\
-\frac{n+1}{2} v \frac{n-1}{2} -s a \frac{n(n+1)}{2} -2s(a^2 - (-1)^s)^{-1} & \text{if } n \text{ is odd} 
\end{cases}$$

by induction. Assume that it holds up to $s-1$ and $s$. If $n$ is even, from (6)

$$r_{0,s+1} = u \frac{n}{2} v \frac{n}{2} a \frac{n(n+1)}{2} r_{n,s+1}$$

$$+ \sum_{k=1}^{n/2} u^k v^{k-1} a^k (2k-1) u^{-k-s} v^{-k-s}$$

$$\times a^{-(k+1)(2k+1)-2s(2k+1)-s^2} \prod_{i=1}^{s} (a^{2i} - (-1)^i)^{-1}$$
\[-u^{k-1}v^{k-1}a^{(k-1)(2k-1)}(-1)^{s}u^{-(s+1)}v^{-(s+1)}\]

\[\times a^{-k(2k+1) - 2s + 2k - s^2} \prod_{i=1}^{s} (a^{2i} - (-1)^i)^{-1}\]

\[= u^{\frac{n}{2}} v^{\frac{n}{2}} a^{\frac{n(n+1)}{2}} r_{n,s+1} + u^{-s-1} v^{-s-1} \prod_{i=1}^{s} (a^{2i} - (-1)^i)^{-1} \cdot a^{-(s+1)^2}\]

\[\times (1 - (-1)^{s} a^{2(s+1)}) \sum_{k=1}^{\frac{n}{2}} a^{-4(s+1)k}\]

\[= u^{\frac{n}{2}} v^{\frac{n}{2}} a^{\frac{n(n+1)}{2}} r_{n,s+1}\]

\[+ (-1)^{s+1} u^{-s-1} v^{-s-1} a^{-(s+1)^2} (1 - a^{-2(s+1)n}) \prod_{i=1}^{s+1} (a^{2i} - (-1)^i)^{-1}.\]

Therefore,

\[r_{n,s+1} = (-1)^{s+1} u^{-\frac{n}{2} - s - 1} v^{-\frac{n}{2} - s - 1} a^{-\frac{n(n+1)}{2} - 2(s+1)n - (s+1)^2} \prod_{i=1}^{s+1} (a^{2i} - (-1)^i)^{-1}.\]

If \(n\) is odd, from (7)

\[r_{0,s+1} = u^{\frac{n+1}{2}} v^{\frac{n-1}{2}} a^{\frac{n(n+1)}{2}} r_{n,s+1} - \sum_{k=1}^{\frac{n+1}{2}} u^{k-1} v^{k-1} a^{(k-1)(2k-1)} (-1)^{s} u^{-k-s}\]

\[\times v^{-k-s} a^{-k(2k+1) - 4sk - s^2} \prod_{i=1}^{s} (a^{2i} - (-1)^i)^{-1}\]

\[+ \sum_{k=1}^{\frac{n-1}{2}} u^{k} v^{k} a^{(2k+1)} u^{-k-s-1} v^{-k-s}\]

\[\times a^{-(k+1)(2k+1) - 2s(2k+1) - s^2} \prod_{i=1}^{s} (a^{2i} - (-1)^i)^{-1}\]

\[= u^{\frac{n+1}{2}} v^{\frac{n-1}{2}} a^{\frac{n(n+1)}{2}} r_{n,s+1} + u^{-s-1} v^{-s-1} \prod_{i=1}^{s} (a^{2i} - (-1)^i)^{-1}\]
\[
\begin{align*}
&\times \left( \sum_{k=1}^{n+1} a^{-4(s+1)k} + a^{-(s+1)^2} \sum_{k=1}^{n-1} a^{-4(s+1)k} \right) \\
&= u^{\frac{n+1}{2}} v^{\frac{n-1}{2}} a^{-\frac{n(n+1)}{2}} r_{n,s+1} \\
&= -u^{s-1} v^{s-1} a^{-(s+1)^2} (a^{-2(s+1)n} - (-1)^{s+1}) \prod_{i=1}^{s+1} (a^{2i} - (-1)^i)^{-1}.
\end{align*}
\]

Since from (4)

\[
n_{s+1} = u a r_{1,s+1} - r_{2,s}
\]

\[
n_{s+1} = u a r_{1,s+1} - (-1)^s u^{s-1} v^{s-1} a^{-s^2-4s-3} \prod_{i=1}^{s} (a^{2i} - (-1)^i)^{-1},
\]

by separating the general term including \( n \) from that where \( n = 1 \), we obtain

\[
r_{n,s+1} = u^{\frac{n+1}{2}} v^{\frac{n-1}{2}} a^{-\frac{n(n+1)}{2}} -2(s+1)n-(s+1)^2 \prod_{i=1}^{s+1} (a^{2i} - (-1)^i)^{-1}.
\]

Because

\[
f_0(1) = \sum_{s=0}^{\infty} r_{0,s} = \sum_{s=0}^{\infty} (-1)^s u^{s-1} v^{s-1} a^{-s^2} \prod_{i=1}^{s} (a^{2i} - (-1)^i)^{-1}
\]

and

\[
f_1(1) = \sum_{s=0}^{\infty} r_{1,s} = \sum_{s=0}^{\infty} u^{s-1} v^{s-1} a^{-(s+1)^2} \prod_{i=1}^{s} (a^{2i} - (-1)^i)^{-1},
\]

we have the desired result. \( \Box \)

If we consider the continued fraction expansion

\[
\frac{1}{a_1'} + \frac{1}{a_2'} - \frac{1}{a_3'} + \frac{1}{a_4'} - \ldots = [0; a_1', a_2' - 1, 1, a_3' - 1, a_4' - 1, 1, a_5' - 1, \ldots]
\]
instead of Lemma 1, we have Corollary 1 below. Eqs. (2) and (4) are replaced by
\begin{align*}
fn(z) &= wn^{n+1}f_{n+1}(z) + (-1)^nzf_{n+2}(z) \\
\end{align*}
(2')
and
\begin{align*}
r_{n,s} &= wn^{n+1}r_{n+1,s} + (-1)^nr_{n+2,s-1} \quad (s = 1, 2, \ldots),
\end{align*}
(4')
respectively.

**Corollary 1.**
\begin{align*}
[0; ua, va^{2k} - 1, 1, ua^{2k+1} - 1]_{k=1}^\infty &= \sum_{s=0}^\infty (-1)^s u^{-s-1} v^{-s} a^{-s+1} \prod_{i=1}^s (a^{2i} - (-1)^i)^{-1} \\
&\sum_{s=0}^\infty u^{-s} v^{-s} a^{-s} \prod_{i=1}^s (a^{2i} - (-1)^i)^{-1}.
\end{align*}

3. **Tasoev’s continued fractions** \((m = 2)\)

Tasoev’s continued fractions in the case of \(m = 2\) is given by
\begin{align*}
[0; a, a, a^2, a^2, a^3, a^3, \ldots] &= \frac{\sum_{s=0}^\infty a^{-(s+1)(s+2)/2} \prod_{i=1}^s (a^i - 1)^{-1}}{\sum_{s=0}^\infty a^{-s(s+1)/2} \prod_{i=1}^s (a^i - 1)^{-1}},
\end{align*}
which was also shown in [5]. A more general one obtained in [6] is as follows:
\begin{align*}
[0; ua^k, va^k]_{k=1}^\infty &= \frac{\sum_{s=0}^\infty u^{-s-1} v^{-s} a^{-s+1} \prod_{i=1}^s (a^i - 1)^{-1}}{\sum_{s=0}^\infty u^{-s} v^{-s} a^{-s+1} \prod_{i=1}^s (a^i - 1)^{-1}},
\end{align*}
where \(u\) and \(v\) are rational so that \(ua\) and \(va\) are positive integers.

If we notice the relation
\begin{align*}
u a - \frac{1}{\frac{1}{va}} &= [0; ua-1, 1, va-1, ua^2-1, 1, va^2-1, \ldots],
\end{align*}
we can have an extended variation of this type.
Theorem 2. 

\[ [0; ua^k - 1, 1, va^k - 1]_{k=1}^{\infty} \]

\[ = \sum_{s=0}^{\infty} u^{-s-1}v^{-s}a^{-(s+1)(s+2)/2} \prod_{i=1}^{s}(a^i - (-1)^i)^{-1} \]

\[ \sum_{s=0}^{\infty}(-1)^s u^{-s}v^{-s}a^{-s(s+1)/2} \prod_{i=1}^{s}(a^i - (-1)^i)^{-1}. \]

The proof is similar and omitted. If we use the relation

\[
\frac{1}{ua + \frac{1}{va - \frac{1}{ua^2 + \frac{1}{va^2 - \ldots}}}} = [0; ua, va - 1, 1, ua - 1, va^2 - 1, 1, ua^2 - 1, \ldots]
\]

instead, we have the following.

Corollary 2. 

\[ [0; ua, va^k - 1, 1, ua^{k+1} - 1]_{k=1}^{\infty} \]

\[ = \sum_{s=0}^{\infty}(-1)^s u^{-s-1}v^{-s}a^{-(s+1)(s+2)/2} \prod_{i=1}^{s}(a^i - (-1)^i)^{-1} \]

\[ \sum_{s=0}^{\infty} u^{-s}v^{-s}a^{-s(s+1)/2} \prod_{i=1}^{s}(a^i - (-1)^i)^{-1}. \]

4. Rogers–Ramanujan continued fractions

Rogers–Ramanujan continued fractions have the form

\[ R(x; a) = 1 + \frac{ax}{1 + \frac{ax^2}{1 + \frac{ax^3}{1 + \ldots}}} \]

(see [3, pp. 290–295], e.g.). We shall show the relation between Rogers–Ramanujan continued fractions and Tasoev’s continued fractions in the case \( m = 2 \).

As it is well-known, there exists an identity:

\[ R(x; a) = \frac{\sum_{s=0}^{\infty} a^sx^s \prod_{i=1}^{s}(1-x^i)^{-1}}{\sum_{s=0}^{\infty} a^sx^{s(s+1)} \prod_{i=1}^{s}(1-x^i)^{-1}}. \]
Especially, for $a = 1$

$$R(x; 1) = \frac{\sum_{s=0}^{\infty} x^s \prod_{i=1}^{s} (1 - x^i)^{-1}}{\sum_{s=0}^{\infty} x^{s(s+1)} \prod_{i=1}^{s} (1 - x^i)^{-1}} = \prod_{n=0}^{\infty} \frac{(1 - x^{5n+2})(1 - x^{5n+3})}{(1 - x^{5n+1})(1 - x^{5n+4})}$$

$$= 1 + \sum_{n=1}^{\infty} (-1)^n (x^{n(5n-1)/2} + x^{n(5n+1)/2})$$

By the equivalence transformation in [4, 2.3.23, p. 35], we get

$$R(q^{-1}; u^{-1}) = 1 + \frac{(uq)^{-1}}{1 + \frac{(uq^2)^{-1}}{1 + \frac{(uq^3)^{-1}}{1 + \ldots}}}$$

On the other hand, by setting $v = 1$ in a general Tasoev’s continued fraction in the case $m = 2$, we have

$$1 + \left[0; uq^k, q^k\right]_{k=1}^{\infty} = 1 + \frac{\sum_{s=0}^{\infty} u^{-s-1} q^{-(s+1)(s+2)/2} \prod_{i=1}^{s} (q^i - 1)^{-1}}{\sum_{s=0}^{\infty} u^{-s} q^{-s(s+1)/2} \prod_{i=1}^{s} (q^i - 1)^{-1}}$$

$$= \frac{\sum_{s=0}^{\infty} u^{-s} q^{-s(s-1)/2} \prod_{i=1}^{s} (q^i - 1)^{-1}}{\sum_{s=0}^{\infty} u^{-s} q^{-s(s+1)/2} \prod_{i=1}^{s} (q^i - 1)^{-1}}$$

because

$$\sum_{s=0}^{\infty} u^{-s} q^{-s(s-1)/2} \prod_{i=1}^{s} (q^i - 1)^{-1}$$

$$- \sum_{s=0}^{\infty} u^{-s} q^{-s(s+1)/2} \prod_{i=1}^{s} (q^i - 1)^{-1}$$

$$= \sum_{s=0}^{\infty} u^{-s} q^{-s(s+1)/2} (q^s - 1) \prod_{i=1}^{s} (q^i - 1)^{-1}$$

$$= \sum_{s=0}^{\infty} u^{-s-1} q^{-(s+1)(s+2)/2} \prod_{i=1}^{s} (q^i - 1)^{-1}$$
We have
\[
[0; q, q^2, q^3, q^4, \ldots] = \sum_{s=0}^{\infty} q^{-(s+1)(s+2)/2} \prod_{i=1}^{s} (q^i - 1)^{-1}.
\]

Therefore, we obtain the relation between Rogers–Ramanujan continued fractions and Tasoev’s continued fractions in the case of \(m = 2\)
\[
R(q^{-1}; u^{-1}) = [1; uq^k, q^k]_{k=1}^{\infty}.
\]

Especially, for \(u = 1\) we obtain the following identity:
\[
\sum_{s=0}^{\infty} q^{-(s+1)/2} \prod_{i=1}^{s} (q^i - 1)^{-1} = 1 + \sum_{n=1}^{\infty} (-1)^n (q^{-(5n+1)/2} + q^{-(5n+3)/2})
\]
\[
= \prod_{n=0}^{\infty} \frac{(1 - q^{-(5n+2)})(1 - q^{-(5n+3)})}{(1 - q^{-(5n+1)})(1 - q^{-(5n+4)})}
\]
\[
= 1 + \frac{q^{-1}}{q^{-2}} = R(q^{-1}; 1)
\]
\[
= [1; q, q^2, q^3, q^4, \ldots] = [1; q^k, q^k]_{k=1}^{\infty}.
\]

Next, we shall find a relation between Rogers–Ramanujan continued fractions and Tasoev’s continued fractions in the case of \(m = 1\).

From
\[
R(x^2; ax^{-1}) = 1 + \frac{ax}{1 + \frac{ax^3}{1 + \frac{ax^5}{1 + \ldots}}}
\]
and the equivalence transformation in [4, 2.3.23, p. 35], we get
\[
R(q^{-2}; q) = 1 + \frac{q^{-1}}{q^{-3}} = [1; q, q^2, q^3, \ldots].
\]
On the other hand,

\[
[0; q, q^2, q^3, \ldots] = \frac{\sum_{s=0}^{\infty} q^{-(s+1)^2} \prod_{i=1}^{s} (q^{2^i} - 1)^{-1}}{\sum_{s=0}^{\infty} q^{-s^2} \prod_{i=1}^{s} (q^{2^i} - 1)^{-1}}
\]

\[
= \frac{\sum_{s=0}^{\infty} q^{-s^2+2s} \prod_{i=1}^{s} (q^{2^i} - 1)^{-1} \sum_{s=0}^{\infty} q^{-s^2} \prod_{i=1}^{s} (q^{2^i} - 1)^{-1}}{\sum_{s=0}^{\infty} q^{-s^2} \prod_{i=1}^{s} (q^{2^i} - 1)^{-1} - 1} = R(q^{-2}; q) - 1
\]

because

\[
\sum_{s=0}^{\infty} q^{-s^2+2s} \prod_{i=1}^{s} (q^{2^i} - 1)^{-1} = \sum_{s=0}^{\infty} q^{-s^2} \prod_{i=1}^{s} (q^{2^i} - 1)^{-1}
\]

Therefore,

\[
\frac{F(1)}{F(q^{-1})} - 1 = \frac{\sum_{s=0}^{\infty} q^{-(s+1)^2} \prod_{i=1}^{s} (q^{2^i} - 1)^{-1}}{\sum_{s=0}^{\infty} q^{-s^2} \prod_{i=1}^{s} (q^{2^i} - 1)^{-1}} = [0; q, q^2, q^3, \ldots].
\]

In the same light as [3, pp. 294–296] we can obtain

\[
R(x^2; ax^{-1}) = \frac{\sum_{r=0}^{\infty} (-1)^r a^{4r} x^{2r} \prod_{i=1}^{r} (1 - a^{4i} x^{8i+2}) P_{r} \prod_{i=r}^{\infty} (1 - a^{2i} x^{2i+1})^{-1}}{\sum_{r=0}^{\infty} (-1)^r a^{4r} x^{2r} \prod_{i=1}^{r} (1 - a^{4i} x^{8i+6}) P_{r} \prod_{i=r}^{\infty} (1 - a^{2i} x^{2i+3})^{-1}},
\]

where \( \lambda(r) = r(5r + 1) \) and \( P_{r} = \prod_{i=1}^{r} (1 - x^{2i})^{-1} \). Hence, we also have a different closed form

\[
R(q^{-2}; q) = [1; q, q^2, q^3, \ldots]
\]

\[
= \frac{\sum_{r=0}^{\infty} (-1)^r q^{-r(5r-1)} (1-q^{-8-2r}) \prod_{i=1}^{r} (1-q^{-2i})^{-1} \prod_{i=r}^{\infty} (1-q^{-2i+1})^{-1}}{\sum_{r=0}^{\infty} (-1)^r q^{-r(5r+3)} (1-q^{-8-6r}) \prod_{i=1}^{r} (1-q^{-2i})^{-1} \prod_{i=r}^{\infty} (1-q^{-2i+3})^{-1}}.
\]

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References