Journal of Pure and Applied Algebra 214 (2010) 1301–1305

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

On the residual finiteness of outer automorphisms of relatively hyperbolic groups

V. Metaftsis [a], M. Sykiotis [b,*]

[a] Department of Mathematics, University of the Aegean, Karlovassi, 832 00 Samos, Greece

[b] Department of Mathematics and Statistics, University of Cyprus, P.O. Box 20537, 1678 Nicosia, Cyprus

ARTICLE INFO

Article history:
Received 11 June 2008
Received in revised form 18 June 2009
Available online 27 January 2010
Communicated by M. Sapir

MSC: 20F65; 20F67; 20E26; 20E36; 20F28

ABSTRACT

We show that every virtually torsion-free subgroup of the outer automorphism group of a conjugacy separable relatively hyperbolic group is residually finite. As a direct consequence, we obtain that the outer automorphism group of a limit group is residually finite.

1. Introduction

Relatively hyperbolic groups were introduced by Gromov in [1], in order to generalize notions such as the fundamental group of a complete, non-compact, finite volume hyperbolic manifold and to give a hyperbolic version of small cancellation theory over free groups by adopting the geometric language of manifolds with cusps. This notion has been developed by several authors and, in particular, various characterizations of relatively hyperbolic groups have been given (see [2–4] and the references therein). We should mention here that Farb [5] introduced a weaker notion of relative hyperbolicity for groups, using constructions on their Cayley graphs, as well as the Bounded Coset Penetration property, an additional condition which makes his definition equivalent to the other definitions.

We recall here one of Bowditch's equivalent definitions (in the case of infinite "peripheral" subgroups). A finitely generated group G is *hyperbolic relative to a family of finitely generated subgroups* $\mathcal{G}$ if G admits a proper, discontinuous and isometric action on a proper, hyperbolic path metric space X such that G acts on the ideal boundary of X as a geometrically finite convergence group and the elements of $\mathcal{G}$ are the maximal parabolic subgroups of G.

Besides the fundamental groups of hyperbolic manifolds of finite volume, examples of relatively hyperbolic groups are fundamental groups of finite graphs of finitely generated groups with finite edge groups, which are hyperbolic relative to the family of infinite vertex groups (which may be empty, in which case the group is hyperbolic), since their action on the Bass–Serre tree satisfies Definition 2 in [2].

Another example of relatively hyperbolic groups are limit groups. The notion of a limit group was introduced by Sela [6,7] in his solution to Tarski's problem for free groups. As it turned out, the family of limit groups coincides with that of finitely generated fully residually free groups first introduced by Baumslag [8], and extensively studied by Kharlampovich and Myasnikov [9,10]. In [11], Dahmani showed that limit groups are hyperbolic relative to their maximal non-cyclic abelian subgroups (see also [12]). Note that each group is relatively hyperbolic to itself. So, from now on, in order to avoid this trivial situation, *we assume that all relatively hyperbolic groups properly contain the corresponding maximal parabolic subgroups*.

* Corresponding author.
E-mail addresses: vmet@aegean.gr (V. Metaftsis), msikiot@ucy.ac.cy, msykiot@math.uoa.gr (M. Sykiotis).

0022-4049/$ – see front matter
doi:10.1016/j.jpaa.2009.07.018

In [13], it was proved that the outer automorphism group of a conjugacy separable hyperbolic group is residually finite. This is a far-reaching generalization of a classical result of Grossman [14], which states that the mapping class group of a closed orientable surface is residually finite. The purpose of this note is to prove the following generalization for relatively hyperbolic groups.

Theorem 1.1. *Let G be a conjugacy separable, relatively hyperbolic group. Then each virtually torsion-free subgroup of the outer automorphism group* $\mathrm{Out}(G)$ *of G is residually finite.*

As an application, we obtain:

Theorem 1.2. *Let G be the fundamental group of a finite graph of groups, such that each edge group is finite and each vertex group is polycyclic-by-finite. Then* $\mathrm{Out}(G)$ *is residually finite.*

Guirardel and Levitt [15] showed that the outer automorphism group of a limit group is virtually torsion-free. More recently, Chagas and Zalesskii [16] have shown that limit groups are conjugacy separable. Therefore, from Theorem 1.1, we immediately deduce the following result.

Theorem 1.3. *The outer automorphism group of a limit group is residually finite.*

2. Proofs of the main results

A group G is *conjugacy separable* if for any two non-conjugate elements x and y of G, there is a finite homomorphic image of G in which the images of x and y are not conjugate. An automorphism f of a group G is called *conjugating* if $f(g)$ is conjugate to g for each $g \in G$. The conjugating automorphisms of a group G form a subgroup of $\mathrm{Aut}(G)$, which we denote by $\mathrm{Conj}(G)$. Clearly, $\mathrm{Conj}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$ containing the inner automorphism group $\mathrm{Inn}(G)$ of G. The importance of this notion to the study of residual properties of the outer automorphism group of G, arises from two facts. The first is that if G is finitely generated and conjugacy separable, then the quotient group $\mathrm{Aut}(G)/\mathrm{Conj}(G)$ is residually finite (see [13, Lemma 2.1]). The second is the following short exact sequence

$$1 \to \mathrm{Conj}(G)/\mathrm{Inn}(G) \hookrightarrow \mathrm{Aut}(G)/\mathrm{Inn}(G) \to \mathrm{Aut}(G)/\mathrm{Conj}(G) \to 1. \tag{1}$$

Thus to prove Theorem 1.1, it suffices to show that if G is relatively hyperbolic, then the quotient $\mathrm{Conj}(G)/\mathrm{Inn}(G)$ is finite.

We will need the following lemma whose a more general version in the case of projections on quasiconvex subspaces can be found in [17, Proposition 2.1, Chapter 10].

Lemma 2.1. *Let (X, d) be a δ-hyperbolic metric space, let g be an isometry of X and let N be a positive real number such that the set $Y = \{y \in X : d(y, gy) \le N\}$ is nonempty. Given a point $x \in X$ and a positive number M, choose $y \in Y$ with $d(x, y) \le d(x, Y) + M$. Then either $d(x, gx) \ge 2d(x, y) + d(y, gy) - 2(3\delta + 2M)$ or $d(y, gy) \le 3\delta + 2M$.*

Proof. We consider a geodesic triangle with vertices x, y and gy (see Fig. 1). Let z and w be the points on the geodesics $[x, y]$ and $[y, gy]$, respectively, which are at distance α from y, where α is the Gromov product of x and gy with respect to y. We first note that $w \in Y$. Indeed, $d(w, gw) \le d(w, gy) + d(gy, gw) = d(w, gy) + d(y, w) = d(y, gy) \le N$. Since $w \in Y$, we have $d(w, x) \ge d(x, Y) \ge d(x, y) - M$ and hence $d(x, z) + d(z, y) = d(x, y) \le d(w, x) + M \le d(x, z) + \delta + M$. It follows that $\alpha = d(z, y) \le \delta + M$. Therefore $d(x, gy) = d(x, y) + d(y, gy) - 2\alpha$, where $\alpha \le \delta + M$. Similarly one can show that $d(y, gx) = d(x, y) + d(y, gy) - 2\beta$, where $\beta \le \delta + M$. Now we turn our attention to a geodesic quadrilateral with vertices x, y, gy and gx. There are two cases to consider, as shown in the following figure.

In the first case, by the four point condition we have $d(x, gy) + d(y, gx) \le d(x, gx) + d(y, gy) + 2\delta$. Therefore,

$$\begin{aligned} d(x, gx) &\ge d(x, gy) + d(y, gx) - d(y, gy) - 2\delta \\ &= d(x, y) + d(y, gy) - 2\alpha + d(x, y) + d(y, gy) - 2\beta - d(y, gy) - 2\delta \\ &= 2d(x, y) + d(y, gy) - 2(\alpha + \beta + \delta) \\ &\ge 2d(x, y) + d(y, gy) - 2(3\delta + 2M). \end{aligned}$$

In the second case, using the four point condition again, we have $d(x, gy) + d(y, gx) \le d(x, y) + d(gx, gy) + 2\delta$. Thus

$$d(x, y) + d(y, gy) - 2\alpha + d(x, y) + d(y, gy) - 2\beta \le d(x, y) + d(gx, gy) + 2\delta$$

and hence $d(y, gy) \le \alpha + \beta + \delta \le 3\delta + 2M$. This completes the proof. $\square$

Lemma 2.2. *Let G be a relatively hyperbolic group. Then the inner automorphism group* $\mathrm{Inn}(G)$ *of G is of finite index in* $\mathrm{Conj}(G)$.

Proof. The proof of the lemma is a generalization of the proof of [13, Lemma 2.2]. In this case instead of the Cayley graph of G we use the δ-hyperbolic metric space X (in the sense that every geodesic triangle in X is δ-thin) on which G acts by

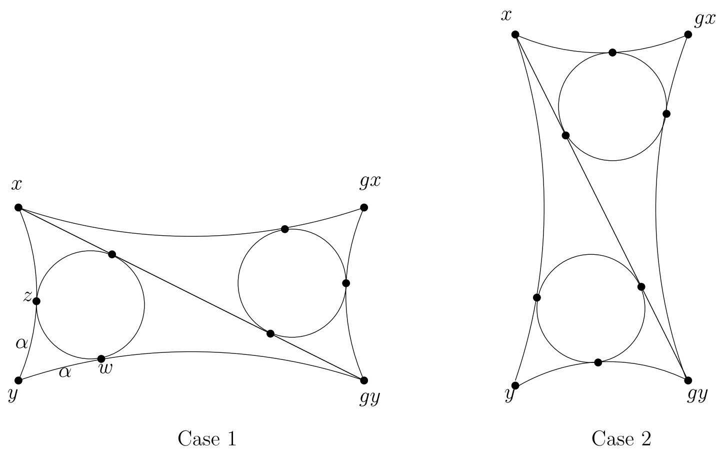

Fig. 1. The two cases of Lemma 2.1.

isometries. One essential difference between the two proofs is the existence of parabolic isometries in the case of relatively hyperbolic groups.

Suppose on the contrary that $\mathrm{Inn}(G)$ is of infinite index in $\mathrm{Conj}(G)$ and fix an infinite sequence $f_1, f_2, \ldots, f_n, \ldots$ of conjugating automorphisms of G representing pairwise distinct cosets of $\mathrm{Inn}(G)$ in $\mathrm{Conj}(G)$. In particular, G is neither finite nor virtually infinite cyclic. Let $\lambda_i = \inf_{x \in X} \max_{s \in S} d(x, f_i(s)x)$, where S is a fixed finite generating set of G closed under inverses, and let $x_i^0 \in X$ such that $\max_{s \in S} d(x_i^0, f_i(s)x_i^0) \le \lambda_i + \frac{1}{i}$. As shown in the proof of Theorem 1.2 in [18], the sequence λ_i converges to infinity. Hence, for a given non-principal ultrafilter ω on $\mathbb{N}$ the based ultralimit $(X_\omega, d_\omega, x_\omega^0)$ of the sequence of based metric spaces (X, d_i, x_i^0), where $d_i = \frac{d}{\lambda_i}$, is an $\mathbb{R}$-tree. Moreover, there is an induced non-trivial isometric G-action on $(X_\omega, d_\omega, x_\omega^0)$ (i.e. G has no a global fixed point in X_ω), given by $g \cdot (x_i) = \big(f_i(g)x_i\big)$.

We shall show again that this action has a global fixed point. Suppose g is an element of G acting as a hyperbolic isometry on X_ω with translation length $\tau_\omega(g)$, and fix an element $x = (x_i) \in X_\omega$ on the axis of g. Then $\lim_\omega d_i\big(f_i(g)^2 x_i, x_i\big) = d_\omega(g^2x, x) = 2\tau_\omega(g) = 2d_\omega(gx, x) = 2\lim_\omega d_i\big(f_i(g)x_i, x_i\big)$ and so $\lim_\omega\Big(2d_i\big(f_i(g)x_i, x_i\big) - d_i\big(f_i(g)^2x_i, x_i\big)\Big) = 0$.

For each index i, let $Y_i = \big\{y \in X : \tau(f_i(g)) \le d(y, f_i(g)y) \le \tau(f_i(g)) + \frac{1}{i}\big\}$, where $\tau(f_i(g))$ is the minimal displacement of $f_i(g)$, and choose $y_i \in Y_i$ such that $d(x_i, y_i) \le d(x_i, Y_i) + 1$.

By Lemma 2.1, there are non-negative constants $C(\delta)$ and $K(\delta)$, depending only on δ, such that $d\big(f_i(g)x_i, x_i\big) \ge 2d(x_i, y_i) + d\big(f_i(g)y_i, y_i\big) - K(\delta)$ whenever $d(y_i, f_i(g)y_i) > C(\delta)$. Let I denote the subset of $\mathbb{N}$ consisting of those indices i for which $d(y_i, f_i(g)y_i) > C(\delta)$. The maximality of ω implies that it contains exactly one of $I, \mathbb{N} - I$.

We consider the two cases separately.

Case 1: $I \in \omega$. In this case for each $i \in I$ we have $d\big(f_i(g)x_i, x_i\big) \ge 2d(x_i, y_i) + d\big(f_i(g)y_i, y_i\big) - K(\delta) \ge 2d(x_i, y_i) + \tau(f_i(g)) - K(\delta)$. On the other hand, $d\big(f_i(g)x_i, x_i\big) \le 2d(x_i, y_i) + d\big(f_i(g)y_i, y_i\big) \le 2d(x_i, y_i) + \tau(f_i(g)) + \frac{1}{i}$. It follows that $|d\big(f_i(g)x_i, x_i\big) - \tau(f_i(g))| \le 2d(x_i, y_i) + K(\delta) + \frac{1}{i}$. Now, it is easy to verify that

$$\begin{aligned} 2d\big(f_i(g)x_i, x_i\big) - d\big(f_i(g)^2x_i, x_i\big) &\ge 4d(x_i, y_i) + 2\tau(f_i(g)) - 2K(\delta) - [2d(x_i, y_i) + 2d\big(f_i(g)y_i, y_i\big)] \\ &\ge 4d(x_i, y_i) + 2\tau(f_i(g)) - 2K(\delta) - \left[2d(x_i, y_i) + 2\tau(f_i(g)) + \frac{2}{i}\right] = 2d(x_i, y_i) - 2K(\delta) - \frac{2}{i}, \end{aligned}$$

and hence $2d(x_i, y_i) \le 2d\big(f_i(g)x_i, x_i\big) - d\big(f_i(g)^2x_i, x_i\big) + 2K(\delta) + \frac{2}{i}$. Finally,

$$\begin{aligned} \left|\tau_\omega(g) - \frac{\tau(f_i(g))}{\lambda_i}\right| &\le \left|\tau_\omega(g) - d_i\big(f_i(g)x_i, x_i\big)\right| + \left|d_i\big(f_i(g)x_i, x_i\big) - \frac{\tau(f_i(g))}{\lambda_i}\right| \\ &\le \left|\tau_\omega(g) - d_i\big(f_i(g)x_i, x_i\big)\right| + 2\frac{d(x_i, y_i)}{\lambda_i} + \frac{K(\delta)}{\lambda_i} + \frac{1}{i\lambda_i} \\ &\le |\tau_\omega(g) - d_i(f_i(g)x_i, x_i)| + 2d_i\big(f_i(g)x_i, x_i\big) - d_i\big(f_i(g)^2x_i, x_i\big) + 3\frac{K(\delta)}{\lambda_i} + \frac{3}{i\lambda_i}, \end{aligned}$$

for all $i \in I$.

Since the ω-limit of the right-hand side of the above inequality is 0 and $\tau(f_i(g)) = \tau(g)$ for all i, $f_i(g)$ being a conjugate of g for each i, it follows that $\tau_\omega(g) = 0$, which contradicts the assumption that $\tau_\omega(g)$ is strictly positive.

Case 2: $\mathbb{N} - I \in \omega$. If $\lim_\omega d_i(x_i, y_i) < \infty$, the sequence $y = (y_i)$ is a point of X_ω fixed by g, since $0 \le d_\omega(y, gy) = \lim_\omega d_i\big(y_i, f_i(g)y_i\big) \le \lim_\omega \frac{C(\delta)}{\lambda_i} = 0$, which contradicts the choice of g. Hence $\lim_\omega d_i(x_i, y_i) = \infty$. For each i, let γ_i :

$[0, d_i(x_i, y_i)] \to X_i$ be a geodesic from x_i to y_i. Since for every $t \geq 0$ the set of indices i for which t lies in the domain of γ_i, is contained in ω, we can define a geodesic ray $\gamma : [0, \infty) \to X_\omega$ by $\gamma(t) = \big(\gamma_i(t)\big)_i$, which is asymptotic to an ideal point $\tilde{y} \in \partial X_\omega$. We will show first that g fixes $\tilde{y}$, and then that $\tilde{y}$ is not one of the points at infinity determined by the axis of g, contradicting the fact that any hyperbolic isometry of a tree fixes exactly two points at infinity.

Claim 1. *g fixes $\tilde{y}$.*

Proof. It suffices to show that the geodesics $g\gamma$ and γ are asymptotic, i.e. that $\sup_t d_\omega(g\gamma(t), \gamma(t)) < \infty$. Let $t \geq 0$. For each i big enough, we consider the quadrilateral defined by the geodesics $f_i(g)\gamma_i$, γ_i, $[x_i, f_i(g)(x_i)]$ and $[y_i, f_i(g)(y_i)]$. Since the space X_i is hyperbolic, there is a non-negative constant $M(\delta)$, depending only on δ, such that the side $f_i(g)\gamma_i$ is contained in the $\frac{M(\delta)}{\lambda_i}$-neighborhood of the union of the other sides. Hence, the side $f_i(g)\gamma_i$ is contained in the R-neighborhood of γ_i, where $R = \frac{M(\delta)}{\lambda_i} + d_i\big(x_i, f_i(g)(x_i)\big) + d_i\big(y_i, f_i(g)(y_i)\big)$. Let $t' \geq 0$ be such that $\gamma_i(t')$ is the projection of $f_i(g)\gamma_i(t)$ on γ_i. Then $t' = d_i\big(\gamma_i(t'), \gamma_i(0)\big) \leq d_i\big(\gamma_i(t'), f_i(g)\gamma_i(t)\big) + d_i\big(f_i(g)\gamma_i(t), f_i(g)\gamma_i(0)\big) + d_i\big(f_i(g)\gamma_i(0), \gamma_i(0)\big)$ and thus $t' - t \leq R + d_i\big(f_i(g)x_i, x_i\big)$. In the same way, we obtain that $t - t' \leq R + d_i\big(f_i(g)x_i, x_i\big)$ and therefore $|t' - t| \leq R + d_i\big(f_i(g)x_i, x_i\big)$. Now

$$\begin{aligned} d_i\big(f_i(g)\gamma_i(t), \gamma_i(t)\big) &\leq d_i\big(f_i(g)\gamma_i(t), \gamma_i(t')\big) + d_i\big(\gamma_i(t'), \gamma_i(t)\big) \\ &\leq R + |t - t'| \\ &\leq 2\frac{M(\delta)}{\lambda_i} + 3d_i\big(x_i, f_i(g)(x_i)\big) + 2d_i\big(y_i, f_i(g)(y_i)\big). \end{aligned}$$

Taking limits, we get $d_\omega\big(g\gamma(t), \gamma(t)\big) \leq 3\tau_\omega(g)$. This proves the claim.

Claim 2. *$\tilde{y}$ is not one of the points at infinity determined by the axis of g.*

Proof. Suppose that $\tilde{y}$ is one of the ends of the axis A_g of g. Since X_ω is a tree, there is $t_0 \geq 0$ such that $\gamma(t) \in A_g$ for all $t \geq t_0$. The assumption that g acts on A_g as a translation of amplitude $\tau_\omega(g)$ implies that either $g\gamma(t) = \gamma\big(t + \tau_\omega(g)\big)$ or $g^{-1}\gamma(t) = \gamma\big(t + \tau_\omega(g)\big)$ for all $t \geq t_0$. Suppose that $g\gamma(t) = \gamma\big(t + \tau_\omega(g)\big)$ for all $t \geq t_0$ (the other case is handled similarly). Then $\lim_\omega d_i\big(f_i(g)\gamma_i(t), \gamma_i(t + \tau_\omega(g))\big) = 0$. Fixing $t \geq t_0$, the geodesic γ_i contains the point $\gamma_i(t) + \tau_\omega(g)$ for each i sufficiently large. Thus $\tau_\omega(g) + d_i\big(\gamma_i(t + \tau_\omega(g)), y_i\big) = d_i\big(\gamma_i(t), y_i\big) = d_i\big(f_i(g)\gamma_i(t), f_i(g)y_i\big) \leq d_i\big(f_i(g)\gamma_i(t), y_i\big) + d_i\big(y_i, f_i(g)y_i\big)$, and so

$$\begin{aligned} \tau_\omega(g) &\leq d_i\big(f_i(g)\gamma_i(t), y_i\big) - d_i\big(\gamma_i(t + \tau_\omega(g)), y_i\big) + d_i\big(y_i, f_i(g)y_i\big) \\ &\leq d_i\big(f_i(g)\gamma_i(t), \gamma_i(t + \tau_\omega(g))\big) + d_i\big(y_i, f_i(g)y_i\big). \end{aligned}$$

It follows that $0 < \tau_\omega(g) \leq \lim_\omega d_i\big(f_i(g)\gamma_i(t), \gamma_i(t + \tau_\omega(g))\big) + \lim_\omega d_i\big(y_i, f_i(g)y_i\big) = 0$. This is a contradiction, proving the claim.

So in all cases, every element of the finitely generated group G fixes some point of X_ω. This implies that the action of G on X_ω has a global fixed point (see [19, Proposition II. 2.15]), which is the desired contradiction. □

Remark 2.3. It follows from the above proof that $\tau_\omega(g) = \lim_\omega \frac{\tau(f_i(g))}{\lambda_i}$ for each $g \in G$ and each sequence (f_i) of automorphisms representing pairwise distinct elements in $\mathrm{Out}(G)$. The proof of the lemma can be simplified if one at the beginning makes use of the hypothesis that each f_i is a conjugating automorphism.

Remark 2.4. In the proof of [13, Lemma 2.2] the points y_i were chosen so that $f_i(g)$ realizes its minimal displacement at y_i. However, this random choice could give y_i for which the inequality $d(f_i(g)x_i, x_i) \geq 2d(x_i, y_i) + d(f_i(g)y_i, y_i) - K(\delta)$ is false. The correct way to proceed with the proof is to choose y_i as above. The fact that in the case of hyperbolic groups the action (on the Cayley graph) is free and cocompact, can be used to avoid Case 2. Indeed, in such an action the translation lengths are bounded away from zero and therefore for each positive number K there is a positive integer m such that $\tau(g^m) > K$ for all group elements g of infinite order. Thus, by replacing each element by its mth power, we can suppose that the translation length of any $f_i(g)$ is big enough.

Proof of Theorem 1.1. We consider the short exact sequence (1). Since the first term is finite, each torsion-free subgroup of $\mathrm{Out}(G)$ embeds in $\mathrm{Aut}(G)/\mathrm{Conj}(G)$, which is residually finite by [13, Lemma 2.1]. Hence, each torsion-free subgroup of $\mathrm{Out}(G)$ is residually finite, from which the theorem follows. □

Remark 2.5. Let $\mathrm{Aut}_n(G)$ be the subgroup of $\mathrm{Aut}(G)$ consisting of automorphisms which fix setwise every normal subgroup of G. Obviously, $\mathrm{Conj}(G) \subseteq \mathrm{Aut}_n(G)$. After this paper appeared as a preprint, Minasyan and Osin [20] proved (using completely different methods than ours) that for any relatively hyperbolic group G, $\mathrm{Inn}(G)$ has finite index in $\mathrm{Aut}_n(G)$ and hence in $\mathrm{Conj}(G)$. Moreover, if G is non-elementary and has no non-trivial finite normal subgroups, then $\mathrm{Aut}_n(G) = \mathrm{Inn}(G)$. Thus, in this case, the hypothesis of virtual torsion freeness can be removed in Theorem 1.1.

To prove Theorem 1.2, we need the following result.

Theorem 2.6 *([15, Corollary 5.3]). Let $G = G_1 * \cdots * G_n * F_k$, where each G_i is finitely generated, freely indecomposable and not infinite cyclic, and F_k is a free group of rank k, with $n+k \geq 2$. Suppose that each factor G_i contains a torsion-free, normal subgroup of finite index H_i such that $\mathrm{Out}(H_i)$ is virtually torsion-free and the quotient $H_i/Z(H_i)$ of H_i by its center $Z(H_i)$ is torsion-free. Then $\mathrm{Out}(G)$ is virtually torsion-free.* □

We also need the following simple lemma, whose proof is left to the reader.

Lemma 2.7. *Every polycyclic-by-finite group G has a normal, torsion-free, finite index subgroup H such that the quotient $H/Z(H)$ of H by its center $Z(H)$ is also torsion-free.* □

Proof of Theorem 1.2. By Dyer's results [21], the class of conjugacy separable groups is closed under finite graphs of groups with finite edge groups. Since polycyclic-by-finite groups are conjugacy separable [22], it follows that G is conjugacy separable.

It remains to show that $\mathrm{Out}(G)$ is virtually torsion-free. By [13, Lemma 2.4], it suffices to find a normal subgroup N of finite index in G with trivial center and virtually torsion-free outer automorphism group. Since G is conjugacy separable (and so residually finite), it has a normal subgroup of finite index N which intersects each edge group trivially. This means that N acts non-trivially on the corresponding tree of G with trivial edge stabilizers and therefore N admits a non-trivial free product decomposition $N_1 * \cdots * N_k$ into freely indecomposable, polycyclic-by-finite factors. In particular, the center of N is trivial. To see that $\mathrm{Out}(N)$ is virtually torsion-free, note first that the outer automorphism group of a polycyclic-by-finite group is virtually torsion-free being finitely generated and isomorphic to a subgroup of $GL_n(\mathbb{Z})$, for some positive integer n (see [23]). Thus, in view of Lemma 2.7, the hypotheses of Theorem 2.6 are satisfied and so $\mathrm{Out}(N)$ is virtually torsion-free. □

Acknowledgement

We thank the referee of the previous version for pointing out a gap in the proof of Lemma 2.2.

References

[1] M. Gromov, Hyperbolic groups, in: S.M. Gersten (Ed.), Essays in Group Theory, in: MSRI Series, vol.8, Springer, 1987, pp. 75–263.
[2] B.H. Bowditch, Relatively hyperbolic groups, http://www.maths.soton.ac.uk/staff/Bowditch/preprints.html.
[3] D. Osin, Relatively hyperbolic groups: Intrinsic geometry, algebraic properties, and algorithmic problems, Mem. Amer. Math. Soc. 179 (843) (2006) vi+100 pp.
[4] C. Drutu, M. Sapir, Tree-graded spaces and asymptotic cones of groups, With an appendix by Denis Osin and Sapir, Topology 44 (5) (2005) 959–1058.
[5] B. Farb, Relatively hyperbolic groups, Geom. Funct. Anal. 8 (5) (1998) 810–840.
[6] Z. Sela, Diophantine geometry over groups. I. Makanin–Razborov diagrams, Publ. Math. Inst. Hautes Études Sci. (93) (2001) 31–105.
[7] Z. Sela, Diophantine geometry over groups. II. Completions, closures and formal solutions, Israel J. Math. 134 (2003) 173–254.
[8] B. Baumslag, Residually free groups, Proc. London Math. Soc. (3) 17 (1967) 402–418.
[9] O. Kharlampovich, A. Myasnikov, Irreducible affine varieties over a free group. I. Irreducibility of quadratic equations and Nullstellensatz, J. Algebra 200 (2) (1998) 472–516.
[10] O. Kharlampovich, A. Myasnikov, Irreducible affine varieties over a free group. II. Systems in triangular quasi-quadratic form and description of residually free groups, J. Algebra 200 (2) (1998) 517–570.
[11] F. Dahmani, Combination of convergence groups, Geom. Topol. 7 (2003) 933–963.
[12] E. Alibegović, A combination theorem for relatively hyperbolic groups, Bull. London Math. Soc 37 (3) (2005) 459–466.
[13] V. Metaftsis, M. Sykiotis, On the residual finiteness of outer automorphisms of hyperbolic groups, Geom. Dedicata 117 (2006) 125–131.
[14] E.K. Grossman, On the residual finiteness of certain mapping class groups, J. London Math. Soc. 2 (9) (1974/75) 160–164.
[15] V. Guirardel, G. Levitt, The outer space of a free product, Proc. Lond. Math. Soc. (3) 94 (3) (2007) 695–714.
[16] S.C. Chagas, P.A. Zalesskii, Limit groups are conjugacy separable, Internat. J. Algebra Comput. 17 (4) (2007) 851–857.
[17] M. Coornaert, T. Delzant, A. Papadopoulos, Geométrie et théorie des groupes. Les groupes hyperboliques de Gromov, in: Lecture Notes in Mathematics, vol. 1441, Springer Verlag, 1990.
[18] I. Belegradek, A. Szczepański, Endomorphisms of relatively hyperbolic groups, With appendix by Oleg V. Belegradek, Internat. J. Algebra Comput. 18 (1) (2008) 97–110.
[19] J. Morgan, P. Shalen, Valuations, trees, and degenerations of hyperbolic structures. I, Ann. of Math. (2) 120 (3) (1984) 401–476.
[20] A. Minasyan, D. Osin, Normal automorphisms of relatively hyperbolic groups, http://arxiv.org/abs/0809.2408.
[21] J.L. Dyer, Separating conjugates in amalgamated free products and HNN extensions, J. Aust. Math. Soc. Ser. A 29 (1) (1980) 35–51.
[22] E. Formanek, Conjugate separability in polycyclic groups, J. Algebra 42 (1) (1976) 1–10.
[23] B.A.F. Wehrfritz, Two remarks on polycyclic groups, Bull. London Math. Soc. 26 (6) (1994) 543–548.