Bessel Functions on Finite Groups

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While each character of a finite group determines the underlying matrix representation, up to change of basis, it is often very difficult to give the representation explicitly even when the character is completely explicit. We shall prove a general result which gives representations from characters. The motivating result is a theorem of S. Gelfand [G] which gives certain “cuspidal” representation of $GL(n, q)$ in terms of so-called “Bessel” functions. We shall, in fact, show that Gelfand’s result is much more general and can be done in a wide context.

Let us fix some notation. Let $U$ be a subgroup of the finite group $G$, let $\vartheta$ be a linear character of $U$, and let $\psi$ be an irreducible character of $G$. The corresponding Bessel function $J (= J(\vartheta, \psi))$ on $G$ is defined by $J(g) = \psi(ge_\vartheta)$ (so is also equal to $\phi(e_\vartheta gf_{\psi})$ as $\phi(ab) = \phi(ba)$ always), where $e_\vartheta$ is the corresponding primitive central idempotent. Since $e_\vartheta = \vartheta(1)|U|^{-1} \sum \vartheta(u^{-1})u$ we have

$$J(g) = |U|^{-1} \sum u \vartheta(u^{-1})\psi(gu).$$

We can now state our main result.

THEOREM. If $\phi$ is an irreducible character of the subgroup $K$ of $G$ which contains $U$ and if $\vartheta$ induced to $K$ equals $\phi$ while $\vartheta$ restricted to $K$ also is $\phi$ then the map of $G$ sending the element $g$ to the $r \times r$ matrix $(J(g, gg^{-1}))$, where $r = |K: U|$ and $g_1, \ldots, g_r$ is a set of (right) coset representatives for $U$ in $K$, is a representation with character $\psi$.

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We begin a character-theoretic proof, similar in parts to Gelfand's argument [G], with a few preliminary results.

**Lemma 1.** Assume that $H$ is a subgroup of $G$ and $\psi$ is a character of $G$ with $\psi_H$ irreducible. It follows that for all elements $s, t$ in $G$ we have the equality

$$\frac{1}{|H|} \sum_h \psi(s h^{-1}) \psi(h t) = \psi(s t) / \psi(1).$$

**Proof.** Let $(a_{ij}(g))$ be the matrix representing $g$ in a representation with character $\psi$. Then

$$\sum_h \psi(s h^{-1}) \psi(h t) = \sum_h \sum_{i, k} a_{ij}(sh^{-1}) a_{k l}(ht)$$

$$= \sum_{i, j, k, l} a_{ij}(s) a_{ik}(h) \sum_h a_{k l}(h^{-1}) a_{i j}(h)$$

$$= \sum_{i, j, k, l} a_{ij}(s) a_{ik}(h) (|H| / \psi(1)) \delta_{ik} \delta_{jl}$$

$$= |H| / \psi(1) \sum_{i, j} a_{ij}(s) a_{ij}(t)$$

$$= (|H| / \psi(1)) \psi(st).$$

Applying the lemma to the case $G = H$ we get the following well-known consequence.

**Corollary.** Assume that $\psi$ is an irreducible character. We have for all $u$ in $G$,

$$\frac{1}{|G|} \sum_g \psi(g^{-1}) \psi(гу) = \psi(u) / \psi(1).$$

Keeping the above hypotheses, let $U$ be a subgroup of $H$ and let $\vartheta$ be a linear character of $U$. Let $J$ be as above.

**Lemma 2.** For all $x, y$ in $G$ we have

$$\frac{1}{|H|} \sum_h J(x h^{-1}) J(h y) = J(xy) / \psi(1).$$

**Proof.** We have

$$\sum_h J(x h^{-1}) J(h y) = \frac{1}{|U|^2} \sum_h \sum_u \vartheta(u^{-1}) \psi(x h^{-1} u) \vartheta(i^{-1}) \psi(h y i).$$
and if we put $u^{-1}v^{-1} = w^{-1}$ so $v = wu^{-1}$,

\[ \begin{align*}
&= 1/|U|^2 \sum_h \sum_u \sum_w \vartheta(w^{-1})\psi(xh^{-1}u)\psi(hywu^{-1}) \\
&= 1/|U|^2 \sum_h \sum_u \sum_w \vartheta(w^{-1})\psi(xh^{-1}u)\psi(u^{-1}hyw) \\
&= (1/|U|^2)|U| \sum_w \vartheta(w^{-1}) \cdot |H|/\psi(1) \cdot \psi(xyw),
\end{align*} \]

by Lemma 1,

\[ \begin{align*}
&= (|H|/\psi(1))J(xy).
\end{align*} \]

**Lemma 3.**  $1/|G|\sum_u J(u)\psi(g^{-1}) = J(1)/\psi(1)$.

**Proof.**

\[ \begin{align*}
1/|G| \sum_u J(u)\psi(g^{-1}) &= 1/|G| \cdot 1/|U| \cdot \sum_u \sum_g \vartheta(u^{-1})\psi(gu)\psi(g^{-1}) \\
&= 1/|U| \sum_u (\vartheta(u^{-1})\psi(u))/\psi(1),
\end{align*} \]

by Lemma 1,

\[ \begin{align*}
&= J(1)/\psi(1).
\end{align*} \]

Note that for all $g$ in $G$ and $u$ in $U$, we have

\[ J(ug) = J(gu) = \vartheta(u)J(g). \]

If $e_\alpha$ is as above, we define for $k$ in $H$ and $g$ in $G$, as in [G],

\[ (e_\alpha k)^{\circ} h = \psi(1)/|H| \sum_h J(kgh^{-1})e_\alpha h, \]

which will lead us to the desired right $G$-modules. Then,

\[ \begin{align*}
((e_\alpha k)^{\circ} x)^{\circ} y &= (\psi(1)/|H|)^2 \sum_{a,b} J(kxa^{-1})J(ayb^{-1})e_\alpha b \\
&= \psi(1)/|H| \sum_b J(kxyb^{-1}), \quad \text{by Lemma 2,} \\
&= (e_\alpha k)^{\circ} (xy).
\end{align*} \]

Now we assume we are in the situation of the Theorem and apply the above to $H = K$. We have, in particular, that $(\psi_U, \vartheta) = 1$ and $\psi(1) = |K: U|$. Thus, $J(u) = 1$ for all $u$ in $U$ while $J(h) = 0$ if $h$ is in $K$ and not
in $U$. Hence, for any $k$ in $K$, we have

$$(e_\phi k) \cdot 1 = \psi(1)/|K| \sum_u J(u^{-1})e_\phi uk = e_\phi k.$$ 

Thus, we have a representation. Take the corresponding $|K: U| \times |K: U|$ matrix to be $M(g)$ for any $g$ in $G$ in the natural way so that $M_{1,1}(g) = J(g)$. Now $M$ has a constituent factor which affords $\psi$ if, and only if,

$$\sum_g M_{1,1}(g)\psi(g^{-1}) \neq 0.$$ 

But

$$\sum_g M_{1,1}(g)\psi(g^{-1}) = \sum_g J(g)\psi(g^{-1}) = |G|/\psi(1),$$

by Lemma 3, so therefore, $M$ has a composition factor which affords $\psi$ and since $M$ and $\psi$ have the same dimension we have that $M$ has character $\psi$.

We shall now give another proof, which is ring-theoretic. We preserve the notation and we prove five preliminary statements. First,

1. $e_\phi CG$ affords the character $\vartheta^G$.

Indeed, the elements $e_\phi t$ are linearly independent as $t$ runs over a set of coset representatives for $U$ in $G$.

2. $e_\phi e_\phi CG$ affords $\psi$.

This follows from $(\psi, \vartheta^G) = (\psi, \vartheta^K) = (\varphi, \varphi) = 1$.

3. If $g$ is in $G$ then $e_\phi ge_\phi \cdot e_\phi e_\phi = \psi(e_\phi ge_\phi)e_\phi e_\phi$.

Before proving this, note that we are saying that $\psi$ is a character of a one-dimensional representation of $e_\phi CGe_\phi$, the endomorphism algebra of $e_\phi CG$—the induced module, by (1)—where the action is left multiplication. In any case, left multiplication by $e_\phi ge_\phi$ is an endomorphism of the simple module $e_\phi e_\phi CG$ so it is a scalar multiplication and we have to prove that the scalar is $\psi(e_\phi ge_\phi)$.

To see that (3) holds, we first claim that $e_\phi e_\phi$ is a rank one idempotent in $e_\phi CG$. If $V_\phi$ is a $CG$-module affording $\vartheta$ then we have that the rank of $e_\phi e_\phi$ is the dimension of $V_\phi e_\phi e_\phi$, that is, of $V_\phi e_\phi$. But this is the dimension of the sum of all the submodules affording $\vartheta$ in $(V_\phi)_U$. But $(\psi_U, \vartheta) = 1$, as we have seen above, so our claim holds.

We have a homomorphism $M$ of $CG$ to matrices corresponding to the module of (2) affording $\psi$ and we can now assume that $M(e_\phi e_\phi)$ is the
matrix unit \( e_{11} \). Thus,

\[
M(e_{\alpha} g e_{\alpha}) = M(e_{\alpha} e_{\alpha} g e_{\alpha}) = e_{11} M(g) e_{11},
\]

since \( M(e_{\alpha}) \) is the identity matrix. Hence, \( M(e_{\alpha} g e_{\alpha}) \) has all entries zero except the 11 entry so this must be the trace of \( e_{\alpha} g e_{\alpha} \) on the module, that is, \( \psi(e_{\alpha} g e_{\alpha}) \). But that is the trace for right multiplication; however, the elements \( e_{\alpha} e_{\phi} \) and \( e_{\alpha} g e_{\alpha} \) commute, so now (3) holds.

Our next result is the following, in the notation of the Theorem.

(4) The elements \( e_{\alpha} e_{\phi} g_i \) are linearly independent and so form a basis of \( e_{\alpha} e_{\phi} CG \).

The second part follows from the first. Since \( \psi(1) = \varphi(1) = |K: U| \) we can express

\[
e_{\phi} = (|K|/|G|) e_{\phi} + \ldots,
\]

where the dots correspond to a linear combination of elements from \( G - K \); for \( \psi_K = \varphi \). Thus, multiplication by \( e_{\alpha} g_i \) and inspection of the terms in \( K \) give the result. The final preliminary statement is the next one. It follows immediately from the formula for \( e_{\phi} \) and the usual formula for induced characters in terms of coset representatives.

(5) \( \sum_j g_j^{-1} e_{\alpha} g_i = e_{\phi} \).

In order to prove the Theorem it remains only to use the module of (2) and the basis of (4) to get the desired matrix. Hence, we wish to prove that for any \( x \) in \( G \),

\[
e_{\phi} e_{\alpha} g_i x = \sum_j \psi(e_{\alpha} g_i x g_j^{-1} e_{\alpha}) e_{\alpha} g_j.
\]

But the right-hand side equals, by (3),

\[
\sum_j e_{\alpha} g_i x g_j^{-1} e_{\alpha} \cdot e_{\alpha} e_{\phi} \cdot g_j = \sum_j e_{\alpha} e_{\alpha} g_i x g_j^{-1} e_{\alpha} g_j
\]

\[
= e_{\alpha} e_{\alpha} g_i x \sum_j g_j^{-1} e_{\alpha} g_j
\]

\[
= e_{\alpha} e_{\alpha} g_i x \cdot e_{\phi}
\]

\[
= e_{\alpha} e_{\alpha} g_i x,
\]

since \( e_{\alpha} \) induces the identity transformation on any module affording \( \psi \) since \( \psi \) restricts to \( \varphi \).
Finally, we shall conclude with a short module-theoretic proof, which is certainly the most conceptual of our proofs. We change notation and here let $M = e_\alpha CG$ be the induced module (see (1)) and let $E$ be its endomorphism algebra identified with $e_\alpha CG e_\alpha$ acting on the left. Since $M$ is a left $E$-module and a right $CG$-module, the functor $\text{Hom}_{CG}(M, -)$ sends right $CG$-modules to right $E$-modules and the functor $- \otimes_{\alpha} M$ does the opposite. Since $M$ is semisimple so is $E$ and there is a well-known (see, e.g., [A], [S]) one-to-one correspondence between isomorphism classes of simple summands of the right $CG$-module $M$ and the isomorphism classes of simple summands of the left $E$-module $E$ given by these functors $\text{Hom}_{CG}(M, -)$ and $- \otimes_{\alpha} M$.

Let $M_\phi$ be the (unique) summand of $M$ affording $\psi$. We assert that $\text{Hom}_{CG}(M, M_\phi)$ is a one-dimensional $E$-module where the element $e_\alpha g e_\phi$ acts on the right by multiplication by $\psi(e_\alpha g e_\phi)$. Indeed, write $M = M_\phi \oplus M^\phi$, the unique direct decomposition with $M_\phi$ as a summand (since $M$ has a unique summand isomorphic to $M_\phi$) so the elements of $\text{Hom}_{CG}(M, M_\phi)$ are scalar multiplications on $M_\phi$ and zero on $M^\phi$. However, by [CF] (or the above arguments), $e_\alpha g e_\phi$ acts on $M_\phi$ by multiplication by $\psi(e_\alpha g e_\phi)$ so the claim follows at once.

Let $N_\phi$ be a one-dimensional right module for $E = e_\alpha CG e_\alpha$ with $n \cdot e_\alpha g e_\phi = \psi(e_\alpha g e_\phi) n$ so that we have $N_\phi \otimes_{E} M \cong M_\phi$. We know that the elements $e_\alpha g_i$ span $M$. Therefore, if $n$ is a non-zero element of $N$, then the elements $n \otimes e_\alpha g_i$ span $N_\phi \otimes_{\alpha} M$. But the dimension of $M_\phi$ is equal to the number of these coset representatives $g_i$, so these elements are a basis of $N_\phi \otimes_{\alpha} M$. Hence, if $x$ is in $G$, it suffices, by the definition of $J$, to prove that

$$\sum_j (n \otimes e_\alpha g_i) x = \sum_j \psi(e_\alpha g_i x g_j^{-1} e_\alpha) n \otimes e_\alpha g_j.$$  

However, the right-hand side equals

$$\sum_j n \cdot e_\alpha g_i x g_j^{-1} e_\alpha \otimes e_\alpha g_j = \sum_j n \otimes e_\alpha g_i x g_j^{-1} e_\alpha \cdot e_\alpha g_j = n \otimes e_\alpha g_j x \sum_j e g_j^{-1} e_\alpha g_j = n \otimes e_\alpha g_j x e_\phi,$$

since $e_\phi$ is the identity on $M$ as $M_K$ affords $\phi$.

Other proofs have been subsequently shown to us by I. M. Isaacs and R. Knorr: the hypotheses of the Theorem force the representation afford-
ing $\psi$ to be unique as an extension of the representation affording $\varphi$; the formula in the Theorem can be deduced from this uniqueness. Our last proof is quite natural as it shows the representation arising in a very natural fashion. We suspect that this last proof is not only conceptual but can be generalized: constructions of representations of Hecke algebras can be turned into group representations using the tensor product functor.

REFERENCES


