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Approximation of nonlinear dynamic systems by rational series

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Abstract

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Given an analytic system, we compute a bilinear system of minimal dimension which approximates it up to order k (i.e. the outputs of these two systems have the same Taylor expansion up to order k). The algorithm is based on noncommutative series computation: let s be the generating series of the analytic system; then a rational series g is constructed, whose coefficients are equal to those of s, for all words of length smaller than or equal to k. These words are digitally encoded, in order to simplify the computations of the Hankel matrices of s and g. We then associate with g, a bilinear system, which is a solution to our problem. Another method may be used for computing a bilinear system which approximates a given analytic system (S). We associate with (S) an $\mathbb{R}$-automaton of vector fields and build the truncated automaton by cancelling all the states which have the following property: the length of the shortest successful path labelled by a word that gets through this state is strictly greater than k. Then, the number of states of this truncated automaton yields the dimension (not necessarily minimal) of the state-space.

1. Introduction

Several methods may be used for determining the input-output behavior of a dynamic system: transfer functions, functional expansions (Volterra series) [6], and generating power series [5]. With all these descriptions, we have the following problem: is it possible to construct a suitable approximation of the input-output behavior of any dynamic system using a more elementary system?

For single input systems, the transfer function can be used to find a linear approximation by means of Padé approximants [2, 3]. Nevertheless it is not possible in general to approximate nonlinear systems by linear ones.

Another way for determining approximations of dynamic systems is to compute its first Volterra kernels [6], but in this case one must solve the new problem of the realization of this truncated Volterra series [12].

The formal power series in several noncommutative variables are the most efficient tools for dealing with functional expansions. One cannot approximate any nonlinear functional with a linear system (i.e a rational series in one variable). But it is possible to approximate any nonlinear functional with a bilinear system (i.e. with a polynomial or a rational series in several noncommutative variables).

In this paper, a new algorithm is presented which enables us to compute a minimal rank bilinear system (B) which approximates a given dynamic system (S)

$$(\mathrm{S})\qquad \begin{cases} \dot q(t) = A_0(q) + \sum_{i=1}^{m-1} u_i(t) A_i(q), \\ y(t) = f(q), \end{cases}$$

up to order k (i.e. the Taylor expansions of the output of (S) and (B) are equal up to order k). As Fliess showed [5, 6], the input–output behavior of a dynamic system (S), can be coded by a noncommutative formal power series, called the generating series of the system (S).

A generating power series is known to correspond to a finite-dimensional bilinear system if and only if it is rational. Furthermore it is known that a series s is rational if and only if the rank of its Hankel matrix $H(s)$ is finite.

The first method developed here is the following: given a generating series s of (S), one builds a rational series g (associated with bilinear system (B)) which coincides with s for all words of length $\leqslant k$ and is of minimal rank.

The second method to compute a bilinear system (B) which approximates a given dynamic system (S) is as follows: with the given system (S), we associate an $\mathbb{R}$-automaton $\mathscr{A}$. As a series is rational if and only if it is recognized by a finite $\mathbb{R}$-automaton [11], then we build a truncated automaton τ (associated with the bilinear system (B)), from the $\mathbb{R}$-automaton $\mathscr{A}$ associated with (S).

2. Notations [1]

Let $X = \{x_1, \ldots, x_n\}$ be a finite, nonempty set called alphabet. The set of finite sequences $x_{j1} \ldots x_{jl}$ of elements of X is the free monoid X^* (with concatenation and neutral element 1).

An element of X^* is a word $w = x_{j1} \ldots x_{jl}$; the length l of which is denoted by $|w|$.

A formal series s with coefficients in $\mathbb{R}$ is a function

$$X^* \to \mathbb{R}, \qquad \omega \to (s, \omega)$$

where (s, ω) is called the coefficient of ω in s.

We note by $\mathbb{R}\langle\langle X\rangle\rangle$ the set of formal series over X with coefficients in $\mathbb{R}$. A structure of a semiring is defined on $\mathbb{R}\langle\langle X\rangle\rangle$ (sum and Cauchy product). The set of polynomials is denoted by $\mathbb{R}\langle X\rangle$.

A formal series $s \in \mathbb{R}\langle\langle X \rangle\rangle$ is proper if the coefficient of the empty word is equal to 0. The sum of the locally finite family $(s^n)_{n \geq 0}$ is denoted by s^* and is called the star of s:

$$s^* = \sum_{n \geq 0} s^n$$

The rational operations in $\mathbb{R}\langle\langle X \rangle\rangle$ are the sum, the Cauchy product, the two external products of $\mathbb{R}$ on $\mathbb{R}\langle\langle X \rangle\rangle$ and the star operation.

The smallest subset containing a subset E of $\mathbb{R}\langle\langle X \rangle\rangle$ and which is rationally closed (i.e. closed for the rational operations) is called the rational closure of E.

A formal series is rational if it belongs to the rational closure of $\mathbb{R}\langle X \rangle$.

A formal series $s \in \mathbb{R}\langle\langle X \rangle\rangle$ is called recognizable if there exists an integer $n \geq 1$, a morphism of monoids

$$\mu : X^* \to \mathbb{R}^{n \times n}$$

and two matrices $\lambda \in \mathbb{R}^{1 \times n}$ and $\gamma \in \mathbb{R}^{n \times 1}$ such that

$$(s, w) = \lambda \mu \omega \gamma$$

In this case, the triple (λ, μ, γ) is called a linear representation of s, and n is its dimension. According to the Fundamental Theorem of Schutzenberger [11], a formal series is recognizable if and only if it is rational.

The Hankel matrix of a formal series s is the matrix $H(s)$ indexed over $X^* \times X^*$ and defined by

$$H_{u,v}(s) = (s, uv), \quad \forall u, v \in X^*$$

and $\mathrm{rk}(s)$ is the rank of its Hankel matrix.

A series is known to be rational if and only if its rank is finite. In this case, its rank is equal to the minimum of the dimensions of the linear representations of s [4].

3. Approximation by a minimal rank bilinear system

3.1 Approximation of a formal series by a rational series [7]

Let s be a given formal series, we want to build a rational series of minimal rank, which approximates s up to order k.

For that purpose, we compute the Hankel matrix $H(p)$ of the polynome p obtained by truncating s by cancelling the coefficients of the words of length greater than k.

Example 3.1. See fig. 1 for $k = 2$.

Then, we try to find recursively a linearly dependent relation between one column and the precedent columns, by substituting nonzero values for some zeros, while preserving the Hankel matrix structure.

	1	x_0	x_1	x_0^2	x_0x_1	x_1x_0	x_1^2
1	d	a	1	ab	b	0	0
x_0	a	ab	b				
x_1	1	0	0				
x_0^2	ab						
x_0x_1	b				O		
x_1x_0	0						
x_1^2	0						

Fig. 1.

Let us denote $H_{\leq l,\leq c}$ the Hankel matrix of p, restricted to the rows indexed by the words u of length $\leq l$, as well as to the columns indexed by the words v of length $\leq c$.

In the previous example, the column vector C_1 indexed by 1 and the column vector C_{x_1} indexed by x_1, are linearly independent in $H_{\leq 2,\leq 0}$, $H_{\leq 1,\leq 1}$, $H_{\leq 0,\leq 2}$, the other column vectors being linearly dependent in these blocks:

$$C_{x_0x_1} = bC_{x_1},$$

$$C_{x_1x_0} = C_{x_1^2} = 0,$$

$$C_{x_0^2} = bC_{x_0},$$

$$C_{x_0} = aC_{x_1}.$$

In this way, we obtain a minimal rank (=2) rational series even though the rank of $H(p)$ is equal to 3 (Fig. 2).

	1	x_0	x_1	x_0^2	x_0x_1	x_1x_0	x_1^2	x_0^3 ......
1	d	a	1	ab	b	0	0	ab^2 ......
x_0	a	ab	b	ab^2	b^2	0	0	ab^3 ......
x_1	1	0	0	0	0	0	0	.
x_0^2	ab	ab^2	b^2	ab^3	b^3	0	0	.
x_0x_1	b	0	0	0	0	0	0	.
x_1x_0	0	0	0	0	0	0	0	.
x_1^2	0	0	0	0	0	0	0	.
x_0^3	ab^2	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.

Fig. 2.

3.2 Application to the bilinearization of dynamic systems [8]

3.2.1. Method

Consider the dynamic analytic systems, of the form

$$(\mathrm{S})\quad \begin{cases} \dot{q}(t)=A_0(q)+\sum_{i=1}^{m-1} u_i(t)A_i(q), \\ y(t)=f(q). \end{cases}$$

- q is the state which belongs to a finite dimensional $\mathbb{R}$-analytic manifold V,
- vector fields $A_0, A_1, \dots, A_{m-1}$ and output function f are analytic and defined in a neighborhood of the initial state $q(0)$.
- $u(t)={}^t(u_1(t)\dots u_{m-1}(t))$ is the input function where $u_1, \dots, u_{m-1}$ are piece-wise continuous.

We build an approximation of these systems by bilinear ones, under the form

$$(\mathrm{B})\quad \begin{cases} \dot{\underline{q}}(t)=\left(M_0+\sum_{i=1}^{m-1} u_i(t)M_i\right)\underline{q}(t), \\ \underline{y}(t)=\lambda\underline{q}(t). \end{cases}$$

- $\underline{q}$ belongs to a finite dimensional $\mathbb{R}$-vector space Q
- $M_0, \dots, M_{m-1}: Q\to Q$ and $\lambda: Q\to\mathbb{R}$ are $\mathbb{R}$-linear
- Input function $u(t)={}^t(u_1(t)\dots u_{m-1}(t))$ where $u_1, \dots, u_{m-1}$ are piece-wise continuous.

We say that (B) approximates (S) up to order k, if and only if, whatever the input function, the outputs of these two systems have the same Taylor expansion up to order k.

As Fliess showed [5, 6], the input-output behavior of a system (S) can be coded by a noncommutative formal power series s, called the generating series of the system.

For that purpose, with vector fields $A_0, \dots, A_{m-1}$, he associates an alphabet $X=\{x_0, x_1, \dots, x_{m-1}\}$. The generating series is noted

$$s=\sum_w (s, w)w.$$

The generating power series may be rewritten as [6]

$$s=f|_{q(0)}+\sum_{\nu\geq 0}\ \sum_{j_0,\dots,j_\nu=0}^{m-1} A_{j_0}\dots A_{j_\nu}f|_{q(0)}x_{j_\nu}\dots x_{j_0}.$$

A generating power series g corresponds to a finite-dimensional bilinear system if and only if it is rational. The rank of g is also the dimension of the state-space of the reduced system [10].

The approximation of (S) with (B) up to order k amounts to the problem of the approximation of a formal series s with a rational one, g.

3.2.2 Algorithm for the computation of a bilinear system of minimal rank [9]

Consider the following problem. Which is the better coding by successive integers of words $\in X^*$, in order to make a matrix calculation (using arrays) of the Hankel matrix of s?

Consider for $m=3$ the word tree shown in Fig. 3; let pre be the precoding (defined below) in the base m notation, and code the coding: We define first the following preliminary coding:

$$\text{precoding}(1)=1$$

$$\text{precoding}(wx_s)=m\times\text{precoding}(w)+s, \quad \forall w\in X^* \quad \forall x_s\in X.$$

Thus,

$$\text{precoding}(x_{i_1}\ldots x_{i_l})=m^l+i_1m^{l-1}+\cdots+i_lm^0.$$

The word $x_{i_1}\ldots x_{i_l}$ is then precoded, in the base m notation, by $1i_1\ldots i_l$.

It may be noticed that this coding produces some "holes", for $m>2$ (between the last word of length $l-1$ and the first word of length l). We obtain the coding by substracting the number of holes from this precoding:

$$\text{coding }(x_{i_1}\ldots x_{i_l})=m^l+i_1m^{l-1}+\cdots+i_lm^0-(m-2)\frac{m^l-1}{m-1}$$

Moreover, let us emphasize that

$$\text{coding}(x_sw)=\text{coding}(w)+(s+1)m^l.$$

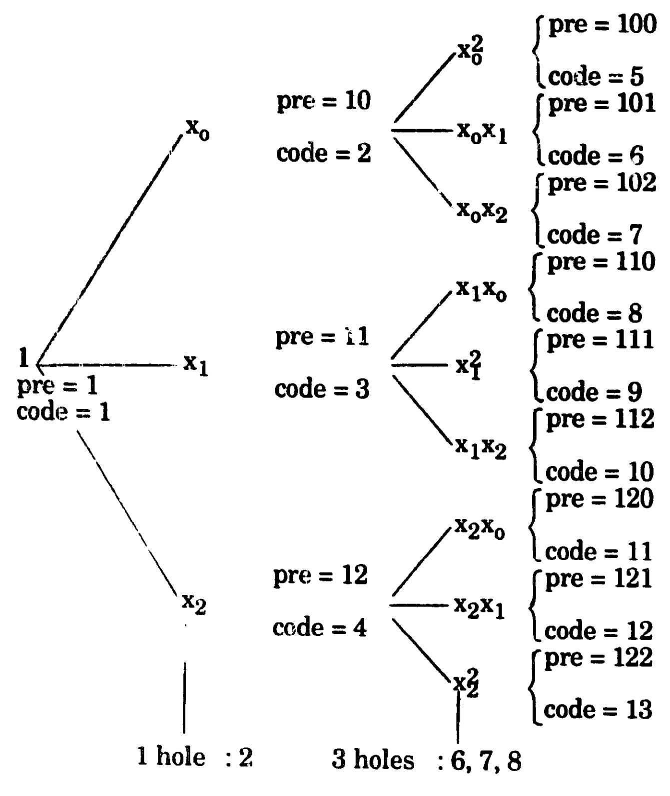

Fig. 3.

This coding of the words allows easy storage of a formal series by the array of its coefficients in $H(s)$.

As the algorithm is written in MACSYMA language, we can make a formal calculation and the generating series s is parameterized by the initial state $q(0)$.

3.2.3 Comparison with the approximation given by the tangent linear system

Example 3.2

$$(\mathrm{S}_2)\quad \begin{cases} A_0 = (-k_1 q_2 - k_2 \sin(q_1))\dfrac{\partial}{\partial q_2} + q_2 \dfrac{\partial}{\partial q_1} \\ A_1 = (1+q_2)\dfrac{\partial}{\partial q_2} \\ y(t) = q_2 \end{cases}$$

A bilinear system (B_2), approaching (S_2) up to order $k = 5$, except for the singular points, can be determined using the previous algorithm:

$$(\mathrm{B}_2)\quad \begin{cases} \dot{\underline{q}}(t) = (M_0 + u_1(t) M_1)\underline{q}(t) \\ \underline{y}(t) = \lambda \underline{q}(t) \end{cases}$$

M_0, M_1, λ being parameterized by the initial state:

$$M_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & \dfrac{q_2+1}{q_2} & \beta b & \mu b & yb \\ 0 & 0 & \dfrac{(2q_2+1)\sin(q_1)k_2 + q_2^2 k_1}{q_2^2+q_2} & \gamma b & \nu b & zb \\ 0 & 1 & 0 & \%r10 & \%r14 & \%r18 \\ 0 & 0 & 1 & \varepsilon b & \pi b & Kb \\ 0 & 0 & 0 & \%r9 & \%r13 & \%r17 \end{pmatrix}$$

$$M_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta c & \mu c & yc \\ 1 & 0 & 0 & \gamma c & \nu c & zc \\ 0 & 0 & 0 & \%r12 & \%r16 & \%r20 \\ 0 & 1 & 0 & \varepsilon c & \pi c & Kc \\ 0 & 0 & 1 & \%r11 & \%r15 & \%r19 \end{pmatrix}$$

$$\lambda = {}^t\begin{pmatrix} q_2 \\ -\sin(q_1)k_2 - q_2 k_1 \\ q_2+1 \\ -k_1(-\sin(q_1)k_2 - q_2 k_1) - q_2\cos(q_1)k_2 \\ -\sin(q_1)k_2 - q_2 k_1 \\ q_2+1 \end{pmatrix}$$

$\%r_i$ are parameters which may be chosen arbitrarily, and the other terms, appearing in M_0 and M_1, are functions of q_1, q_2. The processing is thus generic for every point except for singular ones. At these points, the Hankel matrix is easy to compute numerically. The algorithm yields a numerical minimal bilinear system, approximating the given system, at these points.

The computation of the tangent linear approximant needs the system equation to be rewritten, at every operating point.

In Example 3.2, the generating series expression s of (S_2) and the generating series expression u of the tangent linear system around $q(0)$, show a difference, from length 2 on, except for singular points:

$$(s, x_1x_0) - (u, x_1x_0) = (s, x_1x_0) = -k_1q_2(0) - k_2\sin(q_1(0)),$$

$$(s, x_1^2) - (u, x_1^2) = (s, x_1^2)$$
$$= 1 + q_2(0).$$

Then, from order 2 on, the tangent linear approximant is invalid.

4. Bilinearization via the vector fields automaton

4.1. The vector fields automaton

Recall that the generating series of a dynamic system (S) may be written as

$$s = f|_{q(0)} + \sum_{\nu \geq 0} \sum_{j_0, \ldots, j_\nu = 0}^{m-1} A_{j_0} \ldots A_{j_\nu} f|_{q(0)} x_{j_\nu} \ldots x_{j_0}$$

that is

$$s = \sum_{w \in X^*} A_{\tilde{w}} \circ f|_{q(0)} w$$

where $\tilde{w}$ is the mirror image of w.

Therefore, the problem reduces to checking the differential operators:

$$A_w = \sum_\alpha A_{w,\alpha} D^\alpha$$

with

$$D^\alpha = D_1^{i_1} \ldots D_N^{i_N},$$

$$D_1 = \frac{\partial}{\partial q_1}, \ldots, D_N = \frac{\partial}{\partial q_N}.$$

In order to compute these operators, define the following automaton operating on the right whose

states are D^α,

letter action $D^\alpha * x_l$ is given by (1),

and initial state is I.

Using Leibnitz formula, we can deduce that

$$D^{\alpha}A_l=\sum_{j=1}^{N}\sum_{\beta}C_{\alpha}^{\beta}D_{\alpha}^{\beta}(\theta_l^j)D^{\alpha-\beta}D_j \tag{1}$$

where

$$A_l=\sum_{j=1}^{N}\theta_l^j(q_1,\ldots,q_N)D_j.$$

Moreover, we want to check $A_w\circ f$, that is $\sum_{\alpha}A_{w,\alpha}D^{\alpha}(f)$.

Thus, the observation function f enables us to define the final state vector F, determined by its components $F_{\alpha}=D^{\alpha}f$.

4.2. *The truncated automaton*

In order to compute an approximant bilinear system, whose rank is not necessarily minimal, the truncated vector fields automaton can be used. This automaton is constructed by choosing the states D^{α} that are met along a successful path and the length of which is smaller than of equal to k. Therefore the cut off automaton can be obtained. The number of states appearing in this latter automaton can be easily seen to be the rank of the corresponding bilinear system.

Example 4.1. The Duffing equation is

$$y''+ay'+b^2y+cy^3=u_1(t),$$

that is

$$\dot q(t)=A_0(q)+u_1(t)A_1(q),$$
$$y(t)=q_1(t),$$

with

$$F=-aq_2-b^2q_1-cq_1^3,$$

$$A_0=F\frac{\partial}{\partial q_2}+q_2\frac{\partial}{\partial q_1}=FD_2+q_2D_1,$$

$$A_1=\frac{\partial}{\partial q_2}=D_2.$$

The actions of A_0 and A_1 produced by the letters x_0, x_1 on states $D_1^iD_2^j$, are given by

$$D_1^iD_2^jA_1=D_1^iD_2^{j+1},$$

$$D_1^iD_2^jA_0=FD_1^iD_2^{j+1}+C_i^1F'D_1^{i-1}D_2^{j+1}+C_i^2F''D_1^{i-2}D_2^{j+1}$$
$$+C_i^3F'''D_1^{i-3}D_2^{j+1}-ajD_1^iD_2^j+q_2D_1^{i+1}D_2^j+jD_1^{i+1}D_2^{j-1}.$$

They enable us to obtain the vector fields automaton cell shown in Fig. 4 and the automaton shown in Fig. 5.

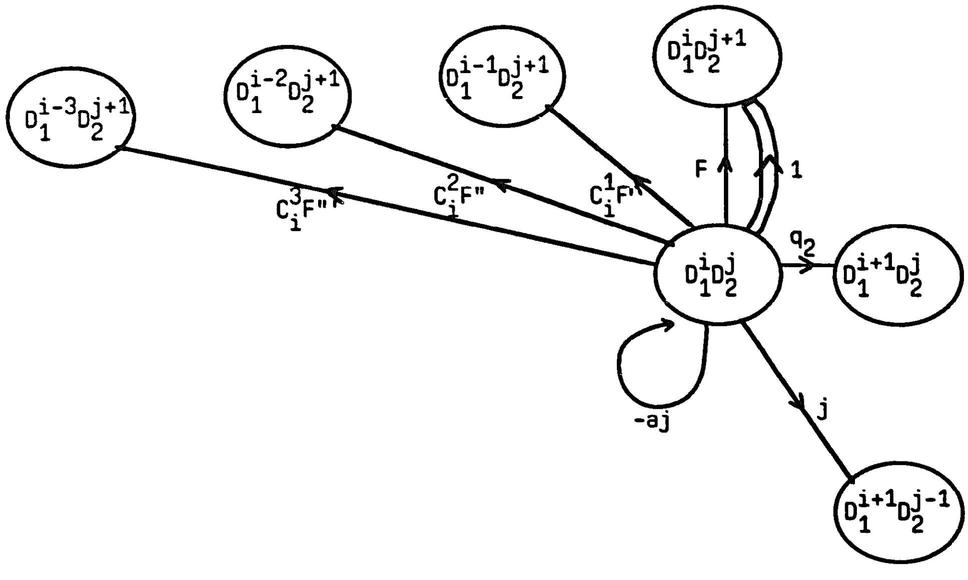

Fig. 4.

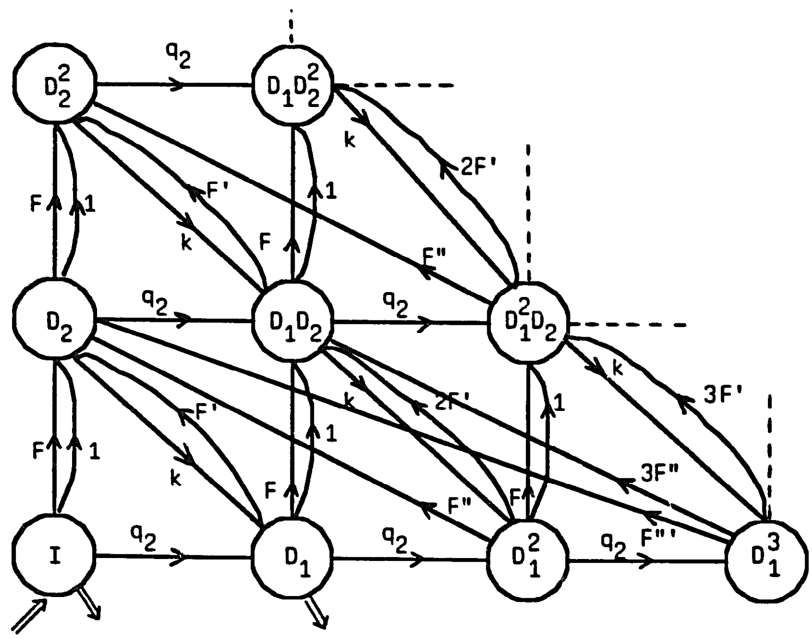

Fig. 5

The truncated automaton can be determined by picking out from these states, the states D^α which are met along a successful path and the length of which is smaller than or equal to k.

Thus we compute

j_{acc} = length of the shortest path between I and D^α,

$j_{\text{co-acc}}$ = length of the shortest path between D^α and a final state.

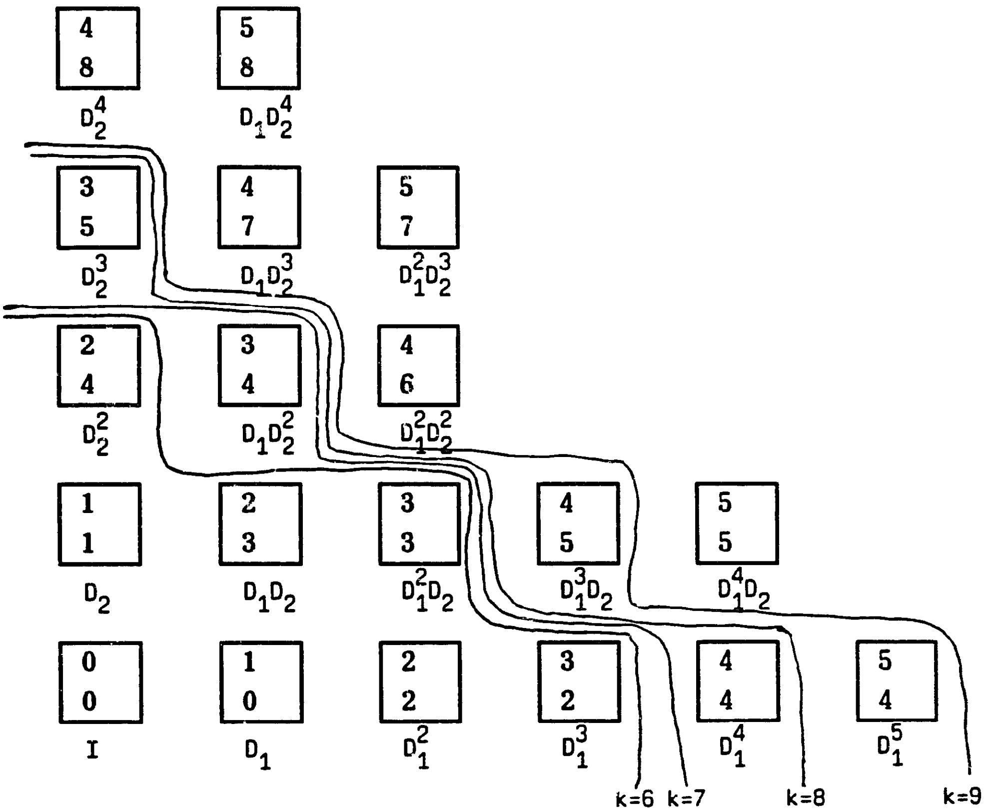

Fig. 6.

	I	D_1	D_2	D_1^2	D_1D_2	D_2^2	D_1^3	$D_1^2D_2$	$D_1D_2^2$
1	q_1	1	0						0
x_o	q_2	0	1	0					0
x_1	0								0
x_o^2	F	F'	-a	F''	0	0	F'''	0	0
x_ox_1	0								0
x_1x_o	1	0							0
x_1^2	0								0
x_o^3	$-aF+q_2F'$	$q_2F''-aF'$	a^2+F'	$q_2F'''-aF''$	F''	0	-aF'''	F'''	0
x_1^3	0								0
x_o^4	$F(a^2+F')$ $+q_2(q_2F''-aF')$	$F'(a^2+F')$ $+FF''$ $+q_2(q_2F'''-aF'')$	$-a(a^2+2F')$ $+2q_2F''$	$F''(a^2+3F')$ $+F'''(F-aq_2)$	$-2aF''$ $+2q_2F'''$	2F''	$3F''^2$ $+F'''(a^2+4F')$	-2aF'''	2F'''
$x_1x_o^3$	a^2+F'	F''	0	F'''	0				0

Fig. 7.

Let $\boxed{\begin{smallmatrix} j_{\mathrm{acc}} \\ j_{\mathrm{co\text{-}acc}} \end{smallmatrix}}$ denote the pair $(j_{\mathrm{acc}}, j_{\mathrm{co\text{-}acc}})$ for every state D^α.

For instance, for $k=7$, a truncated automaton containing 9 states, is obtained this way (Fig. 6).

By truncating the vector fields automaton, a rational series realization is obtained, which approximates the given series up to order k.

The truncated automaton is completely accessible but generally, it is not completely observable. Let $M = (D^\alpha \ast \cdot \circ f)$. M is a matrix whose rows are indexed by words w and whose columns are indexed by the states D^α. The element of row w, column D^α is $D^\alpha A_w f$. In this matrix (Fig. 7), the columns indexed by the states $D_1 D_2^2$ and $D_1^2 D_2$ can be readily seen to be linear combinations of the other columns.

A minimal automaton may be obtained in this way, whose rank, equal to 7, is minimal.

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