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Character Sum Identities in Analogy with Special Functions Identities

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In the beginning of the century Barnes found the identity

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma(a+s)\,\Gamma(b-s)\,\Gamma(c+s)\,\Gamma(d-s)\,ds$$
$$=\frac{\Gamma(a+b)\,\Gamma(b+c)\,\Gamma(c+d)\,\Gamma(a+d)}{\Gamma(a+b+c+d)}$$

For a, b, c, d complex numbers such that none of $a+b$, $b+c$, $c+d$, $a+d$ is 0 or a negative integer (pole of the gamma-function); the path of integration is chosen such that the poles of $\Gamma(a+s)\,\Gamma(c+s)$ are separated from those of $\Gamma(b-s)\Gamma(d-s)$.

For a proof see [Ba, HS, S, or WW].

Replacing the gamma-function Γ by the Gaussian-sum-function G we obtain the "finite Barnes identity"

$$\frac{1}{q-1}\sum_{S}\mathrm{G}(AS)\,\mathrm{G}(BS^{-1})\,\mathrm{G}(CS)\,\mathrm{G}(DS^{-1})$$
$$=\frac{\mathrm{G}(AB)\,\mathrm{G}(BC)\,\mathrm{G}(CD)\,\mathrm{G}(AD)}{\mathrm{G}(ABCD)},$$

where A, B, C, D are elements of the group X of multiplicative characters of the finite field F_q of q elements such that the product $ABCD$ is not the trivial character; the summation is extended over all S in X; the Gaussian-sum-function is defined on X and the values are complex numbers; for T in X we have

$$\mathrm{G}(T)=\sum T(a)\,\psi(a)$$

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summing over a in F_q with $a \neq 0$; here ψ is a fixed non-trivial additive character of F_q.

For a short direct proof of the finite Barnes identity see [HP 91], for a simultaneous proof of the classical and the finite identity see [HS], for a connection with the principal series representations of the finite group $\mathrm{GL}(2, F_q)$ (see [HP 86, LS, and HP 90]); in the last reference you can also find some twisted analogues of the finite identity, twisted in the sense of Galois theory; one of these twisted identities corresponds to the discrete series of representations of $\mathrm{GL}(2, F_q)$.

The Dixon summation formula can be expressed as

$$\sum_s \frac{(-1)^s}{(a+s)!\,(a-s)!\,(b+s)!\,(b-s)!\,(c+s)!\,(c-s)!} = \frac{(a+b+c)!}{a!\,b!\,c!\,(a+b)!\,(b+c)!\,(a+c)!}$$

with a, b, c, s integers, see [Z 87 and K].

The following character sum identity has been proved by R. J. Evans:

$$\frac{1}{q-1}\sum_S S(-1)\,\mathrm{G}(AS)\,\mathrm{G}(AS^{-1})\,\mathrm{G}(BS)\,\mathrm{G}(BS^{-1})\,\mathrm{G}(CS)\,\mathrm{G}(CS^{-1})$$
$$= \mathrm{G}(AB)\,\mathrm{G}(BC)\,\mathrm{G}(AC)\left(\frac{\mathrm{G}(A)\,\mathrm{G}(B)\,\mathrm{G}(C)}{\mathrm{G}(ABC)} + Q(-1)\,\frac{\mathrm{G}(AQ)\,\mathrm{G}(BQ)\,\mathrm{G}(CQ)}{\mathrm{G}(ABCQ)}\right)$$

with A, B, C, S elements of X such that $A^2B^2C^2$ is not the trivial character; the characteristic of the finite field F_q is supposed to be different from 2 (if not the formula simplifies) and Q denotes the quadratic character of the multiplicative group of F_q. For a proof see [E and G].

The twisted Barnes identity corresponding to the discrete series of the group $\mathrm{GL}(2, F_q)$ has the following form

$$-\frac{1}{q-1}\sum_S \mathrm{G}(AS)\,\mathrm{G}(BS)\,\mathrm{G}(\rho(S^{-1}N)) = \frac{\mathrm{G}(\rho(AN))\,\mathrm{G}(\rho(BN))}{\mathrm{G}(AB\rho')}$$

with A, B, S multiplicative characters of F_q, ρ a multiplicative character of the quadratic extension F_{q^2} such that ρ is different from ρ^q; the restriction ρ' of ρ to the multiplicative group of F_q is supposed to be such that $AB\rho'$ is not the trivial character; for x in F_{q^2} the norm $N(x)$ is xx^q; for any C in

X the composition CN of C with the norm N is a multiplicative character of F_{q^2} and the Gaussian sum $\mathrm{G}(\rho(CN))$ is defined by

$$\mathrm{G}(\rho(CN)) = \sum_x \rho(x)\, C(N(x))\, \psi(\mathrm{Tr}(x))$$

with the summation extended over all x in F_{q^2} except $x=0$; note that the composition of the nontrivial additive character ψ of F_q with the trace map Tr from F_{q^2} to F_q is a nontrivial additive character of F_{q^2} and that $\mathrm{G}(\rho(CN))$ is the Gaussian sum of the multiplicative character $\rho(CN)$ with respect to F_{q^2} and this nontrivial additive character.

Let us mention that there are several identities corresponding to the discrete series of representations of the group $\mathrm{GL}(3, F_q)$ and let us give one of them

$$\frac{1}{q-1} \sum_S S(-1)\, \mathrm{G}(AS)\, \mathrm{G}(BS)\, \mathrm{G}(CS)\, \mathrm{G}(\rho(S^{-1}N))$$
$$= \rho'(-1)(q^{-2}\mathrm{G}(\rho(AN))\, \mathrm{G}(\rho(BN))\, \mathrm{G}(\rho(CN)) - q^2);$$

here A, B, C, S are multiplicative characters of F_q while ρ is a multiplicative character of the cubic extension F_{q^3} of F_q; again ρ' denotes the restriction of ρ to the subfield F_q; for x in F_{q^3} the norm $N(x)$ is $xx^q x^{q^2}$; the Gaussian sum $\mathrm{G}(\rho(DN))$ is defined by

$$\mathrm{G}(\rho(DN)) = \sum_x \rho(x)\, \mathrm{D}(N(x))\, \psi(x + x^q + x^{q^2}),$$

where the summation is extended over all x in F_{q^3} except zero; the above identity is valid under the assumption that $ABC\rho' = 1$. For a proof of the identity via the representations and characters of $\mathrm{GL}(3, F_q)$ (see [HP 82], where you can find also some more complicated identities).

Note that all the preceding character sum identities are evaluations of "Mellin Barnes sums," i.e., expressions of the type

$$\sum_S S(x) \frac{\mathrm{G}(A_1 S) \cdots \mathrm{G}(A_r S)\, \mathrm{G}(B_1 S^{-1}) \cdots \mathrm{G}(B_t S^{-1})}{\mathrm{G}(C_1 S) \cdots \mathrm{G}(C_u S)\, \mathrm{G}(D_1 S^{-1}) \cdots \mathrm{G}(D_v S^{-1})}$$

for $x = 1$ or $x = -1$; here $A_1, ..., A_r$, $B_1, ..., B_t$, $C_1, ..., C_u$, $D_1, ..., D_v$ are multiplicative characters of some finite field F, the summation is extended over all multiplicative characters S and x is an element of F.

There are many other interesting identities to be treated; for a discussion of the famous Dyson identity and its character sum analogues; see for example [HP 92] and the work of R. J. Evans cited there.

Much remains to be explored for finite analogues of the Macdonald–Morris identities more generally.

It would be very interesting to find an analogue for D. Zeilberger's approach via holonomic systems.

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