Total domination and matching numbers in graphs with all vertices in triangles

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A set $M$ of edges of a graph $G$ is a matching if no two edges in $M$ are incident to the same vertex. The matching number of $G$ is the maximum cardinality of a matching of $G$. A set $S$ of vertices in $G$ is a total dominating set if every vertex of $G$ is adjacent to some vertex in $S$. The minimum cardinality of a total dominating set of $G$ is the total domination number of $G$. We prove that if all vertices of $G$ belong to a triangle, then the total domination number of $G$ is bounded above by its matching number. We in fact prove a slightly stronger result and as a consequence of this stronger result, we prove a Graffiti conjecture that relates the total domination and matching numbers in a graph.

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1. Introduction

In this paper we continue the study of relations between the total domination number and the matching number of a graph. For notation and graph theory terminology we in general follow [5]. Specifically, let $G = (V, E)$ be a graph with vertex set $V$ of order $n(G) = |V|$ and edge set $E$ of size $m(G) = |E|$, and let $v$ be a vertex in $V$. The open neighborhood of a vertex $v \in V$ is $N_G(v) = \{u \in V \mid uv \in E(G)\}$ and its closed neighborhood is the set $N_G[v] = N_G(v) \cup \{v\}$. For a set $S \subseteq V$, its open neighborhood is the set $N(S) = \bigcup_{v \in S} N(v)$ and its closed neighborhood is the set $N[S] = N(S) \cup S$. The degree of $v$ is $d_G(v) = |N_G(v)|$. We denote the minimum degree of the graph $G$ by $\delta(G)$. If $d_G(v) = k$ for every vertex $v \in V$, we say that $G$ is a $k$-regular graph. If the graph $G$ is clear from the context, we simply write $N(v)$, $N[v]$, $N(S)$, $N[S]$ and $d(v)$ rather than $N_G(v)$, $N_G[v]$, $N_G(S)$, $N_G[S]$ and $d_G(v)$, respectively.

For a subset $S \subseteq V$, the subgraph induced by $S$ is denoted by $G[S]$. If $S \subseteq V$, then by $G - S$ we denote the graph obtained from $G$ by deleting the vertices in the set $S$ (and all edges incident with vertices in $S$). If $S = \{v\}$, then we also denote $G - \{v\}$ simply by $G - v$. A component in $G$ is a maximal connected subgraph of $G$.

A total dominating set, abbreviated as TD-set, of a graph $G = (V, E)$ with no isolated vertex is a set $S$ of vertices of $G$ such that every vertex is adjacent to a vertex in $S$. The total domination number of $G$, denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set. The literature on this subject has been surveyed and detailed in the domination book by Haynes et al. [5]. A recent survey on total domination in graphs can be found in [6]. If $U \subseteq V$, then a set $W$ is said to dominate the set $U$ if $U \subseteq N[W]$, while $S$ totally dominates $U$ if $Y \subseteq N(W)$. In particular, if $S$ dominates $V$, then $S$ is a dominating set in $G$ and the minimum cardinality of a dominating set in $G$ is the domination number of $G$, denoted by $\gamma(G)$.

Two edges in a graph $G$ are independent if they are vertex disjoint in $G$. A set of pairwise independent edges of $G$ is called a matching in $G$, while a matching of maximum cardinality is a maximum matching. The number of edges in a maximum matching of $G$ is called the matching number of $G$ which we denote by $\alpha'(G)$. Matchings in graphs are extensively studied in the literature (see, for example, the survey articles by Plummer [8] and Pulleyblank [9]).
A path covering of a graph $G$ is a collection of vertex disjoint paths of $G$ that partition $V(G)$. The minimum cardinality of a path covering of $G$ is the path covering number of $G$, denoted by $\text{pc}(G)$.

Let $D$ be a digraph with vertex set $V(D)$ and arc set $A(D)$. By a path in $D$ we mean a directed path. If $v \in V(D)$, then we denote the set of all in-neighbors of $v$ in $D$ by $N^-(v) = \{u \in V(D) \mid vu \in A(D)\}$ and the set of all out-neighbors of $v$ in $D$ by $N^+(v) = \{u \in V(D) \mid vu \in A(D)\}$. Further we denote the in-degree of $v$ in $D$ by $d^-(v) = |N^-(v)|$ and the out-degree of $v$ in $D$ by $d^+(v) = |N^+(v)|$. The maximum in-degree of $D$ we denote by $\Delta^-(D)$, and so $\Delta^-(D) = \max\{d^-(v) \mid v \in V(D)\}$. If $Y \subseteq V(D)$, then a set $X$ is said to dominate the set $Y$ in $D$ if $Y \subseteq N^+(X)$. If the underlying graph of $D$ is connected, then we say that $D$ is connected. If $T$ is a component in the underlying graph of $D$, then we call $T$ a component in $D$. If the digraph $D$ is clear from the context, we simply write $N^-(v), N^+(v), d^+(v)$ and $d^-(v)$ rather than $N^-(v), N^+(v), d^+(v), d^-(v)$, respectively.

1.1. Relating the total domination and matching numbers

Bounds relating the domination number and the matching number are studied, for example, in [1,2]. As a consequence of a result due to Bollobás and Cockayne [1], the domination number of every graph with no isolated vertex is bounded above by its matching number.

**Theorem 1 ([1]).** For every graph $G$ with no isolated vertex, $\gamma'(G) \leq \alpha'(G)$.

DeLaViña et al. [4] established the following general relation between the total domination and matching numbers.

**Theorem 2 ([4]).** If $G$ is a connected graph of order $n \geq 2$, then $\gamma_1(G) \leq \alpha'(G) + \text{pc}(G)$, and this bound is sharp.

Since the ends of the edges in a maximum matching in a graph $G$ with no isolated vertex form a TD-set in the graph, we observe that for every graph $G$ with no isolated vertex, $\gamma_1(G) \leq 2\alpha'(G)$. However unlike the domination number, in general the total domination number and the matching number of a graph are incomparable, even for arbitrarily large, but fixed (with respect to the order of the graph), minimum degree.

**Theorem 3 ([7]).** For every integer $\delta \geq 2$, there exist graphs $G$ and $H$ with $\delta(G) = \delta(H) = \delta$ satisfying $\gamma_1(G) > \alpha'(G)$ and $\gamma'_1(H) < \alpha'(H)$.

It would be of interest to determine for which graph classes $\mathcal{G}$ it is true that $\gamma'_1(G) \leq \alpha'(G)$ holds for all graphs $G \in \mathcal{G}$. As a partial answer to this question, the authors in [7] answered this question in the affirmative when $\mathcal{G}$ is the class of claw-free graphs or the class of regular graphs with minimum degree at least 3.

**Theorem 4 ([7]).** Let $G$ be a graph. Then the following holds.

(a) If $G$ is claw-free and $\delta(G) \geq 3$, then $\gamma_1(G) \leq \alpha'(G)$.

(b) If $G$ is $k$-regular and $k \geq 3$, then $\gamma_1(G) \leq \alpha'(G)$.

2. Main result

In this paper we prove a conjecture from Graffiti [3]. In fact we prove a stronger statement than the original conjecture in Theorem 8. In order to obtain this result we determine another graph class with the property that every graph in the class has total domination number at most its matching number. Our main result shows that if all vertices, except for possibly one vertex, of a connected graph on at least four vertices belong to a triangle, then the total domination number is bounded above by its matching number. We shall prove the following result, a proof of which is given in Section 3.

**Theorem 5.** Let $G$ be a connected graph on at least four vertices. If all vertices, except for possibly one vertex, of $G$ belong to a triangle, then $\gamma_1(G) \leq \alpha'(G)$.

As a special case of Theorem 5, we have the following result.

**Corollary 6.** If all vertices in a connected graph $G$ of order at least 4 belong to a triangle, then $\gamma_1(G) \leq \alpha'(G)$.

That the bound in Theorem 5 (and Corollary 6) is sharp may be seen as follows. Let $\mathcal{F}$ be the family of all graphs that can be obtained from a connected graph $F$ in which every vertex belongs to a triangle as follows: for each vertex $v$ of $F$, add a 3-cycle and join $v$ to one vertex of this cycle. Let $G$ denote the resulting graph. Then, $\gamma_1(G) = \alpha'(G) = |V(G)|/2$. A graph $G$ in the family $\mathcal{F}$ is illustrated in Fig. 1 (here the graph $F$ is a complete graph $K_3$).

We remark that Theorem 5 is almost best possible in the sense that there exist connected graphs $G$ of arbitrarily large order with all vertices of $G$, except for four vertices, that belong to a triangle but satisfying $\gamma_1(G) > \alpha'(G)$. For example, for $k \geq 1$ let $G$ be obtained from $k$ vertex disjoint copies of $K_3$ by adding a path $P_4$ on four vertices and joining one of its ends to one vertex from each copy of $K_3$. Then, $\gamma_1(G) = k + 3$ and $\alpha'(G) = k + 2$, and so $\gamma_1(G) > \alpha'(G)$.

As a consequence of Theorem 5, we prove in Section 4 a conjecture of Graffiti.pc on total domination.
3. Proof of main result

In order to prove our main result, we in fact need to prove a stronger result. For this purpose, we introduce some additional notation. Given a graph $G = (V, E)$ and subsets $X, Y \subseteq V$, we call a set $T$ in $G$ a $(G, X, Y)$-TDS if and only if $X \subseteq T$ and $T$ totally dominates $V \setminus Y$. The minimum cardinality of a $(G, X, Y)$-TDS will be denoted by $\gamma_t(G; X, Y)$. We remark that if $X = Y = \emptyset$, then a $(G, X, Y)$-TDS is precisely a TD-set in $G$ and $\gamma_t(G; X, Y) = \gamma_t(G)$. Hence our main result, namely Theorem 5, follows from the following more general result.

**Theorem 7.** Let $G = (V, E)$ be a connected graph and let $X, Y \subseteq V$ be arbitrary subsets. If all vertices in $V \setminus (X \cup Y)$, except for possibly at most one vertex, belong to a 3-cycle, then $\gamma_t(G; X, Y) \leq |X| + \alpha'(G - (X \cap Y))$, unless $G$ has order one and $Y = \emptyset$ or $G$ is a 3-cycle and $X = Y = \emptyset$.

**Proof.** Assume that the theorem is false and let $G = (V, E)$ be a counter-example to the theorem with minimum $\theta(G; X, Y) = |E(G)| + |X| + |Y \setminus X|$. We will now prove a number of claims. □

**Claim A.** $Y \setminus X = \emptyset$.

**Proof of Claim A.** Assume for the sake of contradiction that $Y \setminus X \neq \emptyset$ and let $y \in Y \setminus X$ be arbitrary. Let $G' = G - y$.

**Claim A.1.** The following holds.

(a) The vertex $y$ does not belong to a 3-cycle.
(b) If $G'$ contains a component that is an isolated vertex, then such a vertex belongs to $Y$.
(c) If $G'$ contains a component $C$ that is a 3-cycle, then $|C \cap (X \cup Y)| \geq 1$.

**Proof.** (a) Suppose that $y$ belongs to a 3-cycle in $G$. Let $Y' = Y \setminus \{y\}$. Since $|Y'| \setminus X = |Y \setminus X| - 1$, we note that $\theta(G; X, Y') = \theta(G; X, Y) - 1$. Further we note that all vertices in $V \setminus (X \cup Y')$, except for possibly at most one vertex, belong to a 3-cycle. Hence the minimality of $\theta(G; X, Y)$ implies that $\gamma_t(G; X, Y') \leq |X| + \alpha'(G - (X \cap Y'))$. Since $X \cup Y = X \cup Y'$, we therefore have that $\gamma_t(G; X, Y') \leq |X| + \alpha'(G - (X \cap Y))$. However every $(G; X, Y')$-TDS is a $(G; X, Y)$-TDS, and so $\gamma_t(G; X, Y) \leq \gamma_t(G; X, Y')$, implying that $\gamma_t(G; X, Y) \leq |X| + \alpha'(G - (X \cap Y))$, contradicting the fact that $G$ is a counter-example to the theorem.

(b) Suppose that there is a component in $G'$ that is an isolated vertex but that does not belong to $Y$. Let $r$ be the vertex in such a component. Possibly, $r \in X$. Since $G$ is connected, we note that $N_G(r) = \{y\}$. Let $G^* = G - r$, let $X^* = (X \cup \{y\}) \setminus \{r\}$ and let $Y^* = Y$. Then, $|E(G^*)| = |E(G)| - 1$, $|X^*| \leq |X| + 1$, $|Y^* \setminus X| = |Y \setminus X| - 1$, and so $\theta(G^*; X^*, Y^*) < \theta(G; X, Y)$. Since we do not remove any 3-cycles from $G$ when constructing $G^*$ and since $(X \cup Y) \setminus \{r\} \subseteq X^* \cup Y^*$, all vertices in $V(G^*) \setminus (X^* \cup Y^*)$, except for possibly at most one vertex, belong to a 3-cycle in $G^*$. Hence the minimality of $\theta(G; X, Y)$ implies that $\gamma_t(G^*; X^*, Y^*) \leq |X^*| + \alpha'(G^* - (X^* \cap Y^*))$. Since $X^* \cap Y^* = (X \cap Y) \cup \{y\}$, every matching in $G^* - (X^* \cap Y^*)$ can be extended to a matching in $G - (X \cap Y)$ by adding to it the edge $ry$, implying that $\alpha'(G^* - (X^* \cap Y^*)) \leq \alpha'(G - (X \cap Y)) - 1$, and so $\gamma_t(G^*; X^*, Y^*) \leq |X^*| + \alpha'(G - (X \cap Y)) - 1$.

On the other hand, suppose that $r \in X$. Then, $|X^*| = |X|$, whence $\gamma_t(G^*; X^*, Y^*) \leq |X| + \alpha'(G - (X \cup Y)) - 1$. In this case, every $(G^*; X^*, Y^*)$-TDS can be extended to a $(G; X, Y)$-TDS by adding to it the vertex $r$, and so $\gamma_t(G; X, Y) \leq \gamma_t(G^*; X^*, Y^*) + 1 \leq |X| + \alpha'(G - (X \cap Y))$. On the other hand, suppose that $r \notin X$. Then, $|X^*| = |X| + 1$, whence $\gamma_t(G^*; X^*, Y^*) \leq |X| + \alpha'(G - (X \cup Y))$. In this case, every $(G^*; X^*, Y^*)$-TDS is a $(G; X, Y)$-TDS, and so $\gamma_t(G; X, Y) \leq \gamma_t(G^*; X^*, Y^*) \leq |X| + \alpha'(G - (X \cap Y))$. In both cases, $\gamma_t(G; X, Y) \leq |X| + \alpha'(G - (X \cap Y))$, a contradiction. This completes the proof of Part (b).

(c) Suppose $G'$ contains a component $C$ that is a 3-cycle but such that $C \cap (X \cup Y) = \emptyset$. Let $V(C) = \{u_1, u_2, u_3\}$. Since $G$ is connected, the vertex $y$ is adjacent to a vertex in $C$. Renaming vertices of $C$, if necessary, we may assume that $yu_1 \in E(G)$. Let $G^* = G - \{u_1, u_2, u_3\}$ and let $X^* = X \cup \{y\}$ and let $Y^* = Y$. Since we do not remove any 3-cycles in $G$ from a vertex in $V(G^*)$ when constructing $G^*$ and since $X^* \cup Y^* = X \cup Y$, all vertices in $V(G^*) \setminus (X^* \cup Y^*)$, except for possibly at most one vertex, belong to a 3-cycle in $G^*$. Hence the minimality of $\theta(G; X, Y)$ implies that $\gamma_t(G^*; X^*, Y^*) \leq |X^*| + \alpha'(G^* - (X^* \cap Y^*))$. Since $X^* \cap Y^* = (X \cap Y) \cup \{y\}$, every matching in $G^* - (X^* \cap Y^*)$ can be extended to a matching in $G - (X \cap Y)$ by adding to it the two edges $u_2u_3$ and $u_1y$, implying that $\alpha'(G^* - (X^* \cap Y^*)) \leq \alpha'(G - (X \cap Y)) - 2$, and so $\gamma_t(G^*; X^*, Y^*) \leq |X^*| + \alpha'(G - (X \cap Y)) - 2$. Every $(G^*; X^*, Y^*)$-TDS can be extended to a $(G; X, Y)$-TDS by adding to it the vertex $u_1$, and so $\gamma_t(G; X, Y) \leq \gamma_t(G^*; X^*, Y^*) + 1 \leq |X^*| + \alpha'(G - (X \cap Y)) - 1. Since |X^*| = |X| + 1, we therefore have that $\gamma_t(G; X, Y) \leq |X| + \alpha'(G - (X \cap Y))$, a contradiction. This completes the proof of Part (c). □
We now return to the proof of Claim A. By Claim A.1(a), we do not remove any 3-cycles from $G$ when constructing $G'$. Hence letting $Y' = Y \setminus \{y\}$, we have that all vertices in $V(G') \setminus (X \cup Y')$, except for possibly at most one vertex, belong to a 3-cycle. By Claim A.1(b), no component in $G'$ is an isolated vertex that does not belong to $Y'$. By Claim A.1(c), no component, $C$, in $G'$, is a 3-cycle such that $C \cap (X \cup Y') = \emptyset$. Hence applying the minimality of $\theta(G; X, Y)$ to each component of $G'$, the additive properties of the total domination number and the matching number readily implies that $\gamma_1(G'; X, Y') \leq |X| + \alpha'(G' - (X \cup Y'))$. Since $G' - (X \cup Y')$ is a subgraph of $G - (X \cup Y)$, we have that $\alpha'(G' - (X \cup Y')) \leq \alpha'(G - (X \cup Y))$, implying that $\gamma_1(G'; X, Y') \leq |X| + \alpha'(G - (X \cup Y))$. Since $y \in Y$, every $(G'; X, Y')$-TDS is a $(G, X, Y)$-TDS, and so $\gamma_1(G; X, Y) \leq \gamma_1(G'; X, Y') \leq |X| + \alpha'(G - (X \cup Y))$, contradicting the fact that $G$ is a counter-example to the theorem. This completes the proof of Claim A.

Claim B. $Y = \emptyset$.

Proof of Claim B. Assume for the sake of contradiction that $Y \neq \emptyset$. By Claim A, we have $Y \setminus X = \emptyset$, which implies that $X \cap Y = Y \setminus X = X \setminus Y$. Let $G' = G - Y$ and let $Y' = N(Y) \setminus Y$ and let $X' = X \setminus Y$. We note that any vertex belonging to a 3-cycle in $G$ but not in $G'$ either belongs to $Y$ or belongs to $Y'$ (and therefore does not belong to $G$ at all). Hence all vertices in $V(G') \setminus (X' \cup Y')$, except for possibly at most one vertex, belong to a 3-cycle in $G'$. Furthermore every component, $C$, in $G'$ contains a vertex from $Y'$, and so no component in $G'$ is an isolated vertex that does not belong to $Y'$ or $X'$ or is a 3-cycle such that $C \cap (X' \cup Y') = \emptyset$. Since $|X'| = |X| - |Y'|$ and $|Y' \setminus X'| = |Y'| - 1 = |Y \setminus X|$ and $|E(G')| = |E(G)| - |Y'|$, we have that $\theta(G'; X', Y') < \theta(G; X, Y)$. Hence the minimality of $\theta(G; X, Y)$ implies that $\gamma_1(G'; X', Y') \leq |X'| + \alpha'(G' - (X' \cup Y'))$.

Every $(G'; X', Y')$-TDS can be extended to a $(G; X, Y)$-TDS by adding to it the set $Y$, and so $\gamma_1(G; X, Y) \leq \gamma_1(G'; X', Y') + |Y|$. By definition, we have $|X| = |X| + |Y'| = |X'| + |Y|$, implying that $\gamma_1(G; X, Y) \leq \gamma_1(G'; X', Y') + |X| = |X| + \alpha'(G' - (X' \cup Y'))$. Since $G' - (X' \cup Y')$ is a subgraph of $G$ and since $G' = G - Y = G - (X \cup Y)$, we have that $\alpha'(G' - (X' \cup Y')) \leq \alpha'(G - (X \cup Y))$. Therefore, $\gamma_1(G; X, Y) \leq |X| + \alpha'(G - (X \cup Y))$, a contradiction. This completes the proof of Claim B.

By Claim B, we have that $X \cap Y = \emptyset$, and therefore $\alpha'(G - (X \cup Y)) = \alpha'(G)$. Hence in what follows we can show that $\gamma_1(G; X, Y) \leq |X| + \alpha'(G)$, then we contradict the fact that $G$ is a counter-example to the theorem.

Claim C. $X = \emptyset$.

Proof of Claim C. Assume for the sake of contradiction that $X \neq \emptyset$ and let $x \in X$ be arbitrary. Let $G' = G - x$ and let $Y' = N_C(x)$.

Suppose that $X \cap Y' = \emptyset$. Let $r \in Y'$ be arbitrary and let $X' = (X \cup \{r\}) \setminus \{x\}$. We note that any vertex belonging to a 3-cycle in $G$ but not in $G'$ either belongs to $X'$ or is the vertex $x$ (and therefore does not belong to $G$ at all). Further every component in $G'$ contains a vertex from $X$, and so all vertices in $V(G') \setminus (X' \cup Y')$, except for possibly at most one vertex, belong to a 3-cycle in $G'$. Since $|X'| = |X|, |Y' \setminus X'| = |Y'| - 1 = |Y \setminus X|$ and $|E(G')| = |E(G)| - |Y'|$ (recall that by Claim A, $|Y \setminus X| = 0$ and $|E(G)| = |E(G)|$), we have that $\theta(G'; X', Y') < \theta(G; X, Y)$. Hence the minimality of $\theta(G; X, Y)$ implies that $\gamma_1(G'; X', Y') \leq |X'| + \alpha'(G' - (X' \cup Y'))$. Every $(G'; X', Y')$-TDS can be extended to a $(G; X, Y)$-TDS by adding to it the vertex $x$, and so $\gamma_1(G; X, Y) \leq \gamma_1(G'; X', Y') + 1$. Furthermore since $G' - (X' \cup Y')$ contains neither the vertex $r$ nor the vertex $x$, every matching in $G' - (X' \cup Y')$ can be extended to a matching in $G$ by adding to it the edge $rx$, implying that $\alpha'(G' - (X' \cup Y')) \leq \alpha'(G) - 1$. Therefore, $\gamma_1(G; X, Y) \leq \gamma_1(G'; X', Y') + 1 \leq |X'| + \alpha'(G - (X' \cup Y')) + 1 \leq |X| + \alpha'(G)$, a contradiction. Hence, $|X \cap Y'| \geq 1$.

We now let $X = X \setminus \{x\}$. Proceeding analogously as above, we have that $\gamma_1(G'; X', Y') \leq |X'| + \alpha'(G' - (X' \cup Y'))$, $\gamma_1(G; X, Y) \leq \gamma_1(G'; X', Y') + 1$, and that $\alpha'(G' - (X' \cup Y')) \leq \alpha'(G) - 1$. Since $|X'| = |X| - 1$, we therefore have that $\gamma_1(G; X, Y) \leq \gamma_1(G'; X', Y') + 1 \leq |X'| + \alpha'(G' - (X' \cup Y')) + 1 < |X| + \alpha'(G)$, a contradiction. This completes the proof of Claim C.

By Claims B and C, we have that $X = Y = \emptyset$. Thus, $\gamma_1(G) = \gamma_1(G; X, Y)$. Hence it suffices for us to show that $\gamma_1(G) \leq \alpha'(G)$, since then we contradict the fact that $G$ is a counter-example to the theorem.

Claim D. If $uv \in E(G)$, then $d(u) \neq 2$ or $d(v) \neq 2$.

Proof of Claim D. Assume for the sake of contradiction that $uv \in E(G)$ but $d(u) = d(v) = 2$. As there is at most one vertex in $G$ not belonging to a 3-cycle, we note that there exists a vertex $q$, such that $quq$ is a 3-cycle in $G$. By Claims B and C, $X = Y = \emptyset$, and so since $G$ is not a 3-cycle, we note further that $n(G) \geq 4$. Let $G' = G - [u, v]$ and let $X = \{q\}$ and $Y' = \emptyset$. Then, $G'$ is a connected graph on at least two vertices and if $G'$ is a 3-cycle, then it contains a vertex of $X' \cup Y'$. Every vertex in $V(G') \setminus X'$ belongs to a 3-cycle in $G'$ and if only if it belongs to a 3-cycle in $G$. Since $|E(G')| = |E(G)| - 3$, $|X'| = |X| + 1 = 1$ and $|Y' \setminus X'| = |Y \setminus X| = 0$, we have that $\theta(G'; X', Y') < \theta(G; X, Y)$. Hence the minimality of $\theta(G; X, Y)$ implies that $\gamma_1(G'; X', Y') \leq |X'| + \alpha'(G' - (X' \cup Y')) = |X| + \alpha'(G)$. Every $(G'; X', Y')$-TDS is a $(G; X, Y)$-TDS, and so $\gamma_1(G; X, Y) \leq \gamma_1(G'; X', Y')$. Furthermore since $G'$ contains neither the vertex $u$ nor the vertex $v$, every matching in $G'$ can be extended to a matching in $G$ by adding to it the edge $uv$, implying that $\alpha'(G) \leq \alpha'(G) - 1$. Therefore, $\gamma_1(G; X, Y) \leq \gamma_1(G'; X', Y') \leq |X'| + \alpha'(G) \leq (|X| + 1) + (\alpha'(G) - 1) = |X| + \alpha'(G)$, a contradiction. This completes the proof of Claim D.
We now return to the proof of Theorem 7. By Claims B and C, \( X = Y = \emptyset \), implying that all vertices in \( V(G) \), except for possibly at most one vertex, belong to a 3-cycle in \( G \). In what follows, we define the graph \( G' \) as follows. If there is a vertex in \( G \) that does not belong to a 3-cycle in \( G \), then we call such a vertex \( w \) and we let \( G' = G - w \); otherwise, if all vertices in \( V(G) \) belong to a 3-cycle in \( G \), let \( G' = G \). For every edge \( uv \in E(G') \), we let \( G'_{uv} = G' - uv \).

Claim E. For every edge \( uv \in E(G') \), there is a vertex in \( G'_{uv} \) that does not belong to a 3-cycle.

Proof of Claim E. Assume for the sake of contradiction that \( uv \in E(G') \) but every vertex in \( G'_{uv} \) belongs to a 3-cycle. Then every vertex in \( G - uv \), except possibly for the vertex \( u \), belong to a 3-cycle. Since \( w \) is not incident with the edge \( z \) and since \( z \) is not isolated in \( G \), we note that there are no isolated vertices in \( G - uv \). Further by Claim D, no component in \( G - uv \) is a 3-cycle. Thus since \( \theta(G - uv; X, Y) < \theta(G, X, Y) \), the minimality of \( \theta(G; X, Y) \) implies that \( \gamma_t(G - uv; X, Y) \leq |X| + \alpha'((G - uv) - (X \cap Y)) = |X| + \alpha'(G - uv) \leq |X| + \alpha'(G) \), a contradiction as \( \gamma_t(G) \leq \gamma_t(G - uv) \) holds for all graphs. This completes the proof of Claim E. \( \square \)

We now define the digraph \( D' \) as follows. Let \( V(D') = V(G') \) and let \( A(D') \) be defined as follows. For every edge \( uv \in E(G') \), we do the following.

(D.1) If \( v \) does not belong to a 3-cycle in \( G'_{uv} \), then add an arc from \( u \) to \( v \) in \( D' \).

(D.2) If \( u \) does not belong to a 3-cycle in \( G'_{uv} \), then add an arc from \( v \) to \( u \) in \( D' \).

(D.3) If both \( u \) and \( v \) belong to 3-cycles in \( G'_{uv} \), then, by Claim E, some other vertex \( z_{uv} \) does not belong to a 3-cycle in \( G'_{uv} \). In this case we do not add any arc between \( u \) and \( v \) in \( D' \).

This completes the definition of \( D' \). We remark that if \( uv \in E(G') \), then it is possible to have arcs from \( u \) to \( v \) and from \( v \) to \( u \). Further if both \( u \) and \( v \) belong to 3-cycles in \( G'_{uv} \), and \( z_{uv} \) is as defined in (D.3), then we note that \( uvz_{uv} \) is the only 3-cycle containing \( z_{uv} \), and therefore by (D.1) and (D.2) we add an arc from \( u \) to \( z_{uv} \) and an arc from \( v \) to \( z_{uv} \). We proceed further by establishing properties of the digraph \( D' \) that will prove useful.

Claim F. The following hold in the digraph \( D' \).

(a) \( D' \) is transitive.

(b) \( \Delta^-(D') \leq 2 \).

(c) There is no path of length 3 in \( D' \).

(d) If \( uv \in E(G') \) and there is no arc between \( u \) and \( v \) in \( D' \), then \( d^-(u) = d^-(v) = 0 \).

(e) If \( N^-(x) = \{u, v\} \), then \( uv \in E(G') \).

Proof of Claim F. (a) Suppose that \( uvw \) is a path in \( D' \). We need to show that \( uv \in A(D') \). Since \( uv \in A(D') \), every 3-cycle in \( G' \) containing \( v \) uses the edge \( uv \). Analogously since \( vw \in A(D') \), every 3-cycle in \( G' \) containing \( w \) uses the edge \( vw \). Therefore every 3-cycle using the edge \( wv \) also uses the edge \( uv \), which implies that the only 3-cycle containing \( w \) is \( uvwuv \). Hence the vertex \( w \) does not belong to a 3-cycle in \( G'_{uv} \), and so \( uv \in A(D') \).

(b) For the sake of contradiction assume that \( \Delta^-(D') \geq 3 \). Then there exists a vertex \( x \) in \( D' \) such that \( d^-(X) \geq 3 \). Let \( xuw \) be any 3-cycle in \( G' \) and let \( z \in N^-(x) \) \( \{u, v\} \) be arbitrary. Since \( xz \in A(D') \), the vertex \( x \) does not belong to a 3-cycle in \( G'_{xz} \). But deleting the edge \( xz \) from \( D' \) does not remove the 3-cycle \( xuwx \), a contradiction.

(c) For the sake of contradiction assume that there is a path \( u_1u_2u_3u_4 \) in \( D' \). By Part (a), we note that \( u_1u_4, u_2u_4 \in A(D') \), which implies that \( d^-(u_4) \geq 3 \), contradicting Part (b).

(d) Suppose that \( uv \in E(G') \) and that there is no arc between \( u \) and \( v \) in \( D' \). Thus both \( u \) and \( v \) belong to 3-cycles in \( G'_{uv} \). Further there exists a vertex \( z_{uv} \) as defined in (D.3) such that all 3-cycles containing \( z_{uv} \) in \( G' \) use the edge \( uv \). In other words, \( uuvz_{uv} \) is the only 3-cycle in \( G' \) containing \( z_{uv} \) and therefore by (D.1) and (D.2), we have that \( uvz_{uv}, uz_{uv} \in A(D') \). For the sake of contradiction assume that \( d^-(u) \geq 1 \) and let \( q \in N^-(u) \) be arbitrary. Since there is no arc between \( u \) and \( v \) in \( D' \), we note that \( q \neq v \). If \( q \neq z_{uv} \), then \( quvz_{uv} \) is a path in \( D' \), and so by Part (a) the transitivity of \( D' \) implies that \( quvz_{uv} \in A(D') \). But then \( d^-(z_{uv}) \geq 3 \), contradicting Part (b). Therefore, \( q = z_{uv} \), and so \( vzuuv \) is a path in \( D' \). Thus by Part (a) the transitivity of \( D' \) implies that \( uuvz_{uv} \in A(D') \), contradicting the fact that there was no arc between \( u \) and \( v \) in \( D' \). Therefore, \( d^-(u) = 0 \). Analogously, \( d^-(v) = 0 \).

(e) Suppose that \( x \) is a vertex in \( D' \) with \( N^-(x) = \{u, v\} \). Then every 3-cycle containing \( x \) also contains the edges \( ux \) and \( vx \), implying that \( uvw \) is a 3-cycle in \( G' \). In particular, \( uv \in E(G') \). This completes the proof of Claim F. \( \square \)

Let \( S = \{v \in V(D') \mid d^-(v) = 0\} \), and so \( S \) is the set of all vertices in \( D' \) with in-degree zero. We proceed further with the following three claims.

Claim G. If \( T \) is a component in \( D' - S \) and there is an arc from \( s \in S \) to some vertex in \( T \), then \( s \) dominates \( V(T) \).

Proof. Let \( T \) be a component in \( D' - S \) and suppose there is an arc from \( s \in S \) to some vertex in \( T \). Since \( T \) is a component in \( D' - S \), there is a path in \( G'[V(T)] \) between every two vertices in \( V(T) \). Since there is an arc from \( s \in S \) to some vertex in \( T \), there is therefore a path from \( s \) to every vertex in \( V(T) \) in \( G'[V(T) \cup \{s\}] \). For the sake of contradiction assume that there is a vertex in \( T \) not dominated by \( s \) in \( D' \). Among all such vertices, let \( u \) be chosen to be at minimum distance from \( s \) in \( G'[V(T) \cup \{s\}] \). Let \( t \) be a neighbor of \( u \) on a shortest path from \( u \) to \( s \) in \( G'[V(T) \cup \{s\}] \). By our choice of \( u \), we note that
Claim H. If two vertices in $S$ are connected in $G$, then they are connected in $G'[S]$. 

Proof. Suppose that $u$ and $v$ are two arbitrary vertices in $S$ that are connected in $G'$. For the sake of contradiction assume that $u$ and $v$ are not connected in $G'[S]$. Then there exists a partition $(S', S'')$ of $S$ such that there is no edge between $S'$ and $S''$ in $G'[S]$ and $u, v \in S'$. However since there is a path from $u$ to $v$ in $G'$, there is a path between $S'$ and $S''$ in $G$. Let $P = p_0, p_1, \ldots, p_{t-1}$ be a shortest such path in $G$, where $p_0 \in S'$ and $p_t \in S''$. We choose by our choice of the path $P$, we note that the vertices $W$ all belong to the same component in $G - S$. Since $p_0 \in S$, we have that $d^-(p_0) = 0$. Further since $p_t \notin S$, we have $d^-(p_t) > 0$. Hence since $p_0 p_t \in E(G')$, we have by Claim F(d) that $p_0 p_t \in A(D')$. Analogously, $p_1 p_{t-1} \in A(D')$. Therefore by Claim G, $p_0$ and $p_t$ both have arcs to all vertices in $W$ in $D'$. Hence by Claim F(e), $p_0 p_t \in E(G')$, contradicting the fact that there is no edge between $S'$ and $S''$. Therefore, $u$ and $v$ are connected in $G'[S]$. □

Claim I. If $C$ is a component in $G'$ and $r \in V(C)$, then there exists a TD-set, $T_C$, in $C$ such that $r \in T_C$ and $|T_C| \leq \alpha'(C)$.

Proof. Let $C$ be a component in $G'$ and let $r \in V(C)$ be arbitrary. Let $S_C = S \cap V(C)$. We proceed further with two subclaims.

Claim I.1. $|V(C)| \geq 4$.

Proof. For the sake of contradiction, assume that $|V(C)| \leq 3$. Since all vertices in $G'$ belong to 3-cycles, this implies that $C$ is a 3-cycle. Since $G$ is not a 3-cycle and $G$ is connected, we must have deleted a vertex from $G$ to obtain $G'$. Using our earlier notation, there is therefore a vertex in $G$ that does not belong to a 3-cycle in $G$, which we called $w$, and $G' = G - w$. Since $G$ is connected, the vertex $w$ is joined in $G$ to a vertex in $V(C)$. However as $w$ does not belong to any 3-cycle in $G$, at most one vertex in $V(C)$ is joined to $w$ in $G$. Consequently, exactly one vertex in $V(C)$ is joined to $w$ in $G$, implying that there are two adjacent vertices in $C$ both having degree 2 in $G$, contradicting Claim D. □

Claim I.2. Let $v \in V(C)$. If in the digraph $D'$ we have $d^-(v) > 0$ and $d^-(u) > 0$ for all $u \in N^-(v)$, then the desired result holds.

Proof. Let $v \in V(C)$ and suppose that $d^-(v) > 0$ and $d^-(u) > 0$ for all $u \in N^-(v)$. We show then that the desired result holds; that is, there exists a TD-set, $T_C$, in $C$ such that $r \in T_C$ and $|T_C| \leq \alpha'(C)$. Let $u \in N^-(v)$ be arbitrary.

We show first that $N^-(u) = \{v\}$. By assumption, $d^-(u) > 0$. Let $zu \in A(D')$ be arbitrary. Suppose that $z \neq v$. Then, $zv$ is a path in $D$. By Claim F(a) the transitivity of $D'$ implies that $zv \in A(D')$, and so $z \in N^-(v)$. By assumption, $d^-(z) > 0$. Let $z'z \in A(D')$ be arbitrary. If $z' \notin \{u, v\}$, then we get a contradiction to Claim F(c). Hence, $z' \in \{u, v\}$. By transitivity we deduce that all possible arcs exist between vertices in $\{u, v, z\}$. By Claim F(b) and Claim F(d), the component $C$ is a 3-cycle, a contradiction Claim I.1. Therefore, $z = v$, implying that $N^-(u) = \{v\}$.

Since $uw \in A(D')$ (and $uw \in A(D')$), there is a 3-cycle $uwy$, where $y \in V(C)$. Since $d^-(u) > 0$ and $uw \in E(G')$, we have by Claim F(d) that there is an arc between $u$ and $q_1$ in $D'$. However, $d^-(u) = 1$, implying that $uwq_1 \in E(D')$. Thus, $uwq_1$ is a path in $D'$. By Claim F(a) the transitivity of $D'$ implies that $q_1 w \in A(D')$. To show next that $d^+(q_1) = 0$. Assume, to the contrary, that $d^+(q_1) > 1$. Then there is some vertex $q'$ such that $q_1 q' \in A(D')$. If $q' \notin \{u, v\}$, then $uvq_1$ is a path of length 3 in $D'$, contradicting Claim F(c). Hence, $q' \in \{u, v\}$. If $q' = u$, then $d^-(u) = 1$, and $d^-(q_1) = 1$. If $q' = v$, then $q_1 uv$ is a path in $D'$, and by Claim F(a) the transitivity of $D'$ implies that $q_1 u \in A(D')$ and once again $d^-(u) > 1$. Both cases produce a contradiction. Therefore, $d^+(q_1) = 0$. Furthermore, by Claim F(b) and Claim F(d), we have that $d_G(q_1) = 2$.

Since $uvq_1u$ is a 3-cycle in $C$ and $d_G(q_1) = 2$, by Claim I.1 there exists some vertex $q_2 \in V(C) \setminus \{u, v, q_1\}$ which is adjacent to $u$ or $v$ in $C$. We show that $uvq_2 \in A(D')$. On the one hand, suppose that $uvq_2 \in E(G')$. Then by Claim F(d) there is an arc between $q_2$ and $u$ in $D'$. However, $d^-(u) = 1$, implying that $uvq_1 \in E(D')$. Thus, $uvq_2$ is a path in $D'$, and by Claim F(a) the transitivity of $D'$ implies that $q_2 u \in A(D')$. On the other hand, suppose that $uvq_2 \in E(G')$. Then by Claim F(d) there is an arc between $q_2$ and $v$ in $D'$. If $q_2 v \in A(D')$, then $q_2 uv$ is a path in $D'$, and by Claim F(a) the transitivity of $D'$ implies that $q_2 u \in A(D')$. But then $d^+(u) \geq 2$, a contradiction. Hence, $uvq_2 \in E(D')$. Thus, $uwq_1 \in A(D')$, and by Claim F(a) the transitivity of $D'$ implies that $uwq_1 \in A(D')$. In both cases, $uvq_2 \in A(D')$. An analogous argument as with the vertex $q_1$ shows that $d^+(q_2) = 0$ and $d_G(q_2) = 2$.

If $V(C) \neq \{u, v, q_1, q_2\}$, then there exists a vertex $q_3$ adjacent to $u$ or $v$. We continue this process until $V(C) = \{u, v, q_1, q_2, \ldots, q_t\}$ for some $t \geq 2$. Note that $E(C)$ consists of the edge $uw$, together with all edges between $\{u, v\}$ and $\{q_1, q_2, \ldots, q_t\}$. We note that $\gamma(C) = 2 = \alpha'(C)$. Let $T_C$ contain the vertex $r$ (which is defined in the statement of Claim I) and any vertex in $\{u, v\} \setminus \{r\}$. Then, $T_C$ is a TD-set of $C$ and $|T_C| = 2 = \alpha'(C)$, and the desired result follows. □

We now return to the proof of Claim I. We may assume that the following claim holds, for otherwise we are done by Claim I.2.
Claim I.3. We may assume that if \( v \in V(C) \), then in the digraph \( D' \) either \( d^-(v) = 0 \) or there exists an arc \( uv \in A(D') \) such that \( d^-(u) = 0 \).

By Claim I.3, we have that \( |S_C| \geq 1 \). Suppose that \( |S_C| = 1 \). Let \( S_C = \{s\} \). By our assumption, we have \( V(C) = N^+(s) \cup \{s\} \). Hence, \( \gamma_C(C) = 2 \). Furthermore by Claim I.1, \(|V(C)| \geq 4\). Thus since every vertex of \( C \) belongs to a 3-cycle, we note that \( \alpha'(C) \geq 2 \). If \( s = r \), let \( T_C \) contain \( s \) and any vertex in \( V(C) \setminus \{s\} \). If \( s \neq r \), let \( T_C = \{r, s\} \). Then, \( T_C \) is a TD-set of \( C \) and \( |T_C| = 2 \geq \alpha'(C) \). Hence we may assume that \( |S_C| \geq 2 \), for otherwise the desired result follows.

Let \( S_C = \{s_1, s_2, \ldots, s_k\} \) by some \( b \geq 2 \). By Claim F(b) and Claim I.3, for every vertex \( v \in V(C) \setminus S_C \), we have \(|N^-(v) \cap S_C| \in \{1, 2\} \). For \( i \in \{1, 2, \ldots, b\} \), let \( T_i = \{y \in V(C) \setminus S_C \mid N^-(y) \cap S_C = \{s_i, s_j\}\} \). Further for \( 1 \leq i < j \leq b \), let \( T_{ij} = \{y \in V(C) \setminus S_C \mid N^-(y) \cap S_C = \{s_i, s_j\}\} \). As observed earlier, every vertex \( v \in V(C) \setminus S_C \) belongs to exactly one \( T_i \) for some \( i \) where \( 1 \leq i \leq b \), or exactly one \( T_{ij} \) for some \( i \) and \( j \) where \( 1 \leq i < j \leq b \). By transitivity we note that there are no arcs between different \( T_i \)'s or \( T_{ij} \)'s and by Claim F(d) there is no edge in \( G' \) between such sets either. By Claim H, the graph \( G[S_C] \) is connected. Let \( R \) be a spanning tree in \( G[S_C] \). We note that \( R \) has order \( b \geq 2 \). Renaming vertices of \( S_C \), if necessary, we may assume that \( s_1 \) and \( s_2 \) are leaves of the tree \( R \). We proceed further with the following subclaims.

Claim I.4. If \( s_i, s_j \in E(G') \) for some \( i \) and \( j \) where \( 1 \leq i < j \leq b \), then \( |T_{ij}| \geq 1 \).

Proof. Suppose that \( s_i, s_j \in E(G') \) where \( 1 \leq i < j \leq b \). The reason there is no arc between \( s_i \) and \( s_j \) in \( D' \) that is when we deleted the edge \( s_i s_j \) from \( G' \) some third vertex \( q \) does not lie in a 3-cycle. Therefore by construction of \( D' \), we have that \( s_i q \) and \( s_j q \) are arcs in \( D' \), implying that \( q \in T_{ij} \). Hence, \( |T_{ij}| \geq 1 \).

Now consider the following four cases. Let \( T_1 \neq \emptyset \) and \( T_2 \neq \emptyset \). Since the vertices in \( T_1 \) lie on a 3-cycle in \( G' \), we have by Claim F(d) that there is at least one edge in \( G[T_1] \). Analogously, there is a 3-cycle in \( G[T_2] \). Since \( R \) is a spanning tree in \( G[S_C] \), there are \( |S_C| - 1 \) edges in \( R \). By Claim I.4, for each such edge, say \( s_is_j \), in \( R \) we have that \( |T_{ij}| \geq 1 \), and these sets can easily be used to obtain a matching of cardinality \( |S_C| - 1 \) in \( C \). Adding an edge from \( T_1 \) and an edge from \( T_2 \) to such a matching produces a matching of size \( |S_C| + 1 \) in \( C \). We now let \( T_C = S_C \cup \{r\} \). Then, \( T_C \) is a TD-set of \( C \), and so \( \gamma_C(C) \leq |T_C| \leq |S_C| + 1 \leq \alpha'(C) \), and the desired result follows.

Case 2. \( T_1 = \emptyset \) and \( T_2 = \emptyset \). Proceed analogously as in Case 1, except in this case we let \( T_C = (S_C \cup \{r\}) \setminus \{s\} \). Further in this case we add an edge from \( T_2 \) to a matching of size \( |S_C| - 1 \) in \( C \) corresponding to the \( |S_C| - 1 \) edges in the tree \( R \) to obtain a matching of size \( |S_C| \) in \( C \). Thus, \( \gamma_C(C) \leq |T_C| \leq |S_C| \leq \alpha'(C) \), and the desired result follows.

Case 3. \( T_1 = \emptyset \) and \( T_2 \neq \emptyset \). This case is handled analogously to Case 2.

Case 4. \( T_1 = \emptyset \) and \( T_2 = \emptyset \). If \( |S_C| \geq 4 \), then we proceed analogously as in Case 1, except that in this case we let \( T_C = (S_C \cup \{r\}) \setminus \{s_1, s_2\} \) and note that \( \gamma_C(C) \leq |T_C| \leq |S_C| - 1 \leq \alpha'(C) \), and the desired result follows.

If \( |S_C| = 3 \), the vertex \( s_3 \) is adjacent to all vertices in \( C \) to all vertices in \( V(C) \setminus \{s_1, s_2\} \). Hence in this case we let \( T_C = \{r, s_3\} \). Then, \( T_C \) is a TD-set of \( C \) and \( |T_C| = 2 \leq \alpha'(C) \), and the desired result follows.

If \( |S_C| = 2 \), then \( E(C) \) contains the edge \( s_1 s_2 \), as well as all edges between \( S_C \) and \( V(C) \setminus S_C \), implying that \( \gamma(C) = 2 = \alpha'(C) \) since \( |V(C)| \geq 4 \). In this case, we let \( T_C \) contain the vertex \( r \) and any vertex in \( \{s_1, s_2\} \setminus \{r\} \). Then, \( T_C \) is a TD-set of \( C \) and \( |T_C| = 2 \leq \alpha'(C) \), and the desired result follows. This completes the proof of Claim I. □

We now return to the proof of Theorem 7 one last time. As remarked earlier, with our earlier claims it suffices for us to show that \( \gamma(G) \leq \alpha'(G) \), since then we contradict the fact that \( G \) is a counter-example to the theorem. If every vertex of \( G \) belongs to a 3-cycle, then the desired result follows from Claim I. Hence we may assume that there is a vertex \( w \) in \( G \) that does not belong to a 3-cycle in \( G \), and so \( G' = G - w \). Since \( X = Y = \emptyset \) by Claims B and C, we note that the vertex \( w \) is the only vertex in \( G \) that does not belong to a 3-cycle. Let \( G_1, \ldots, G_k \) denote the components of \( G' \). By the connectivity of \( G \), the vertex \( w \) is joined in \( G \) to a vertex from each component of \( G' \). Let \( r \) be a vertex in the component \( G_1 \) which is adjacent to \( w \) in \( G \). By Claim I, there is a TD-set \( T_1 \) in \( G_1 \) such that \( r \in T_1 \) and \( |T_1| \leq \alpha'(G_1) \). Further if \( k \geq 2 \), then by Claim I there is a TD-set \( T_i \) in \( G_i \) such that \( |T_i| \leq \alpha'(G_i) \). Let \( T = \bigcup_{i=1}^k T_i \). Then, \( T \) is a TD-set in \( G \), implying that \( \gamma(G) \leq |T| \leq \sum_{i=1}^k |T_i| \leq \sum_{i=1}^k \alpha'(G_i) \leq \alpha'(G) \), a contradiction. This completes the proof of Theorem 7. □

4. Closing remarks

Much interest in total domination in graphs has arisen from a computer program Graffiti.pc that has generated several hundred conjectures on total domination. Graffiti.pc is a conjecture-making program written by Ermelinda DeLaViña, which was inspired by Siemon Fajtlowicz’s conjecture-making program Graffiti. DeLaViña remarks on her web page (see [3]) that “both programs utilize Fajtlowicz’s Dalmatian heuristic, however each has its individual implementations.” A numbered, annotated list of Graffiti.pc’s conjectures on total domination and their current status is posted on DeLaViña’s web page.

Most of the Graffiti conjectures on total domination have now been proved or disproved, but some remain open, despite much interest in the area. As a consequence of our main result, namely Theorem 7, we can prove conjecture #285 of Graffiti.pc on total domination, which has been open since 2007. In order to state this conjecture, we introduce some additional notation. In a graph \( G \), let \( T(v) \) be the number of 3-cycles containing \( v \) and let \( FT_3(G) \) be the frequency of the minimum value of \( T(v) \). That is, if \( T^\min(G) = \min\{T(v) \mid v \in V(G)\} \), then \( FT_3(G) = \{|v \in V(G) \mid T(v) = T^\min(G)\} \). The following is Conjecture #285 of Graffiti.pc on the total domination conjectures.
Conjecture 1. If $G$ is a simple connected graph on at least two vertices, then $\gamma_t(G) \leq \alpha'(G) \times FT_3(G)$.

We will prove the following stronger result and then show that this implies that Conjecture 1 holds.

Theorem 8. If $G$ is a simple connected graph on at least two vertices, then
\[ \gamma_t(G) \leq \alpha'(G) + FT_3(G) - 1. \]

Proof. Adopting our earlier notation, we let $T_{\min}(G) = \min(T(v) \mid v \in V(G))$. If $T_{\min}(G) > 0$, then by Theorem 7 we have $\gamma_t(G) \leq \alpha'(G) \leq \alpha'(G) + FT_3(G) - 1$, as we always have $FT_3(G) \geq 1$. Hence we may assume that $T_{\min}(G) = 0$, for otherwise the desired result follows. If $FT_3(G) = 1$, then Theorem 7 implies that $\gamma_t(G) \leq \alpha'(G) = \alpha'(G) \leq \alpha'(G) + FT_3(G) - 1$, and we are done as before. Hence we may assume that $FT_3(G) \geq 2$. Let $\ell = FT_3(G) - 1$ and note that $\ell \geq 1$.

Let $Q = \{q_1, q_2, \ldots, q_t\}$ be any set of $\ell$ vertices in $G$, such that no vertex in $Q$ belongs to a 3-cycle in $G$. Let $A = \{a_1, a_2, \ldots, a_t\}$ and $B = \{b_1, b_2, \ldots, b_t\}$. Let $G'$ be obtained from $G$ by adding the $2\ell$ new vertices in $A \cup B$ and for each $i = 1, 2, \ldots, \ell$, adding the edges $a_ib_i, a_iq_i$ and $b_iq_i$ (so that $a_ib_iq_i$ is a 3-cycle in $G'$). Let $D$ be a $\gamma_t(G')$-set. If $q_i \notin D$ for some $i, 1 \leq i \leq \ell$, then $[a_i, b_i] \subseteq D$ in order to totally dominate $[a_i, b_i]$ in $G'$. In this case, we replace $[a_i, b_i]$ in the set $D$ with the vertex $q_i$ and a neighbor of $q_i$ in $G$. Hence we may assume that $Q \subseteq D$ and that $|D \cap [a_i, b_i]| \leq 1$ for each $i, 1 \leq i \leq \ell$.

If $a_i \in D$ for some $i, 1 \leq i \leq \ell$, then we replace the vertex $a_i$ in the set $D$ with a neighbor of $q_i$ in $G$. Analogously, if $b_i \in D$ for some $i$, then we replace the vertex $b_i$ in the set $D$ with a neighbor of $q_i$ in $G$. Hence we may assume that $(A \cup B) \cap D = \emptyset$. But then $D$ is a 1D-set of $G$, implying that $\gamma_t(G) \leq |D| = \gamma_t(G')$. Moreover we can choose a maximum matching, $M'$ say, in $G$ so that $a_i, b_i \in M$ for each $i, 1 \leq i \leq \ell$. Removing these $\ell$ edges from $M'$ produces a matching in $G$, implying that $\alpha'(G) \geq |M'| - \ell = \alpha'(G') - \ell$.

Since $G$ is a connected graph, so too is $G'$. By construction, all vertices, except for exactly one vertex, of $G'$ belong to a triangle. Further, $G'$ contains at least $3\ell + 1 \geq 4$ vertices. Applying Theorem 7 to the graph $G'$, we have that $\gamma_t(G') \leq \alpha'(G')$.

Hence,
\[ \gamma_t(G) \leq \gamma_t(G') \leq \alpha'(G') \leq \alpha'(G) + \ell = \alpha'(G') + FT_3(G) - 1, \]
which establishes the desired upper bound. □

As an immediate consequence of Theorem 8, we have the following result.

Corollary 9. Conjecture 1 holds.

Proof. Let $G$ be a simple connected graph on at least two vertices. Since $\alpha'(G) \geq 1$ and $FT_3(G) \geq 1$, we have that
\[ 0 \leq (\alpha'(G) - 1)(FT_3(G) - 1) = \alpha'(G) \times FT_3(G) - \alpha'(G) - FT_3(G) + 1. \]
Hence by Theorem 8 we have that $\gamma_t(G) \leq \alpha'(G) + FT_3(G) - 1 \leq \alpha'(G) \times FT_3(G)$. □

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