Tame automorphisms of $\mathbb{C}^3$ with multidegree of the form $(3, d_2, d_3)$

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ABSTRACT

In this note we prove that the sequence $(3, d_2, d_3)$, where $d_1 \geq d_2 \geq 3$, is the multidegree of some tame automorphism of $\mathbb{C}^3$ if and only if $3|d_2$ or $d_3 \in 3\mathbb{N} + 2d_2\mathbb{N}$.

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1. Introduction

The multidegree of a polynomial mapping $F = \langle F_1, \ldots, F_n \rangle : \mathbb{C}^n \to \mathbb{C}^n$, denoted by $\text{mdeg } F$, is defined to be the sequence $(\deg F_1, \ldots, \deg F_n)$. It is of interest for which sequences $(d_1, \ldots, d_n)$ there are (polynomial) automorphisms or tame automorphisms of $\mathbb{C}^n$ with $\text{mdeg } F = (d_1, \ldots, d_n)$. Let us recall that a tame automorphism is a composition of linear and triangular automorphisms, where a triangular automorphism is a mapping of the form

$$T : \mathbb{C}^n \ni \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ x_2 + f_2(x_1) \\ \vdots \\ x_n + f_n(x_1, \ldots, x_{n-1}) \end{bmatrix} \in \mathbb{C}^n,$$

where $f_2 \in \mathbb{C}[x_1], \ldots, f_n \in \mathbb{C}[x_1, \ldots, x_{n-1}]$.

We will denote by $\text{Tame}(\mathbb{C}^n)$ the group of all tame automorphisms of $\mathbb{C}^n$, by $\text{Aut}(\mathbb{C}^n)$ the group of all polynomial automorphisms of $\mathbb{C}^n$, and by $\text{mdeg }$ the mapping from the set of all polynomial endomorphisms of $\mathbb{C}^n$ into the set $\mathbb{N}^n$.

It is easy to see that

$$\text{mdeg}(\text{Tame}(\mathbb{C}^1)) = \text{mdeg}(\text{Aut}(\mathbb{C}^1)) = \{1\}.$$

In the two-dimensional case from Jung [2] and van der Kulk [8] we know that

$$\text{mdeg}(\text{Tame}(\mathbb{C}^2)) = \text{mdeg}(\text{Aut}(\mathbb{C}^2)) = \{(d_1, d_2) : d_1|d_2 \text{ or } d_2|d_1\}.$$

In [3] it was proven that $(3, 4, 5), (3, 5, 7), (4, 5, 7), (4, 5, 11) \notin \text{mdeg}(\text{Tame}(\mathbb{C}^3))$ and that for all $d_3 \geq 2, (2, d_2, d_3) \in \text{mdeg}(\text{Tame}(\mathbb{C}^3))$. Next in [4] it was proven that if $d_3 \geq d_2 > d_1 > 2$, and $d_1, d_2$ are prime numbers, then $(d_1, d_2, d_3) \in \text{mdeg}(\text{Tame}(\mathbb{C}^3))$ if and only if $d_3 \notin d_1\mathbb{N} + d_2\mathbb{N}$. In this paper we investigate the set

$$\{(3, d_2, d_3) : 3 \leq d_2 \leq d_3 \} \cap \text{mdeg}(\text{Tame}(\mathbb{C}^3))$$

We prove the following theorem.

**Theorem 1.1.** If $3 \leq d_2 \leq d_3$, then $(3, d_2, d_3) \in \text{mdeg}(\text{Tame}(\mathbb{C}^3))$ if and only if $3|d_2$ or $d_3 \in 3\mathbb{N} + 2d_2\mathbb{N}$.

Since for all permutations $\sigma$ of the set $\{1, 2, 3\}$, $(d_1, d_2, d_3) \in \text{mdeg}(\text{Tame}(\mathbb{C}^3))$ if and only if $(d_{\sigma(1)}, d_{\sigma(2)}, d_{\sigma(3)}) \in \text{mdeg}(\text{Tame}(\mathbb{C}^3))$, the assumption $3 \leq d_2 \leq d_3$ is not restrictive.

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2. Some useful results

For the convenience of the reader we collect in this section all results that we will need in the proof of Theorem 1.1. The first is the following result from number theory.

**Theorem 2.1** (See e.g. [1]). If \(a, b\) are positive integers such that \(\gcd(a, b) = 1\), then for every integer \(k \geq (a - 1)(b - 1)\) there are \(k_1, k_2 \in \mathbb{N}\) such that

\[ k = k_1a + k_2b. \]

Moreover, \((a - 1)(b - 1) - 1 \notin a\mathbb{N} + b\mathbb{N}\).

The second one is the following easy proposition.

**Proposition 2.2** ([3], Proposition 2.2). If for a sequence of integers \(1 \leq d_1 \leq \ldots \leq d_n\) there is \(i \in \{1, \ldots, n\}\) such that

\[ d_i = \sum_{j=1}^{i-1} k_jd_j \quad \text{with} \quad k_j \in \mathbb{N}, \]

then there exists a tame automorphism \(F\) of \(\mathbb{C}^n\) with \(m\deg F = (d_1, \ldots, d_n)\).

**Proof.** Consider the mappings \(h = (h_1, \ldots, h_n) : \mathbb{C}^n \to \mathbb{C}^n\) and \(g = (g_1, \ldots, g_n) : \mathbb{C}^n \to \mathbb{C}^n\) given by the formulas

\[ h_k(x_1, \ldots, x_n) = \begin{cases} x_k & \text{for } k = i, \\ x_k + x_i^a & \text{for } k \neq i. \end{cases} \]

and

\[ g_k(u_1, \ldots, u_n) = \begin{cases} u_k + u_i^a & \text{for } k = i, \\ u_k & \text{for } k \neq i. \end{cases} \]

Now it is easy to see that for the automorphism \(F = g \circ h\) we have \(m\deg F = (d_1, \ldots, d_n)\). \(\Box\)

We will also use the following notions and results from the papers of Shestakov and Umirbayev [5,6].

**Definition 2.1** ([5], Definition 1). A pair \(f, g \in \mathbb{C}[X_1, \ldots, X_n]\) is called \(*\)-reduced if

(i) \(f, g\) are algebraically independent;

(ii) \(f, g\) are algebraically dependent, where \(\overline{h}\) denotes the highest homogeneous part of \(h\); and

(iii) \(\overline{f} \notin \mathbb{C}[\overline{g}]\) and \(\overline{g} \notin \mathbb{C}[\overline{f}]\).

**Definition 2.2** ([5], Definition 1). Let \(f, g \in \mathbb{C}[X_1, \ldots, X_n]\) be a \(*\)-reduced pair with \(\deg f < \deg g\). Put \(p = \frac{\deg f}{\gcd(\deg f, \deg g)}\). In this situation the pair \(f, g\) is called \(p\)-reduced.

**Theorem 2.3** ([5], Theorem 2). Let \(f, g \in \mathbb{C}[X_1, \ldots, X_n]\) be a \(p\)-reduced pair, and let \(G(x, y) \in \mathbb{C}[x, y]\) with \(\deg G(x, y) = pq + r, \quad 0 \leq r < p\). Then

\[ \deg G(f, g) \geq q \left(p \deg g - \deg f - \deg f + \deg G(f, g)\right) + r \deg g. \]

In the above theorem \([f, g]\) means the Poisson bracket of \(f\) and \(g\), defined as the following formal sum:

\[ \sum_{1 \leq i < j \leq n} \left( \frac{\partial f}{\partial X_i} \frac{\partial g}{\partial X_j} - \frac{\partial f}{\partial X_j} \frac{\partial g}{\partial X_i} \right) [X_i, X_j] \]

and:

\[ \deg [f, g] = \max_{1 \leq i < j \leq n} \left( \left( \frac{\partial f}{\partial X_i} \frac{\partial g}{\partial X_j} - \frac{\partial f}{\partial X_j} \frac{\partial g}{\partial X_i} \right) [X_i, X_j] \right), \]

where by definition \(\deg [X_i, X_j] = 2\) for \(i \neq j\) and \(\deg 0 = -\infty\).

From the definition of the Poisson bracket we have

\[ \deg [f, g] = \deg f + \deg g \]

and by Proposition 1.2.9 of [7],

\[ \deg [f, g] = 2 + \max_{1 \leq i < j \leq n} \left( \frac{\partial f}{\partial X_i} \frac{\partial g}{\partial X_j} - \frac{\partial f}{\partial X_j} \frac{\partial g}{\partial X_i} \right) \]

for \(f, g\) algebraically independent and \([f, g] = 0\) for \(f, g\) algebraically dependent.

Notice also that the estimate from Theorem 2.3 is true even if the condition (ii) of Definition 2.1 is not satisfied. Indeed, if \(G(x, y) = \sum u_{i,j} x^{u_i} y^{u_j}\), then, by the algebraic independence of \(\overline{f}\) and \(\overline{g}\) we have

\[ \deg G(f, g) = \max_{i,j} \deg (a_{i,j} f^g) \geq \deg_x G(x, y) \cdot \deg g \]

\[ = (qp + r) \deg g \geq q(p \deg g - \deg f - \deg g + \deg G(f, g)) + r \deg g. \]

The last inequality is a consequence of (1).

Thus we have the following proposition.
Proposition 2.4. Let \( f, g \in \mathbb{C}[X_1, \ldots, X_n] \) satisfy the conditions (i) and (iii) of Definition 2.1. Assume that \( \deg f < \deg g \), put
\[
p = \frac{\deg f}{\gcd (\deg f, \deg g)},
\]
and let \( G(x, y) \in \mathbb{C}[x, y] \) with \( \deg_y G(x, y) = pq + r, \ 0 \leq r < p \). Then \( \deg G(f, g) \geq q (p \deg g - \deg g - \deg f + \deg(f, g)) + r \deg g \).

The last result we will need is the following theorem.

Theorem 2.5 ([5], Theorem 3). Let \( F = (F_1, F_2, F_3) \) be a tame automorphism of \( \mathbb{C}^3 \). If \( \deg F_1 + \deg F_2 + \deg F_3 > 3 \) (in other words, if \( F \) is not a linear automorphism), then \( F \) admits either an elementary reduction or a reduction of one of types \( I \)–\( IV \) (see [5], Definition 2–4).

Let us also recall that an automorphism \( F = (F_1, F_2, F_3) \) admits an elementary reduction if there exists a polynomial \( g \in \mathbb{C}[x, y] \) and a permutation \( \sigma \) of the set \( \{1, 2, 3\} \) such that \( \deg (F(\sigma(1)) - g(F(\sigma(2)), F(\sigma(3)))) < \deg F(\sigma(1)) \); in other words, if there exists an elementary automorphism \( \tau : \mathbb{C}^3 \to \mathbb{C}^3 \) such that \( \text{md} \tau = \text{md} F \), where \((d_1, \ldots, d_3) < (k_1, \ldots, k_3)\) means that \( d_i \leq k_i \) for all \( i \in \{1, 2, 3\} \) and \( d_i < k_i \) for at least one \( i \in \{1, 2, 3\} \). Recall also that a mapping \( \tau = (\tau_1, \ldots, \tau_n) : \mathbb{C}^n \to \mathbb{C}^n \) is called an elementary automorphism if there exists \( i \in \{1, \ldots, n\} \) such that
\[
\tau_j (x_1, \ldots, x_n) = \begin{cases} x_j & \text{for } j \neq i, \\ x_j + g(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) & \text{for } j = i. \end{cases}
\]

3. Proof of the theorem

Proof. By Proposition 2.2, if \( 3 \mid d_2 \) or \( 3 \mid d_3 \in 3\mathbb{N} + d_2 \mathbb{N} \), then there exists a tame automorphism \( F : \mathbb{C}^3 \to \mathbb{C}^3 \) such that \( \text{md} F = (3, d_2, d_3) \). Thus in order to prove Theorem 1.1 it is enough to show that if \( 3 \nmid d_2 \) and \( 3 \nmid d_3 \), then there is no tame automorphism of \( \mathbb{C}^3 \) with multidegree \( (3, d_2, d_3) \). So from now on we will assume that \( 3 \nmid d_2 \) and \( 3 \nmid d_3 \).

Since \( 3 \nmid d_2 \), we have \( \gcd (3, d_3) = 1 \). Hence Theorem 2.1 implies that for all \( k \geq (3-1)(d_2-1) = 2d_2 - 2 \) we have \( k \in 3\mathbb{N} + d_2 \mathbb{N} \). Thus, since \( d_3 \notin 3\mathbb{N} + d_2 \mathbb{N} \), we have
\[
d_3 < 2d_2 - 2. \tag{2}
\]
Assume that \( F = (F_1, F_2, F_3) \) is an automorphism of \( \mathbb{C}^3 \) such that \( \text{md} F = (3, d_2, d_3) \). Our aim is to prove that \( F \) cannot be tame. By Theorem 2.5 it is enough to show that \( F \) admits neither a reduction of any of types \( I \)–\( IV \) (see [5], Definition 2–4) nor an elementary reduction.

Let us recall ([5], Definition 2) that if \( F \) admits a reduction of type \( I \) then there is a permutation \( \sigma \) of \( \{1, 2, 3\} \) and \( n \in \mathbb{N} \setminus \{0\} \) such that \( \deg F(\sigma(1)) = 2n \), \( \deg F(\sigma(2)) = ns \), where \( s \geq 3 \) is odd and \( 2n < \deg F(\sigma(3)) \leq ns \). Also, recall ([5], Definition 3) that if \( F \) admits a reduction of type \( II \), then there is a permutation \( \sigma \) of \( \{1, 2, 3\} \) and \( n \in \mathbb{N} \setminus \{0\} \) such that \( \deg F(\sigma(1)) = 2n \), \( \deg F(\sigma(2)) = 3n \), and \( \frac{3}{2} n < \deg F(\sigma(3)) \leq 2n \).

Thus if we assume that \( F \) admits a reduction of type \( I \) or \( II \), then since \( 3 \leq d_2 \leq d_3 \) we have \( n > 1 \), since otherwise \( 2n = 2 < 3 \). Next, for type \( I \), \( 2n \) is the smallest degree of \( F_i \). So, if we assume that \( F \) admits a reduction of type \( I \), then we obtain \( 2n = 3 \), a contradiction.

If \( F \) admits a reduction of type \( II \), then \( \deg F(\sigma(2)) = 3n \) is divisible by \( 3 \), which contradicts the above restrictions (since \( n > 1 \), we have \( 3n \neq 3 \)).

Now, recall ([5], Definition 4) that if \( F \) admits a reduction of type \( III \) or \( IV \), then there is a permutation \( \sigma \) of \( \{1, 2, 3\} \) and \( n \in \mathbb{N} \setminus \{0\} \) such that \( \deg F(\sigma(1)) = 2n \), and either
\[
\deg F(\sigma(2)) = 3n, \quad n < \deg F(\sigma(3)) \leq \frac{3}{2} n, \tag{3}
\]
or
\[
\frac{5}{2} n < \deg F(\sigma(2)) \leq 3n, \quad \deg F(\sigma(3)) = \frac{3}{2} n. \tag{4}
\]
If (3) is satisfied, then \( \deg F(\sigma(2)) = 3n \) means that \( n = 1 \), and so \( 1 < \deg F(\sigma(3)) \leq \frac{3}{2} \), a contradiction. If (4) is satisfied, then \( \deg F(\sigma(3)) = \frac{3}{2} n \) means that \( 3 | \deg F(\sigma(3)) \), so \( \deg F(\sigma(3)) = 3 \) and \( n = 2 \). But then \( 5 < \deg F(\sigma(2)) \leq 6 \), so \( \deg F(\sigma(2)) = 6 \) and is divisible by \( 3 \). This is a contradiction with \( 3 \nmid d_2 \) and \( d_3 \neq 3\mathbb{N} + d_2 \mathbb{N} \).

Now, assume that \( (F_1, F_2, F_3 - g(F_1, F_2)) \), where \( g \in \mathbb{C}[x, y] \), is an elementary reduction of \( (F_1, F_2, F_3) \). Hence we have \( \deg g(F_1, F_2) = \deg F_3 = d_3 \). Since \( \gcd (d_3, 3) = 1 \), by Proposition 2.4 (notice that the condition (i) Definition 2.1 is satisfied because \( F_1, F_2 \) are components of the automorphism \( (F_1, F_2, F_3) \), and (iii) is satisfied because \( 3 \nmid d_2 \) and \( d_2 \nmid 3 \)), we have
\[
\deg g(F_1, F_2) \geq q(3d_2 - d_2 - 3 + \deg [F_1, F_2]) + rd_2,
\]
where \( \deg g(x, y) = 3q + r \) with \( 0 \leq r < 3 \). Since \( F_1, F_2 \) are algebraically independent, we have \( \deg [F_1, F_2] \geq 2 \) and so \( 3d_2 - d_2 - 3 + \deg [F_1, F_2] \geq 2d_2 - 1 \). Then (2) implies that \( q = 0 \). Also by (2) we must have \( r < 2 \). Thus
\[ g(x, y) = g_0(x) + g_1(x)y. \] Since \( 3N \cap (d_2 + 3N) = \emptyset \), we deduce that \( d_3 = \deg g(F_1, F_2) \in 3N \cup (d_2 + 3N) \subset 3N + d_2N \), contrary to assumption.

Now, assume that \((F_1, F_2 - g(F_1, F_3), F_2)\), where \( g \in \mathbb{C}[x, y] \), is an elementary reduction of \((F_1, F_2, F_3)\). Then \( \deg g(F_1, F_3) = d_2 \). Since \( d_3 \not\in 3N + d_2N \), it follows that \( \gcd(3, d_3) = 1 \). Then by Proposition 2.4 we have
\[ \deg g(F_1, F_3) \geq q(3d_3 - d_3 - 3 + \deg(F_1, F_2)) + rd_3, \]
where \( \deg g(x, y) = 3q + r \) with \( 0 \leq r < 3 \). Since \( 3d_3 - d_3 - 3 + \deg(F_1, F_3) \geq 2d_3 - 1 > d_2 \), we infer that \( q = 0 \). Since also \( d_3 > d_2 \) (because \( d_3 \geq d_2 \) and \( d_3 \not\in 3N + d_2N \)), we see that \( r = 0 \). Thus \( g(x, y) = g(x) \), and \( d_2 = \deg g(F_1, F_3) = \deg g(F_1) \in 3N \), a contradiction.

Finally, assume that \((F_1 - g(F_2, F_3), F_2, F_3)\), where \( g \in \mathbb{C}[x, y] \), is an elementary reduction of \((F_1, F_2, F_3)\). Then \( \deg g(F_2, F_3) = 3 \). Let
\[ p = \frac{d_2}{\gcd(d_2, d_3)}. \]
Since \( d_3 \not\in 3N + d_2N \), we obtain \( d_2 \nmid d_3 \), and hence \( p > 1 \). By Proposition 2.4 we have
\[ \deg g(F_2, F_3) \geq q(pd_3 - d_3 - d_2 + \deg(F_2, F_3)) + rd_3, \]
where \( \deg g(x, y) = qp + r \) with \( 0 \leq r < p \). Since \( d_3 > 3 \), it follows that \( r = 0 \). Consider the case \( p \geq 3 \).

Then \( pd_3 - d_3 - d_2 + \deg(F_2, F_3) \geq d_3 + \deg(F_2, F_3) > 3 \). Thus we must have \( q = 0 \). Hence \( g(x, y) = g(x) \), and \( 3 = \deg g(F_2, F_3) = \deg g(F_2) \in 2N \). This contradicts \( d_2 \neq 3 \) (we have assumed that \( 3 \nmid d_2 \)).

Consider now the case \( p = 2 \). Since \( p = 2 \), we have, for some \( n \in \mathbb{N} \), \( d_2 = 2n \) and \( d_3 = ns \), where \( s \geq 3 \) is odd. Since also \( d_2 > 3 \), it follows that \( n \geq 2 \). This means that \( d_3 - d_2 \geq 2 \), and \( 2d_3 - d_3 - d_2 + \deg(F_2, F_3) = d_3 - d_2 + \deg(F_2, F_3) \geq 4 > 3 \). Thus, also in this case we have \( q = 0 \). As before this leads to a contradiction. \( \square \)

4. Question

Theorem 1.1 and the results of [4] suggest the following question/conjecture.

Conjecture 4.1. If \( p \leq d_2 \leq d_3 \), where \( p > 2 \) is a prime number, then \((p, d_2, d_3) \in \text{mdeg}(\text{Tame}(\mathbb{C}^3))\) if and only if \( p \mid d_2 \) or \( d_3 \in pN + d_2N \).

References