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A proof of the simple connectivity at infinity of *Out* F_4

John Rickert

Department of Mathematics, University of Dallas, Irving, TX 75062, USA

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Abstract

A finitely presented group G is said to be simply connected at infinity if, for any compact set C in the universal cover $\tilde{X}$ for the standard 2-complex for G, there exists a compact set D such that any loop in $\tilde{X} \setminus D$ is homotopically trivial in $\tilde{X} \setminus C$. Suppose that F_4 is a free group on four generators, $Aut\, F_4$ its automorphism group, and $Inn\, F_4$ the subgroup of inner automorphisms. We use direct, elementary means to show that the outer automorphism group of rank 4, $Aut\ F_4/Inn\ F_4$ is simply connected at infinity.

MSC: 20F32

0. Introduction

Suppose that G is a group with finite presentation $\langle A; R\rangle$. Let X be the standard 2-complex corresponding to this presentation and $\tilde{X}$ its universal cover. The 1-skeleton of $\tilde{X}$ is the Cayley graph corresponding to the presentation $\langle A; R\rangle$. If G is infinite, then $\tilde{X}$ is infinite and therefore may be said to have "end properties". In particular, $\tilde{X}$ is *simply connected at infinity* if, for any compact set $C \subset \tilde{X}$, there is a compact set D containing C such that any loop in $\tilde{X} \setminus D$ is homotopically trivial in $\tilde{X} \setminus C$. If $\tilde{X}$ is also 1-ended, then $\tilde{X}$ is said to be 1-connected at infinity. It can be shown that simple connectivity at infinity does not depend on the presentation chosen for the finitely presented group G [7], and thus the property is said to apply to G itself.

The group we consider is the outer automorphism group of a free group of rank 4, *Out* F_4. *Out* F_2 has infinitely many ends, and Vogtmann [8] showed that *Out* F_n is 1-ended for $n \geq 3$ and simply connected at infinity for $n \geq 5$. Our Main Theorem is that *Out* F_4 is simply connected at infinity. It should be reasonably clear that our argument extends to the cases where $n > 4$, but we leave details of this to the reader. Recently, Bestvina et al. [1] have completely answered the question of the higher connectivity

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at infinity of *Out* F_n by using a Morse-theoretic approach on 'Outer Space'. The proof we offer here we believe to be conceptually simple and concrete, using elementary methods of geometric group theory.

1. Preliminaries

Let F_4 be a free group of rank 4, *Aut* F_4 its automorphism group, *Inn* F_4 its inner automorphism group, and *Out* $F_4 = Aut\ F_4/Inn\ F_4$ its outer automorphism group. In order to show simple connectivity at infinity for *Out* F_4, it is sufficient to establish this property for some group of finite index.

Consider the exponent sum homomorphism $\hat{\sigma}: Aut\ F_4 \to GL_4Z$ defined in the following way [5]. Since any homomorphism from a free group is determined by its action on the generators, each automorphism ϕ of F_4 may be described as a quadruple $(a_1 \mapsto W_1(a_1,\dots,a_4),\dots,a_4 \mapsto W_4(a_1,\dots,a_4))$, where $W_i(a_1,\dots,a_4)$ is a word in the generators $a_1,\dots,a_4$ for $1 \le i \le 4$. Then the ij entry of the matrix $\hat{\sigma}(\phi)$ is defined to be the exponent sum of a_i in $W_j(a_1,\dots,a_4)$. Composing $\hat{\sigma}$ with the homomorphism $det: GL_4Z \to Z_2$ yields a map $\hat{\tau}$; the kernel of $\hat{\tau}$ is a subgroup of index 2 in *Aut* F_4 and is called the *special automorphism group*, $S\,Aut_4$. As $Inn\ F_4 \subset S\,Aut_4$, there are induced maps $\sigma: Out\ F_4 \to GL_4Z$ and $\tau: Out\ F_4 \to Z_2$. The kernel of τ has index 2 in *Out* F_4 and will be called the group of *special outer automorphisms*, $SOut_4$. It will be shown that $SOut_4$ is simply connected at infinity.

Definitions and conventions. Assume $F_4 = \langle a,b,c,d \rangle$. The inverse of a is denoted $\bar{a}$; similarly for the other generators. Except where noted, i, j, k, etc. mean variables with values ranging over the generators of F_4 and their inverses. For $j \ne i,\bar{\imath}$, E_{ij} is the automorphism which sends $i \mapsto ij$, leaving all other generators $(\ne \bar{\imath})$ fixed. (These are called "Nielsen automorphisms".) Occasionally, the subscripts will be dropped, for ease of notation. Automorphisms will be composed from right to left, in contrast to many writers, e.g., [2]. The commutator is taken to be $[\alpha,\beta] = \alpha\beta\alpha^{-1}\beta^{-1}$. Generators of F_4 will be said to be *positive*, barred generators *negative*. E_{ij} is called *positive* if i, j have the same sign. Observe that if E is positive, then $\sigma(E)$ has only nonnegative entries. The same is true for any product of positive generators of *Out* F_n with positive exponents.

The group $SOut_4$ acts on $\tilde{X}$ on the left as a group of covering transformations, so that X is the quotient space of $SOut_4\backslash\tilde{X}$. We will also be interested in an intermediate cover $IO_4\backslash\tilde{X}$ of X, where IO_4 is a normal subgroup of $SOut_4$ to be described below.

Lemma 1.1. *$SOut_4$ has presentation $\langle E_{ij}; R\rangle$, where $i, j \in \{a,b,c,d\}$, $j \ne \bar{\imath}$ and R is the following set of relations*:

$$E_{ij}^{-1} = E_{i\bar{j}}, \tag{1}$$

$$[E_{ij}, E_{kl}] = 1 \quad \text{if } k \notin \{i, j, \bar{j}\} \quad \textit{and} \quad l \notin \{i, \bar{\imath}\}, \tag{2}$$

$$[E_{i\bar{j}}, E_{j\bar{k}}] = E_{i\bar{k}}, \ k \notin \{i, \bar{\imath}\}, \tag{3}$$

$$w_{ij} = w_{\bar{\imath}\bar{j}}, \quad \textit{where } w_{ij} = E_{\bar{j}i} E_{\bar{\imath}j} E_{ji}, \tag{4}$$

$$w_{ij}^4 = 1, \tag{5}$$

$$\prod_{j \neq i} E_{ji} E_{\bar{j}i} = 1, \qquad \textit{with } j \textit{ positive.} \tag{6}$$

Proof. Gersten [2] showed that $SAut_4$ has presentation given by $\langle E_{ij}; R' \rangle$, where R' consists of relations (1)–(5). Relation (6) trivializes the inner automorphisms. □

Remark. The 1-skeleton of the universal cover $\tilde{X}$ of the standard 2-complex for $SOut_4$ with presentation given by Lemma 1.1 is the Cayley graph for $SOut_4$ relative to the generators $\{E_{ij}\}$. The vertices of the 1-skeleton correspond to elements of $SOut_4$, and the edges are labeled by generators. Hence, $\sigma : SOut_4 \to SL_4Z$ may be regarded as mapping from the vertices of $\tilde{X}$ to SL_4Z.

Main Theorem. *Out F_4 is simply connected at infinity.*

Proof. It is sufficient to establish simple connectivity at infinity for $SOut_4$, a subgroup of index 2 in $Out\,F_4$. Let $\tilde{X}$ be the universal cover of the standard 2-complex for the presentation in Lemma 1.1. Suppose that $C \subset \tilde{X}$ is compact. The first part of Mihalik's proof of Jackson's theorem applies: A subspace $[IO]_M$ for some positive integer M and a compact set D exist such that any loop γ'' in $\tilde{X} \backslash D$ is homotopic in $\tilde{X} \backslash C$ to a loop γ' in $[IO]_M$ (Lemmas 2.1 and 2.2). (If IO_4 were known to be finitely presented, the theorem would now follow at once.) By Lemma 2.2 γ' is homotopic in $\tilde{X} \backslash C$ to a loop γ which is in turn homotopic in $\tilde{X} \backslash C$ to each of its translates by E^n_{ab}, $n \geq 0$. Finally, Lemmas 4.1–5.3 show that each of these translates of γ bounds a disk, and that for all sufficiently large n, these disks lie in $\tilde{X} \backslash C$. □

Corollary. *Out F_4 is 1-connected at infinity.*

For, *Out* F_4 is connected at infinity as well.

2. Reduction to the subspace $[IO]_M$

A preliminary simplification will allow us to assume that the loop to be contracted lies in a certain kind of subspace of $\tilde{X}$.

The exact sequence

$$1 \to IA_4 \to Aut\ F_4 \to GL_4Z \to 1 \tag{7}$$

induces the exact sequence:

$$1 \to IO_4 \to SOut_4 \to SL_4Z \to 1, \tag{8}$$

where IA_4 and IO_4 are the kernels of $\hat{\sigma}$ and σ, respectively. Magnus [4] found that IA_4, and hence IO_4 is generated by the set of all x_{ij} and x_{ijk}, where $x_{ij} = E_{ij}E_{\bar{i}j}$ and $x_{ijk} = [E_{ij}, E_{ik}]$.

In what immediately follows, we assume that the presentation for $SOut_4$ has been expanded to include the x_{ij}, x_{ijk} generators above and their defining relations. Also added are relations of the form $E_{i\bar{j}}x_{kl}E_{ij} = r_{ijkl}$ and $E_{i\bar{j}}x_{klm}E_{ij} = r_{ijklm}$, where r_{ijkl} and r_{ijklm} are products of x_{ij} and x_{ijk}. This will not affect simple connectivity at infinity but aids in the construction of homotopies. For the time being, we pass to the universal cover of the space corresponding to the expanded presentation. This universal cover is obtained from the previous one by adding only finitely many 1- and 2-cells at each vertex of the previous universal cover.

Definition. The subspace $[IO] \subset \tilde{X}$ consists of the vertices in the coset IO_4 and the edges $\{x_{ij}, x_{ijk}\}$ connecting these vertices. The subspace $[IO]_k$ is $E_{ab}^k[IO]$, the translate of $[IO]$ by E_{ab}^k. Note that every vertex in $[IO]_k$ maps under σ to the same matrix, that $[IO]_k \cap [IO]_m = \emptyset$ unless $k = m$, and $[IO]$ is a Cayley graph for IO_4 with generating set $\{x_{ij}, x_{ijk}\}$.

In [6], Mihalik proves the following extension of a theorem of Jackson [3].

Theorem. *If $1 \to H \to G \to K \to 1$ is a short exact sequence of infinite, finitely generated groups with G finitely presented, K 1-ended and H contained in a finitely presented subgroup L of infinite index in G, then G is simply connected at infinity.*

In (8), IO_4 is finitely generated, SL_4Z 1-ended, and $SOut_4$ finitely presented. It is not known, however, whether IO_4 is contained in a finitely presented subgroup of infinite index in $SOut_4$. But Mihalik's proof of his Lemma 8 in the proof of the above theorem immediately implies the following:

Lemma 2.1. *Let $C' \subset \tilde{X}$ be compact. There exist a compact set D containing C' and a positive integer K such that, for every $k \geq K$ and every loop γ'' in $\tilde{X} \backslash D$, γ'' is homotopic to a loop γ' in $[IO]_k$ by a homotopy whose image misses C'.*

For $n \geq 1$, $\sigma(E_{ab}^n)$ is an element of infinite order in SL_4Z. Now recall that a *ray* is a continuous map with domain $[0, \infty)$ and a map is *proper* if the inverse image of every compact set is compact. Thus, for $n \geq 1$, $ray(E_{ab}) \equiv ray(E_{ab}^n)$ is proper in $\tilde{X}$ and projects to a proper ray in the quotient space $IO_4/\tilde{X}$.

Definition. The *standard ray* is the ray E^n_{ab}, $n \geq 1$ in $\tilde{X}$.

Now since C has compact image in $SL_4 Z$, there is an integer M' such that the maximum entry of any matrix in $\sigma(C)$ is less than M'. The subspace $[IO]_m$ is therefore disjoint from C for all $m \geq M'$.

Stipulation. Let C be a compact subcomplex of $\tilde{X}$. Let D be determined by Lemma 2.1, and let $M = \max\{K, M'\}$, where K is given by the preceding lemma and M' is given by the discussion above. Hence, any loop γ'' in $\tilde{X} \backslash D$ is homotopic to a loop γ' in $[IO]_M$ by a homotopy in $\tilde{X} \backslash C$, and $[IO]_M$ is disjoint from C.

The proof of the Main Theorem will be completed by establishing the following claim.

Claim 2.1. *With D given by the previous stipulation, any loop in $\tilde{X} \backslash D$ is homotopically trivial in $\tilde{X} \backslash C$.*

Lemma 2.2. (1) *Let any loop γ' in $[IO]_M$ be given. Then there is a homotopy in $\tilde{X} \backslash C$ from γ' to a loop γ such that γ is an edge path, each of whose directed edges are labeled by members of $\{E_{ij} : j \neq i, \bar{\imath}\}$, and such that every vertex in the homotopy projects under σ to a matrix with maximum value at least M.*

(2) *Let v be a vertex in $\tilde{X}$, and let $|\sigma(v)|$ be the maximum value of the entries in $\sigma(v)$. If $\sigma(v)$ has only nonnegative entries and $|\sigma(v)| \geq M$, and if P is any product of positive generators, then $|\sigma(vP)| \geq M$. Hence, $vP \notin C$.*

(3) *Let v be a vertex such that $\sigma(v)$ has only nonnegative entries. For any positive number N, there exists a positive number L such that, if P is any product of positive generators with length at least L, then $|\sigma(vP)| > N$.*

Proof. (1) Now γ' is a loop with edge labels in $\{x_{ij}^{\pm 1}, x_{ijk}^{\pm 1}\}$. First, it may be assumed that all of the arguments i, j, k in x_{ij} and x_{ijk} are positive. When negative arguments are involved, the following equations show how to convert to positive ones only. With i, j, k positive, we have

$$
\begin{aligned}
&x_{i\bar{\jmath}} = x_{ij}^{-1}, && x_{\bar{\imath}j} = x_{ij}, && x_{\bar{\imath}\bar{\jmath}} = x_{ij}^{-1}, \\
&x_{ij\bar{k}} = x_{jk} x_{ikj} x_{j\bar{k}}, && x_{i\bar{\jmath}k} = x_{kj} x_{ikj} x_{k\bar{\jmath}}, && x_{\bar{\imath}jk} = [x_{ij}, x_{jk}] x_{ikj}, \\
&x_{i\bar{\jmath}\bar{k}} = x_{kj} x_{i\bar{k}j} x_{k\bar{\jmath}}, && x_{\bar{\imath}j\bar{k}} = [x_{ij}, x_{i\bar{k}}] x_{i\bar{k}j}, && x_{\bar{\imath}\bar{\jmath}k} = [x_{i\bar{\jmath}}, x_{ik}] x_{ik\bar{\jmath}}, \\
&x_{\bar{\imath}\bar{\jmath}\bar{k}} = [x_{i\bar{\jmath}}, x_{i\bar{k}}] x_{i\bar{k}\bar{\jmath}}.
\end{aligned}
$$

Then x_{ij} is homotopic to $E_{ij} E_{\bar{\imath}\bar{\jmath}}^{-1}$ and x_{ijk} is homotopic to $E_{ij} E_{ik} E_{ij}^{-1} E_{ik}^{-1}$ with i, j, k all positive. Thus γ' is homotopic to a path γ in $\{E_{ij}\}$ in such a way that $|\sigma(g)| \geq M$ for each vertex g in γ, by a homotopy with image missing C.

(2) Since $\sigma(v)$ has no negative entries and $|\sigma(v)| \geq M$, and since $\sigma(E_{ij})$ adds one column of $\sigma(v)$ to another when i,j have the same sign, it follows that $|\sigma(vP)| \geq M$ for all positive products P.

(3) Each column of $\sigma(v)$ has at least one positive entry and no negative entries, and the fact that each generator E is positive implies that $|\sigma(vE)| \geq |\sigma(v)|$. The statement thus holds if P is long enough to have $N+1$ occurrences of the same generator. □

3. Standard slides

Lemma 2.2 allows us to assume that the loop to be contracted is a product of $\{E_{ij}^{\pm 1}\}$ and that it has a contraction using basic relators not involving any x_{ij}, x_{ijk}. Hence, for the remainder of the proof of the x_{ij}, x_{ijk} and all relators in which they appear are deleted, yielding the presentation for $SOut_4$ of Lemma 1.1.

In order to find a contracting homotopy for γ outside of C, we first describe the process of "sliding" γ, and edges in general, along the standard ray.

Lemma 3.1. *Each E_{ij} commutes with at least one of $E_{ab}, E_{ba}, E_{cd}, E_{dc}$.*

Proof. By direct calculation E_{ij} commutes with E_{kj}, and each of a,b,c,d appears as the second argument in the set $\{E_{ab}, E_{ba}, E_{cd}, E_{dc}\}$. □

Definition. Let $e \in \{E_{ij}\}$ be an edge in $\tilde{X}$ with initial vertex e_0 and terminal vertex e_1, and let $n \geq 1$. The nth *standard slide* of e along E_{ab}^n is the disk chosen to be as follows:

(1) If e commutes with E_{ab}, then the nth standard slide of e is the disk based at e_0 with boundary $E_{ab}^n e E_{ab}^{-n} e^{-1}$ filled in with copies of the 2-cell corresponding to the basic relator $[E_{ab}, e]$.

(2) If e does not commute with E_{ab} but commutes with E_{dc}, then the nth standard slide of e is the disk based at e_0 with boundary

$$E_{ab}^n E_{dc}^n E_{ab}^{-n} e E_{ab}^n E_{dc}^{-n} E_{ab}^{-n} e^{-1},$$

filled in with copies of the 2-cell corresponding to the basic relator $[E_{dc}, e]$ and of the 2-cell corresponding to the basic relator $[E_{ab}, E_{dc}]$ in the obvious way.

(3) Similarly, if e does not commute with E_{ab} or E_{dc} but commutes with E_{cd}, then the nth standard slide of e is the disk based at e_0 with boundary

$$E_{ab}^n E_{cd}^n E_{ab}^{-n} e E_{ab}^n E_{cd}^{-n} E_{ab}^{-n} e^{-1},$$

filled in analogously to the case given in (2).

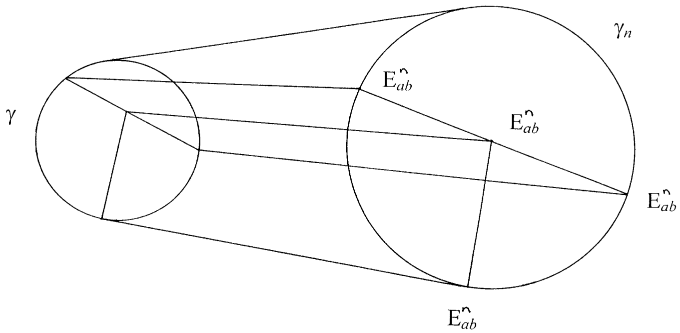

Fig. 1. Sketch of plan for contracting γ outside of C.

(4) Finally, if e commutes with none of E_{ab}, E_{cd}, E_{dc}, then the nth standard slide of e is the disk based at e_0 with boundary

$$E^n_{ab}E^n_{dc}E^{-n}_{ab}E^n_{ba}E^{-n}_{dc}eE^n_{dc}E^{-n}_{ba}E^n_{ab}E^{-n}_{dc}E^{-n}_{ab}e^{-1},$$

filled in also analogously to the case given in (2).

The edge e occurring in the middle of its standard slide is called its *translate along the standard slide*.

The choice of standard slides yields the following.

Lemma 3.2. *If e is an edge in γ, then the standard slide of e along E^n_{ab} is disjoint from C for all $n \geq 1$. Hence, for all $n \geq 1$, the "lateral homotopy" formed by the nth slide of γ is disjoint from C.*

Proof. By Lemma 2.2(1), the vertices of e project under σ to matrices with nonnegative entries whose maximum values are at least M. The standard slides have both arguments positive on each ray used, and hence, if v is a vertex in the slide of e, $v = e_0P$ or $v = e_1P$, where e_0 is the initial vertex of e and e_1 its terminal vertex, and P is a product of positive generators. Lemma 2.2(2) yields that $v \notin C$. □

Using these lemmas the proof of the Main Theorem is reduced to proving Claim 3.1 below. First, a contracting homotopy H is taken for γ using copies of basic relators in the presentation for $SOut_4$. As H is slid along E^n_{ab}, the slide of γ is disjoint from C, according to the theorem just proved. The slides of the basic relators on the interior of H may still intersect C. But for each $n \geq 1$, the nth slide of each of these basic relators along E^n_{ab} can be capped (contracted) by a disk, and these disks are disjoint from C for all sufficiently large n. The caps together with the lateral homotopy provide a contraction for γ in $\tilde{X} \backslash C$. (See Fig. 1.)

Claim 3.1. *Suppose that A is a copy of a 2-cell corresponding to a basic relator in H, and let A_n consist of A and the nth standard slide of each of its edges. Then there* *is a disk* $B_n \subset \tilde{X}$ *with* $\partial B_n = \partial A_n$ *such that, for all sufficiently large* n, B_n *is disjoint from* C.

4. The commutation complex

In order to specify the disks B_n, we make extensive use of the commutation relations in $SOut_4$.

Definition. The *commutation complex* Δ has vertices $\{E_{ij}\colon E_{ij}$ positive$\}$ and an undirected edge between E_{ij} and E_{kl} if E_{ij} and E_{kl} commute.

Remark. Lemma 3.1 shows that Δ is connected.

Definition. The *standard commutation path* from E_{ij} to E_{ab} in Δ is the shortest path obtained by using the edges in $\mathrm{Star}_\Delta(E_{ij})$ and the path $E_{cd} - E_{ab} - E_{dc} - E_{ba}$, going through E_{dc} in preference to E_{cd}.

Definition. If v is a vertex in $\tilde{X}$, $n \geq 1$, and E_1, E_2 commute, then a *simple commutation path in* $\tilde{X}$ *of order* n from vE_1^n to vE_2^n with *origin* v is the path $E_2^n E_1^{-n}$ with initial point vE_1^n. This path will be said to correspond to the edge (E_1, E_2) in Δ. Observe that the resulting loop based at v encloses a disk composed of commutation relations. A *commutation path in* $\tilde{X}$ of order n is a concatenation of simple commutation paths of order n with a common origin. For any vertex $v \in \tilde{X}$, $n \geq 1$, and E_{ij}, the *standard commutation path in* $\tilde{X}$ of order n is the commutation path from vE_{ij}^n to vE_{ab}^n corresponding to the standard commutation path in Δ.

The process of specifying the homotopies B_n mentioned above can now be continued. The following lemma allows a translation of A "orthogonal" to the disk A_n.

Lemma 4.1. *For each basic relation, there is some* E_{xy} *that commutes with all of the generators in the relation. Moreover,* E_{xy} *can be chosen to be positive.*

Proof. For relations (1) and (2) in Lemma 1.1, E_{ij} is sufficient; for (3), E_{lk}; for (4) and (5), E_{kl}; and for (6), E_{ji}. By relation (1), it is possible to adjust the second argument as needed so that both x and y have the same sign. □

Remark. The preceding lemma does not hold when $n = 3$.

Definition. For each basic relator A in H, choose a positive generator $E_{xy}^{(A)}$ such that $E_{xy}^{(A)}$ commutes with each edge in A. Then $E_{xy}^{(A)}$ will be called *orthogonal* to A. When a specific A is understood, the superscript on E_{xy} will be omitted.

Let E_{xy} be orthogonal to A. For each vertex v in A, there is a standard commutation path from vE_{ab}^n to vE_{xy}^n. Hence, the boundary of A_n together with AE_{xy}^n is comprised

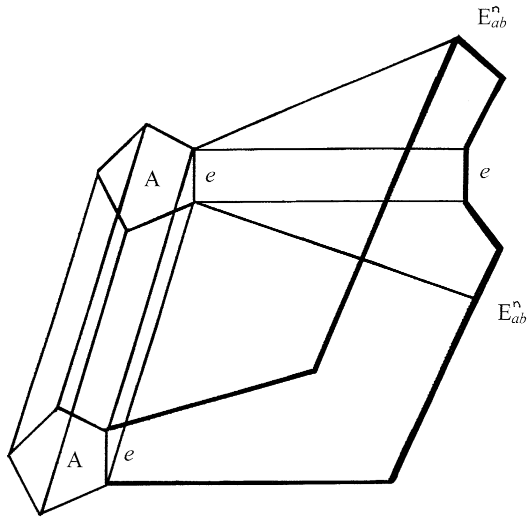

Fig. 2. The translate of A by E^n_{xy} and formation of sectors.

of “sectors” corresponding to each edge e of A. Specifically, if e has initial vertex e_0 and terminal vertex e_1, the sector involving e has boundary eE^n_{xy}, the standard commutation path from $e_1E^n_{xy}$ to $e_1E^n_{ab}$, the edge path given by the standard slide of e from $e_1E^n_{ab}$ to $e_0E^n_{ab}$, and the standard commutation path from $e_0E^n_{ab}$ to $e_0E^n_{xy}$. Because standard commutation paths have been chosen, adjacent sectors attach coherently. (See Fig. 2.)

Claim 4.1. *Let A be a basic relator in H and e an edge in A. Then for all sufficiently large n, the sector involving e with base on A_n is homotopically trivial in $\tilde{X}\backslash C$.*

That is, the translate of A together with the disks for the sectors outside of C yield a disk B_n with $\partial B_n = \partial A_n$ disjoint from C for all n sufficiently large.

To fill in the sectors, we show first that the nth translate of e in its standard slide and the translate of e by E^n_{xy} can be included into the boundary of a disk with a convenient form.

Lemma 4.2. *Suppose that e is a generator and $[e,E_{ij}]=[e,E_{xy}]=1$. Then there exists a sequence of generators $E_{ij}=E_0 - E_1 - \cdots - E_r = E_{xy}$ such that $[e,E_p]=1$ and $[E_p,E_{p+1}]=1$ for all $0 \le p < r$.*

Proof. Without loss of generality, suppose that $e = E_{ab}$, and let $Z(E_{ab})$ be the vertices in $\mathrm{Star}_\Delta(E_{ab})$. The statement holds trivially if either E_{ij} or $E_{xy} = E_{ab}$. In the nontrivial case, the claim holds because the subgraph of Δ restricted to the vertices $Z(E_{ab})\backslash E_{ab}$ is connected. See Fig. 3. □

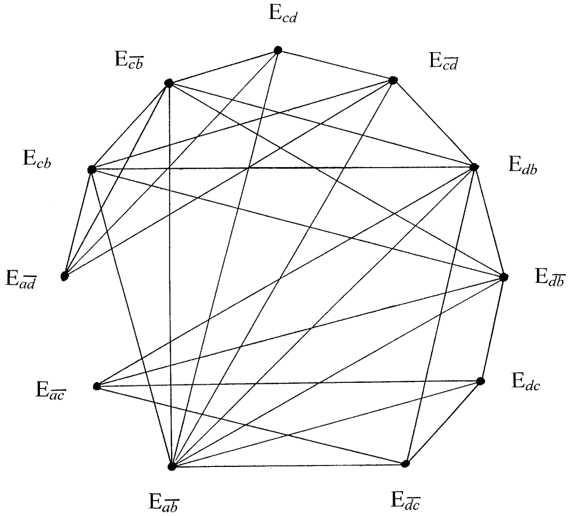

Fig. 3. $Z(E_{ab})\backslash E_{ab}$.

Lemma 4.3. *Suppose e is an edge in A with initial vertex e_0 and terminal vertex e_1. Suppose also that e is translated by E_{wz}^n in its standard slide and that E_{xy} is orthogonal to A. Then there is a commutation path s in $\tilde{X}$ from $e_1E_{wz}^n$ to $e_1E_{xy}^n$ such that e commutes with each edge in s. Hence, the loop based at $e_0E_{wz}^n$ with boundary $ese^{-1}s^{-1}$ can be filled in with basic commutation relations.*

Proof. If E_{xy} commutes with E_{wz}, then e, E_{xy}, and E_{wz} commute pairwise. Hence, the loop based at $e_0E_{wz}^n$ with boundary $eE_{xy}^nE_{wz}^{-n}e^{-1}E_{wz}^nE_{xy}^{-n}$ can be filled in with basic commutation relations. Thus, $s=E_{xy}^nE_{wz}^{-n}$, a simple commutation path, suffices. Observe that this strip can be said to arise from filling in the "box" spanned by e, E_{xy}^n, E_{wz}^n, since the generators commute pairwise. In the general case, there is a commutation path s from E_{wz} to E_{xy} such that each edge in s commutes with e, by Lemma 4.2. The resulting boxes attach coherently and provide the homotopy claimed.

Lemma 4.4. *For all sufficiently large n, the strips given by Lemma* 4.3 *are disjoint from C.*

We prove Lemma 4.4 using the following lemma:

Lemma 4.5. *Let E_1, E_2, E_3 be positive generators that commute pairwise. Suppose that $K\subset\tilde{X}$ is compact, and let B_n be the* 3*-dimensional box spanned by E_1^n, E_2^n, and E_3. For all sufficiently large n, the projection under σ of the strip with base E_1^n and boundary $E_2^nE_1^{-n}E_3E_1^nE_2^{-n}E_3^{-1}$ is disjoint from K.*

Proof of Lemma 4.5. Each generator maps to an elementary matrix whose effect on a matrix T is to add one column of T to another, and all generators are positive. The

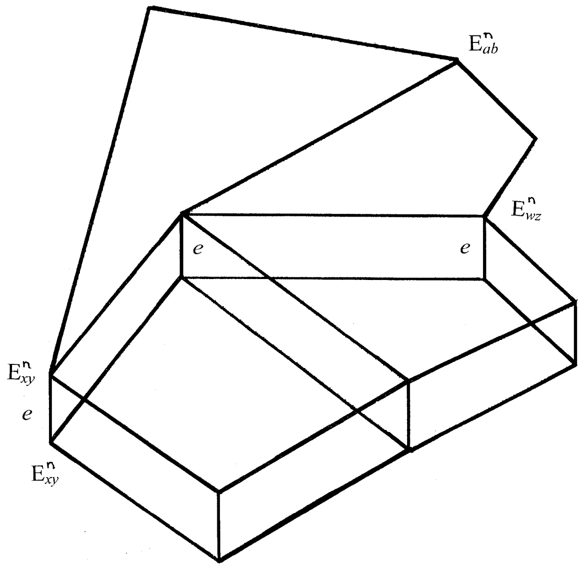

Fig. 4. The remaining loops to consider, commutation loops.

vertices in the strip correspond to $E_1^nE_2^k$, $E_2^nE_1^k$, $E_1^nE_2^kE_3$, and $E_2^nE_1^kE_3$, where $0 \leq k \leq n$. By Lemma 2.2(3) the projection under σ of the nth strip is disjoint from $\sigma(K)$ for all n sufficiently large, and hence the strip itself is disjoint from K. □

Proof of Lemma 4.4. Here an arbitrary vertex $v \in \tilde{X}$ is the origin of the cube instead of $1 \in \tilde{X}$. Premultiplying all vertices by v^{-1} translates v to the identity and is an isometry of $\tilde{X}$. E_{wz} is positive by definition of standard slides, e may be taken in its positive sense, and E_{xy} may be assumed positive by Lemma 4.1. Lemma 4.5 applies to yield a strip disjoint from $v^{-1}C$. Translating back by v is also an isometry of the space, and the translated strip is disjoint from C. □

Observe that the sector under consideration has been partially filled in by disks that are disjoint from C for all n sufficiently large, and that the loops remaining are composed of commutation paths in $\tilde{X}$. Specifically, a commutation path from the terminal vertex vE^n_{wz} of the translate of e in its standard slide to vE^n_{ab}, as part of the standard slide; a path from vE^n_{ab} to vE^n_{xy} by using the standard commutation path; and finally, a commutation path from vE^n_{xy} to vE^n_{wz} as given by the previous proposition. There is likewise a loop based at the initial vertex of the translate of e in its standard slide. (See Fig. 4.)

If these commutation loops can be contracted by disks that are disjoint from C for sufficiently large n, B_n will be specified: B_n consists of the translate of A by E^n_{xy}, the strips from the translates of the edges of A to their translates by E^n_{xy}, and contracting disks for the two commutation loops in each sector.

5. Attaching 2-cells to the commutation complex and construction of homotopies

Lemma 5.1. *Suppose that δ_n is a loop in $\tilde{X}$ with basepoint on the nth slide of A such that δ_n corresponds to a loop in Δ. Then there is a disk ε_n with $\partial\varepsilon_n = \delta_n$ such that ε_n is disjoint from C for all n sufficiently large.*

Proof. First we consider triangles in the commutation complex. If E_1, E_2, E_3 is a loop in the commutation complex, then the generators commute pairwise. If v is the origin of the loop in $\tilde{X}$, then the loop has initial vertex vE_1^n and boundary $E_2^n E_1^{-n} E_3^n E_2^{-n} E_1^n E_3^{-n}$. We let ε_n be the disk with the same boundary and origin $vE_1^n E_2^n E_3^n$. Observe that ε_n is obtainable by filling in the cube spanned by E_1^n, E_2^n, E_3^n with origin v. The argument from Lemma 4.4 shows that for all sufficiently large n, ε_n is disjoint from C. This construction can obviously be extended to any triangulable loop in the commutation complex.

Convention. If a homototopy corresponding to a loop δ in Δ has been specified, a 2-cell is added to Δ with boundary δ. By the previous remarks, each triangle in Δ bounds a disk. Showing that Δ is simply connected in this sense means that a commutation loop of order n in $\tilde{X}$ can be contracted; these contractions will be shown to be disjoint from C for all sufficiently large n.

To complete the specification of homotopies, Δ is regarded as having the following arrangement. For each j positive, $\{E_{ij}: E_{ij}, E_{\bar{i}\bar{j}}\} = K_j$ is a complete graph on six vertices, and $\bigcup_j K_j$ includes all the vertices in Δ. Thus, Δ can be considered a tetrahedron with "vertices" K_j, and loops in Δ decompose into three kinds, up to a homotopy in Δ given by the previous convention:

(1) Loops in a single K_i;
(2) Loops between K_i and K_j, $i \neq j$;
(3) Loops involving K_i, K_j, K_k, i, j, k distinct.

Type (1) loops can be triangulated since K_i is a complete graph.

For Type (2), consider a loop between K_a and K_b. Since these graphs are both complete, the loop may be assumed to have the following form: $E_{1a} - E_{2a} - E_{3b} - E_{4b} - E_{1a}$; $E_{\bar{i}j}$ here indicates the positive generator $E_{\bar{i}\bar{j}}$, its inverse. Now $E_{1a} \neq E_{ba}, E_{\bar{b}a}$ since $[E_{1a}, E_{4b}] = 1$. Suppose that $E_{1a} = E_{ca}$. Then E_{3b} commutes with E_{ca}, allowing a triangulation, unless $E_{3b} = E_{cb}, E_{ab}, E_{\bar{a}b}$; but the last two violate the condition $[E_{3b}, E_{2a}] = 1$. Hence, $E_{3b} = E_{cb} = E_{1b}$. Since E_{2a} commutes with E_{ca} and E_{cb}, it follows that $E_{2a} = E_{\bar{c}a}, E_{da}, E_{\bar{d}a}$. The argument just made can be repeated to show that $E_{2a} = E_{4b}$. When $E_{2a} = E_{\bar{c}a}, E_{\bar{d}a}$ then all elements commute with E_{da}, giving a triangulation; when $E_{2a} = E_{da}$, all elements commute with $E_{\bar{d}\bar{a}}$, also giving a triangulation. Thus, when $E_{1a} = E_{ca}$, the loop can be triangulated, and the same happens if $E_{1a} = E_{\bar{c}a}$, $E_{da}, E_{\bar{d}a}$.

For Type (3), suppose the given the loop to be $E_{1a} - E_{2a} - E_{3b} - E_{4b} - E_{5c} - E_{6c} - E_{1a}$. Now, E_{da} commutes with E_{1a}, E_{2a}; $E_{\overline{db}}$ commutes with E_{3b}, E_{4b}, and E_{da}; E_{dc} commutes

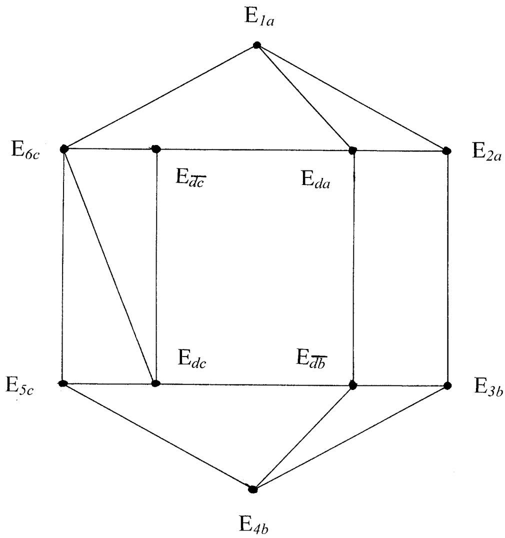

Fig. 5. Reduction of Type (3) loop.

with E_{5c}, E_{6c}; and $E_{\bar{d}\bar{c}}$ commutes with E_{6c}, E_{dc}, and E_{da}. Thus, the hexagon can be subdivided by triangles, squares of Type (2), and the loop $E_{da} - E_{\overline{db}} - E_{dc} - E_{\bar{d}\bar{c}}$. (See Fig. 5.)

Inserting E_{ac} inside this loop yields two more triangles since E_{ac} commutes with all the elements in the loop except E_{da}. The following lemma shows that a suitable ε_n exists for the loop $E_{ac} - E_{\overline{db}} - E_{da} - E_{\bar{d}\bar{c}}$. □

Definition. If E_{ij}, E_{kl} are such that $k = i$ or $k = \bar{\imath}$, then there is a *near commutation relation* between E_{ij}, E_{kl}, denoted $E_{ij} \sim E_{kl}$.

The loop $E_{ac} - E_{\overline{db}} - E_{da} - E_{\bar{d}\bar{c}}$ thus consists of two triangles involving a near commutation relation, $E_{ac} \sim E_{da} - E_{\bar{d}\bar{c}} - E_{ac}$, and $E_{ac} - E_{\overline{db}} - E_{da} \sim E_{ac}$. The first triangle will be treated in detail, since the second can be handled in a similar way.

Lemma 5.2. (1) *The path* $(E_{da}E_{dc}^{-n})^n E_{ac}^{-n}$ *in* $\tilde{X}$ *is a path from* E_{ac}^n *to* E_{da}^n *in* $\tilde{X}$ *with origin* 1, *and may hence be said to correspond to the near commutation relation* $E_{ac} \sim E_{da}$.

(2) *The loop* $(E_{da}E_{dc}^{-n})^n E_{ac}^{-n} E_{\bar{d}\bar{c}}^n E_{da}^{-n} E_{ac}^n E_{\bar{d}\bar{c}}^{-n}$ *is a loop of order n with origin* $1 \in \tilde{X}$ *corresponding to the triangle* $E_{ac} - E_{\bar{d}\bar{c}} - E_{da} \sim E_{ac}$.

Proof. (1) By relation (3) of Lemma 1.1., $E_{ac}^{-1}E_{da}E_{ac} = E_{da}E_{dc}^{-1}$. Since E_{dc} commutes with $E_{ac}, E_{ac}^{-2}E_{da}E_{ac}^2 = E_{da}E_{dc}^{-2}$. It follows that $E_{ac}^{-n}E_{da}E_{ac}^n = E_{da}E_{dc}^{-n}$ and hence that $E_{ac}^{-n}E_{da}^nE_{ac}^n = (E_{da}E_{dc}^{-n})^n$. Thus, a path in $\tilde{X}$ of order n from E_{ac}^n to E_{da}^n is $(E_{da}E_{dc}^{-n})^n E_{ac}^{-n}$.

(2) The first part of the loop is given by part (1). The remainder of the loop is $E_{\bar{d}\bar{c}}^n E_{da}^{-n} E_{ac}^n E_{\bar{d}\bar{c}}^{-n}$ See Fig. 6. □

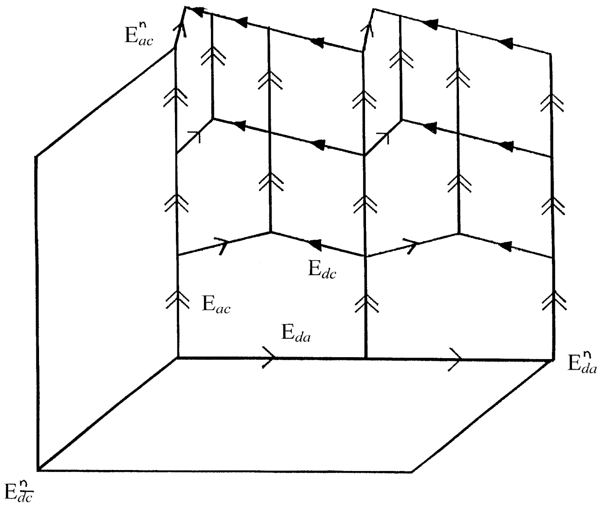

Fig. 6. Loop in $\tilde{X}$ involving near commutation relation.

Lemma 5.3. *Suppose that δ_n is a loop in $\tilde{X}$ with basepoint on A_n and origin v corresponding to the loop $E_{ac} - E_{\overline{db}} - E_{da} - E_{\bar b \bar c} - E_{ac}$ in Δ. Then for all sufficiently large n, δ_n is homotopically trivial in $\tilde{X} \backslash C$.*

Proof. Lemma 5.2(1) gives a near commutation path from vE^n_{ac} to vE^n_{da}. This divides δ_n into two triangles, as indicated before the statement of Lemma 5.2(1). By Lemma 5.2(2) the loop arising from the triangle $E_{ac} - E_{\bar d \bar c} - E_{da} \sim E_{ac}$ involves only E_{da}, E_{dc}, E_{ac}, and $E_{\bar d \bar c}$. Since $E_{\bar d \bar c}$ commutes with E_{ac}, E_{da}, and E_{dc}, ε'_n can be taken to be the disk with the boundary given in Lemma 5.2(2) and origin $E^n_{ac} E_{\bar d \bar c} E^n_{da}$. Observe that ε'_n is obtainable by filling in the "cube" spanned by $E^n_{ac}, E^n_{da}, E^n_{\bar d \bar c}$. To show that this disk is disjoint from C for all n sufficiently large, we note that every vertex p in ε'_n satisfies $p = E^r_{da} E^s_{ac} E^t_{dc} E^u_{\bar d \bar c}$, with at least one of $r, s, u = n$. By the argument from Lemma 4.4 and Lemma 2.3(3), $p \notin C$ for all sufficiently large n.

The case is essentially the same for the triangle $E_{ac} \sim E_{da} - E_{\overline{db}}$, since $E_{\overline{db}}$ commutes with E_{ac}, E_{da}, and E_{dc}, and a disk ε''_n is obtained. The disks $\varepsilon'_n, \varepsilon''_n$ attach coherently and provide a disk ε_n with $\partial \varepsilon_n = \delta_n$ that is disjoint from C for all n sufficiently large. $\square$

The disks B_n have now been specified and are disjoint from C for all sufficiently large n.

5. Conclusion

A brief word is in order about why the arguments given do not resolve the case when $n = 3$. Two key points present some difficulty. First, it is not the case that some generator commutes with all the generators in each basic relator. Thus, it is not possible to cap the slides of basic relators in the way we have done for $n = 4$. Second, the commutation complex for $Out\ F_3$ is connected, but it is not clear whether 2-cells arising from the presentation for $Out\ F_3$ could be attached in such a way as to make the commutation complex simply connected. It is natural to wonder, nevertheless, whether the commutation complex serves as a means of showing simple connectivity at infinity for any other groups.

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