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T_0 *-compactification in the hyperspace

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A *-compactification of a T_0 quasi-uniform space $(X,\mathcal{U})$ is a compact T_0 quasi-uniform space $(Y,\mathcal{V})$ that has a $\mathcal{T}(\mathcal{V}\vee\mathcal{V}^{-1})$-dense subspace quasi-isomorphic to $(X,\mathcal{U})$. In this paper we study when the hyperspace with the Hausdorff–Bourbaki quasi-uniformity is *-compactifiable and describe some of its *-compactifications.

1. Introduction

Fletcher and Lindgren began in Chapter 3 of [3] the study and construction of compactifications for (Hausdorff) quasi-uniform spaces. Further contributions, in this direction, were given in [4] and [13]. In particular, it was proved in [4] that a totally bounded T_1 quasi-uniform space has a T_1-compactification if and only if it is point symmetric, although such a compactification is not unique, in general (see [4, p. 34]).

To avoid this inconvenience, the notion of (T_1) *-compactification for a T_1 quasi-uniform space was introduced in [15], where it was shown, among other results, that each T_1 quasi-uniform space having a T_1 *-compactification has an (up to quasi-isomorphism) unique T_1 *-compactification.

Later on, a study and description of the structure of T_0 *-compactifications of a quasi-uniform space was carried out in [17], while T_1 *-compactifications on the hyperspace were studied in [16], where some characterizations of T_1 *-compactifiability of the hyperspace were given.

Since the existence of a T_1 *-compactification on the hyperspace implies symmetry conditions on the quasi-uniformity of the base space, it is natural to study the existence and description of T_0 *-compactifications on the hyperspace of those quasi-uniformities that lack those symmetry conditions.

In this paper we study and describe a T_0 *-compactification of the hyperspace, as well as its relation with the compactness of the stability space (the bicompletion of the hyperspace), which was recently introduced and discussed by the authors in [11].

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2. Basic notions and preliminary results

Our basic references for quasi-uniform spaces are [3] and [6]. Terms and undefined concepts may be found in [3].

Let us recall that if $\mathcal{U}$ is a quasi-uniformity on a set X, then $\mathcal{U}^{-1} = \{U^{-1}\colon U \in \mathcal{U}\}$ is also a quasi-uniformity on X called *the conjugate* of $\mathcal{U}$. The uniformity $\mathcal{U} \vee \mathcal{U}^{-1}$ will be denoted by $\mathcal{U}^*$. If $U \in \mathcal{U}$, the entourage $U \cap U^{-1}$ of $\mathcal{U}^*$ will be denoted by U^*.

Each quasi-uniformity $\mathcal{U}$ on X induces a topology $\mathcal{T}(\mathcal{U})$ on X, defined as follows: $\mathcal{T}(\mathcal{U}) = \{A \subseteq X\colon$ for each $x \in A$ there is $U \in \mathcal{U}$ such that $U(x) \subseteq A\}$.

A quasi-uniform space $(X, \mathcal{U})$ is said to be *point symmetric* if $\mathcal{T}(\mathcal{U}) \subseteq \mathcal{T}(\mathcal{U}^{-1})$.

A quasi-uniform space $(X, \mathcal{U})$ is called *precompact* [3, 3.13] if for each $U \in \mathcal{U}$ there is a finite subset A of X such that $U(A) = X$. $(X, \mathcal{U})$ is said to be *hereditarily precompact* if any subspace of $(X, \mathcal{U})$ is precompact, and it is *totally bounded* provided that $\mathcal{U}^*$ is a totally bounded uniformity on X.

A filter $\mathcal{F}$ on a quasi-uniform space $(X, \mathcal{U})$ is called *stable* [2] if for each $U \in \mathcal{U}$, $\bigcap_{F \in \mathcal{F}} U(F) \in \mathcal{F}$, and $\mathcal{F}$ is called *doubly stable* if it is stable both for $(X, \mathcal{U})$ and $(X, \mathcal{U}^{-1})$.

A filter $\mathcal{F}$ on a quasi-uniform space $(X, \mathcal{U})$ is *left K-Cauchy* provided that for each $U \in \mathcal{U}$ there exists $F \in \mathcal{F}$ such that $U(x) \in \mathcal{F}$ for all $x \in F$, and $(X, \mathcal{U})$ is called *left K-complete* if every left K-Cauchy filter converges with respect to $\mathcal{T}(\mathcal{U})$ [5,14].

It is well known [14, Propositions 1 and 2] that an ultrafilter on a quasi-uniform space $(X, \mathcal{U})$ is left K-Cauchy if and only if it is stable on $(X, \mathcal{U}^{-1})$, and that $(X, \mathcal{U})$ is left K-complete if and only if every stable ultrafilter on $(X, \mathcal{U}^{-1})$ converges with respect to $\mathcal{T}(\mathcal{U})$. Therefore, we have the following characterization which will be used later on: A quasi-uniform space $(X, \mathcal{U})$ is left K-complete if and only if every left K-Cauchy ultrafilter converges with respect to $\mathcal{T}(\mathcal{U})$.

A quasi-uniform space $(X, \mathcal{U})$ is called *bicomplete* if each $\mathcal{U}^*$-Cauchy filter converges with respect to the topology $\mathcal{T}(\mathcal{U}^*)$, i.e., if the uniform space $(X, \mathcal{U}^*)$ is complete [3, 3.28].

A *bicompletion* of a quasi-uniform space $(X, \mathcal{U})$ is a bicomplete quasi-uniform space $(Y, \mathcal{V})$ that has a $\mathcal{T}(\mathcal{V}^*)$-dense subspace quasi-isomorphic to $(X, \mathcal{U})$; more formally, there is a quasi-uniform embedding $i : (X, \mathcal{U}) \to (Y, \mathcal{V})$.

Each T_0 quasi-uniform space $(X, \mathcal{U})$ has an (up to quasi-isomorphism) unique T_0 bicompletion, which will be denoted by $(\widetilde{X}, \widetilde{\mathcal{U}})$ and will be called the bicompletion of $(X, \mathcal{U})$. The construction and uniqueness up to quasi-isomorphism of $(\widetilde{X}, \widetilde{\mathcal{U}})$ is described in detail in Section 3.2 of [3]. For our purposes here it suffices to recall that $\widetilde{X}$ consists of all minimal $\mathcal{U}^*$-Cauchy filters on X, and that the family $\{\widetilde{U}\colon U \in \mathcal{U}\}$ is a base for $\widetilde{\mathcal{U}}$, where for each $U \in \mathcal{U}$, $\widetilde{U} = \{(\mathcal{F}, \mathcal{G}) \in \widetilde{X} \times \widetilde{X}\colon$ there are $F \in \mathcal{F}$ and $G \in \mathcal{G}$ with $F \times G \subseteq U\}$, and the quasi-uniform embedding $i : (X, \mathcal{U}) \to (\widetilde{X}, \widetilde{\mathcal{U}})$ is given as follows: for $x \in X$, we have that $i(x)$ is the $\mathcal{T}(\mathcal{U}^*)$-neighborhood filter of x.

In the sequel, the restriction of $\widetilde{\mathcal{U}}$ to any subset of $\widetilde{X} \times \widetilde{X}$ will be also denoted by $\widetilde{\mathcal{U}}$, if no confusion arises.

Following [17], a T_0 **-compactification* of a T_0 quasi-uniform space $(X, \mathcal{U})$ is a compact T_0 quasi-uniform space $(Y, \mathcal{V})$ that has a $\mathcal{T}(\mathcal{V}^*)$-dense subspace quasi-isomorphic to $(X, \mathcal{U})$. If $\mathcal{T}(\mathcal{V})$ is a T_1 topology on Y, we say that $(Y, \mathcal{V})$ is a T_1 **-compactification* of $(X, \mathcal{U})$.

In [15, Corollary of Theorem 1] it was proved that if a T_1 quasi-uniform space $(X, \mathcal{U})$ has a T_1 *-compactification, then any T_1 *-compactification of $(X, \mathcal{U})$ is quasi-isomorphic to $(G(X), \widetilde{\mathcal{U}})$, where $G(X)$ denotes the set of closed points in $(\widetilde{X}, \mathcal{T}(\widetilde{\mathcal{U}}))$, while in [15, Theorem 6] it was proved that a T_1 quasi-uniform space is T_1 *-compactifiable if and only if it is point symmetric and its bicompletion is compact. The latter result was extended to T_0 quasi-uniform spaces in [17, Theorem 1] as follows: a T_0 quasi-uniform space is T_0 *-compactifiable if and only if its bicompletion is compact.

3. T_0 *-compactification in the hyperspace

Given a quasi-uniform space $(X, \mathcal{U})$, we denote by $\mathcal{P}_0(X)$ the collection of all nonempty subsets of X. Then, *the Hausdorff–Bourbaki quasi-uniformity* of $(X, \mathcal{U})$ [1,9,12] is the quasi-uniformity $\mathcal{U}_H$ on $\mathcal{P}_0(X)$ which has as a base the family of sets of the form

$$U_H = \{(A, B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X)\colon B \subseteq U(A),\ A \subseteq U^{-1}(B)\},$$

where $U \in \mathcal{U}$.

If $(X, \mathcal{U})$ is a T_0 quasi-uniform space, then $(\mathcal{P}_0(X), \mathcal{U}_H)$ is not necessarily T_0 [9]. This fact, suggests to work on the set $\mathcal{C}_\cap(X) = \{A^!\colon A \in \mathcal{P}_0(X)\}$ where $A^! = Cl_{\mathcal{U}}(A) \cap Cl_{\mathcal{U}^{-1}}(A)$.

Indeed, it is clear that $(\mathcal{C}_\cap(X), \mathcal{U}_H)$ is a T_0 quasi-uniform space whenever $(X, \mathcal{U})$ is a T_0 quasi-uniform space.

In our first result we establish two simple but useful facts, whose easy proofs are omitted.

Proposition 1. *Let $(X, \mathcal{U})$ be a quasi-uniform space and let $A, B \in \mathcal{P}_0(X)$ and $U \in \mathcal{U}$.*

1. *If $B \in U_H(A)$, then $B, B^! \in (U^2)_H(A^!)$ and $B^! \in (U^2)_H(A)$.*
2. *If $B^! \in U_H(A^!)$, then $B^!, B \in (U^2)_H(A)$ and $B \in (U^2)_H(A^!)$.*

Proposition 2. *Let $(X, \mathcal{U})$ be a quasi-uniform space. Then $(\mathcal{C}_\cap(X), \mathcal{U}_H)$ is compact if and only if $(\mathcal{P}_0(X), \mathcal{U}_H)$ is compact.*

Proof. Suppose that $(\mathcal{C}_\cap(X), \mathcal{U}_H)$ is compact, and let (F_λ) be a net in $\mathcal{P}_0(X)$. Then $(F_\lambda^!)$ is a net in $\mathcal{C}_\cap(X)$, so it has a cluster point $C^! \in \mathcal{C}_\cap(X)$. By Proposition 1, it easily follows that C (and $C^!$) is a cluster point of (F_λ).

Conversely, suppose that $(\mathcal{P}_0(X), \mathcal{U}_H)$ is compact, and let $(F_\lambda^!)$ be a net in $\mathcal{C}_\cap(X)$. Then $(F_\lambda^!)$ is a net in $\mathcal{P}_0(X)$, so it has a cluster point $C \in \mathcal{P}_0(X)$. It easily follows from Proposition 1 that $C^!$ is a cluster point of $(F_\lambda^!)$. □

Remark 1. Let $(X, \mathcal{U})$ be a quasi-uniform space. In [10] it is proved that $(\mathcal{P}_0(X), \mathcal{U}_H)$ is compact if and only if $(X, \mathcal{U})$ is compact and $(X_m, \mathcal{U}^{-1})$ is hereditarily precompact, where $X_m = \{y \in X\colon\ y$ is a minimal element in the (specialization) pre-order of the space $(X, \mathcal{U})\}$. Note that X_m is the set of closed points in $(X, \mathcal{T}(\mathcal{U}))$ if $(X, \mathcal{U})$ is T_0.

The next proposition is the first step for the description of the T_0 *-compactification of the hyperspace as the hyperspace of the bicompletion.

Proposition 3. *Let $(X, \mathcal{U})$ be a T_0 quasi-uniform space, and let the map $\phi : (\mathcal{C}_\cap(X), \mathcal{U}_H) \to (\mathcal{C}_\cap(\widetilde{X}), \widetilde{\mathcal{U}}_H)$ be defined by $\phi(A^!) = Cl_{\widetilde{\mathcal{U}}}(A) \cap Cl_{\widetilde{\mathcal{U}}^{-1}}(A)$. Then ϕ is a quasi-isomorphism from $(\mathcal{C}_\cap(X), \mathcal{U}_H)$ onto $(\phi(\mathcal{C}_\cap(X)), \widetilde{\mathcal{U}}_H)$. Furthermore, $\phi(\mathcal{C}_\cap(X))$ is dense in $(\mathcal{C}_\cap(\widetilde{X}), \mathcal{T}((\widetilde{\mathcal{U}}_H)^*))$.*

Proof. Since $\phi(A^!) \cap X = A^!$ it follows that ϕ is injective. We deduce from Proposition 1 that ϕ and ϕ^{-1} are quasi-uniformly continuous.

In order to prove that $\phi(\mathcal{C}_\cap(X))$ is dense in $(\mathcal{C}_\cap(\widetilde{X}), \mathcal{T}((\widetilde{\mathcal{U}}_H)^*))$, let $A \subseteq \widetilde{X}$ and $U \in \mathcal{U}$. Let $A^b = Cl_{\widetilde{\mathcal{U}}}(A) \cap Cl_{\widetilde{\mathcal{U}}^{-1}}(A)$. For each $a \in A$, take $b_a \in X$ such that $a \in \widetilde{U}^*(b_a)$, and let $B = \{b_a\colon\ a \in A\}$. Then it is clear that $A \subseteq \widetilde{U}^*(B)$ and $B \subseteq \widetilde{U}^*(A)$, and hence $A \in (\widetilde{U}_H)^*(B)$. A straightforward computation shows $A^b \in (\widetilde{U}_H)^*(B^!)$, which completes the proof. □

The next result characterizes when the hyperspace of the bicompletion is a T_0 *-compactification of the hyperspace.

Theorem 1. *Let $(X, \mathcal{U})$ be a T_0 quasi-uniform space. Then $(\mathcal{C}_\cap(\widetilde{X}), \widetilde{\mathcal{U}}_H)$ is a T_0 *-compactification of $(\mathcal{C}_\cap(X), \mathcal{U}_H)$ if and only if $(X, \mathcal{U})$ is T_0 *-compactifiable and $(G(X), \widetilde{\mathcal{U}}^{-1})$ is hereditarily precompact.*

Proof. Suppose that $(\mathcal{C}_\cap(\widetilde{X}), \widetilde{\mathcal{U}}_H)$ is a T_0 *-compactification of $(\mathcal{C}_\cap(X), \mathcal{U}_H)$. Then $(\mathcal{C}_\cap(\widetilde{X}), \widetilde{\mathcal{U}}_H)$ is compact and by Remark 1 and Proposition 3 $(\widetilde{X}, \widetilde{\mathcal{U}})$ is compact and $(G(X), \widetilde{\mathcal{U}}^{-1})$ is hereditarily precompact.

Conversely, if $(\widetilde{X}, \widetilde{\mathcal{U}})$ is compact and $(G(X), \widetilde{\mathcal{U}}^{-1})$ is hereditarily precompact, then $(\mathcal{C}_\cap(\widetilde{X}), \widetilde{\mathcal{U}}_H)$ is compact, and by Proposition 3, it is a T_0 *-compactification of $(\mathcal{C}_\cap(X), \mathcal{U}_H)$. □

Lemma 1. *Let $(X, \mathcal{U})$ be a T_0 quasi-uniform space such that $(X, \mathcal{U}^{-1})$ is hereditarily precompact. Then $(X, \mathcal{U})$ is T_0 *-compactifiable if and only if it is precompact.*

Proof. Suppose that $(X, \mathcal{U})$ is T_0 *-compactifiable. By [17, Theorem 1], $(\widetilde{X}, \widetilde{\mathcal{U}})$ is compact, so it is precompact, and by [7, Proposition 4], $(X, \mathcal{U})$ is precompact.

Conversely, by [7, Proposition 4] we have that $(\widetilde{X}, \widetilde{\mathcal{U}})$ is precompact and $(\widetilde{X}, \widetilde{\mathcal{U}}^{-1})$ is hereditarily precompact. Now we observe that $(\widetilde{X}, \widetilde{\mathcal{U}})$ is left K-complete: Indeed, let $\mathcal{F}$ be a left K-Cauchy ultrafilter on $(\widetilde{X}, \widetilde{\mathcal{U}})$; since $(\widetilde{X}, \widetilde{\mathcal{U}}^{-1})$ is hereditarily precompact, it follows that $\mathcal{F}$ is also left K-Cauchy on $(\widetilde{X}, \widetilde{\mathcal{U}}^{-1})$ [14, Corollary 1.1], and hence it is a Cauchy ultrafilter on $(\widetilde{X}, (\widetilde{\mathcal{U}})^*)$, so it converges with respect to $\mathcal{T}(\widetilde{\mathcal{U}})^*$; therefore $(\widetilde{X}, \widetilde{\mathcal{U}})$ is left K-complete. Then, the conclusion follows from the fact that every precompact left K-complete quasi-uniform space is compact [5, Proposition 13]. □

Corollary 1. *Let $(X, \mathcal{U})$ be a precompact T_0 quasi-uniform space such that $(X, \mathcal{U}^{-1})$ is hereditarily precompact. Then $(\mathcal{C}_\cap(X), \mathcal{U}_H)$ is T_0 *-compactifiable and $(\mathcal{C}_\cap(\widetilde{X}), \widetilde{\mathcal{U}}_H)$ is a T_0 *-compactification of $(\mathcal{C}_\cap(X), \mathcal{U}_H)$.*

Proof. By Lemma 1, $(X, \mathcal{U})$ is T_0 *-compactifiable. Moreover, by [7, Proposition 4], $(\widetilde{X}, \widetilde{\mathcal{U}}^{-1})$ is hereditarily precompact, and hence so is $(G(X), \widetilde{\mathcal{U}}^{-1})$. Now, the conclusion follows from Theorem 1. □

The stability space $(S_D(X), \mathcal{U}_D)$ of a quasi-uniform space $(X, \mathcal{U})$ was introduced in [11] to describe the bicompletion of the hyperspace. A generalization of the stability space is the scale of a quasi-uniform space (see [8]), which can also be used to construct the bicompletion of the T_0-reflection of the Hausdorff–Bourbaki quasi-uniformity of a quasi-uniform space.

Next we establish and recall some results which will be used later on.

Remark 2. The following fact is an easy consequence of [11, Theorem 1]: given a T_0 quasi-uniform space $(X, \mathcal{U})$, then $(\mathcal{C}_\cap(X), \mathcal{U}_H)$ is T_0 *-compactifiable if and only if the stability space $(S_D(X), \mathcal{U}_D)$ is compact.

Remark 3. ([11, Propositions 7 and 8]) Given a T_0 quasi-uniform space $(X,\mathcal{U})$, then $(S_D(X),\mathcal{U}_D)$ is precompact if and only if $(X,\mathcal{U})$ is precompact, and $(S_D(X),\mathcal{U}_D)$ is totally bounded if and only if $(X,\mathcal{U})$ is totally bounded. It follows that if $\mathcal{U}$ is a uniformity then $(S_D(X),\mathcal{U}_D)$ is compact if and only if $(X,\mathcal{U})$ is totally bounded.

In the sequel we find necessary conditions for the compactness of the stability space.

The proof of the following proposition is based on [16, Proposition 2.7].

Proposition 4. *Let $(X,\mathcal{U})$ be a T_0 quasi-uniform space and let $A \subseteq X$ be such that for each $a \in A$ and each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ with $V^{-1}(a) \subseteq U(a)$. If $(S_D(X),\mathcal{U}_D)$ is compact then $(A,\mathcal{U}^{-1})$ is hereditarily precompact.*

Proof. Suppose that $(A,\mathcal{U}^{-1})$ is not hereditarily precompact. Then there exist $B \subseteq A$, $U_0 \in \mathcal{U}$ and a sequence $(b_n)_{n\in\mathbb{N}}$ in B such that $b_{n+1} \notin \bigcup_{i=1}^{n} U_0^{-1}(b_i)$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, put $B_n = \{b_i\colon\ i \leqslant n\}$.

Since $(S_D(X),\mathcal{U}_D)$ is compact, there exists a doubly stable filter $\mathcal{F} \in S_D(X)$ such that $(B_n)_{n\in\mathbb{N}}$ clusters to $\mathcal{F}$.

Let $U \in \mathcal{U}$ with $U^2 \subseteq U_0$. Let $k \in \mathbb{N}$ be such that $B_k \in U_D(\mathcal{F})$. Then $B_k \subseteq U(F)$ for each $F \in \mathcal{F}$ and $U^{-1}(B_k) \in \mathcal{F}$.

Let $V \in \mathcal{U}$ with $V^{-1}(b_{k+1}) \subseteq U(b_{k+1})$, and let $n \geqslant k+1$ with $B_n \in V_D(\mathcal{F})$. Then $B_n \subseteq V(F)$ for each $F \in \mathcal{F}$ and $V^{-1}(B_k) \in \mathcal{F}$. It follows that $B_n \subseteq VU^{-1}(B_k)$. In particular $b_{k+1} \in VU^{-1}(B_k)$ and hence $V^{-1}(b_{k+1}) \cap U^{-1}(B_k) \neq \emptyset$. Then $U(b_{k+1}) \cap U^{-1}(B_k) \neq \emptyset$ and $b_{k+1} \in U^{-2}(B_k) \subseteq U_0^{-1}(B_k)$, a contradiction. □

Corollary 2. *Let $(X,\mathcal{U})$ be a point symmetric T_0 quasi-uniform space with $(S_D(X),\mathcal{U}_D)$ compact. Then $(X,\mathcal{U}^{-1})$ is hereditarily precompact.*

Corollary 3. *Let $(X,\mathcal{U})$ be a T_0 *-compactifiable quasi-uniform space with $(S_D(X),\mathcal{U}_D)$ compact. Then $(G(X),\widetilde{\mathcal{U}})$ is point symmetric and $(G(X),\widetilde{\mathcal{U}}^{-1})$ is hereditarily precompact.*

Proof. If $(X,\mathcal{U})$ is T_0 *-compactifiable, then $(\widetilde{X},\widetilde{\mathcal{U}})$ is compact. Take a $\mathcal{U}^*$-Cauchy filter $\mathcal{F} \in G(X)$ and let $U_0 \in \mathcal{U}$. Suppose that $\widetilde{V}^{-1}(\mathcal{F}) \nsubseteq \widetilde{U}_0(\mathcal{F})$ for each $V \in \mathcal{U}$, and let $x_V \in \widetilde{V}^{-1}(\mathcal{F}) \setminus \widetilde{U}_0(\mathcal{F})$. Since $(\widetilde{X},\widetilde{\mathcal{U}})$ is compact, the net (x_V) clusters to some $\mathcal{G} \in \widetilde{X}$. It is easy to prove that $\mathcal{G} \in \widetilde{U}^{-1}(\mathcal{F})$ for each $U \in \mathcal{U}$, and hence $\mathcal{F} = \mathcal{G}$, but $x_V \notin \widetilde{U}_0(\mathcal{F})$, a contradiction. It follows that $(G(X),\widetilde{\mathcal{U}})$ is point symmetric and by Corollary 2 and [7, Proposition 4], $(G(X),\widetilde{\mathcal{U}}^{-1})$ is hereditarily precompact. □

Corollary 4. *Let $(X,\mathcal{U})$ be a T_0 *-compactifiable quasi-uniform space. Then $(\mathcal{C}_\cap(X),\mathcal{U}_H)$ is T_0 *-compactifiable if and only if $(G(X),\widetilde{\mathcal{U}}^{-1})$ is hereditarily precompact. Furthermore, in this case $(\mathcal{C}_\cap(\widetilde{X}),\widetilde{\mathcal{U}}_H)$ is a T_0 *-compactification of $(\mathcal{C}_\cap(X),\mathcal{U}_H)$.*

Finally, for point symmetric quasi-uniform spaces, we give a characterization of the compactness of the stability space, which is equivalent to the existence of a T_0 *-compactification of the hyperspace.

Corollary 5. *Let $(X,\mathcal{U})$ be a point symmetric T_0 quasi-uniform space. The following are equivalent*:

(1) *$(S_D(X),\mathcal{U}_D)$ is compact.*
(2) *$(X,\mathcal{U})$ is T_0 *-compactifiable and $(X,\mathcal{U}^{-1})$ is hereditarily precompact.*
(3) *$(X,\mathcal{U})$ is T_0 *-compactifiable and $(G(X),\widetilde{\mathcal{U}}^{-1})$ is hereditarily precompact.*
(4) *$(\mathcal{C}_\cap(\widetilde{X}),\widetilde{\mathcal{U}}_H)$ is compact.*
(5) *$(\mathcal{C}_\cap(X),\mathcal{U}_H)$ is T_0 *-compactifiable.*
(6) *$(\mathcal{C}_\cap(\widetilde{X}),\widetilde{\mathcal{U}}_H)$ is a T_0 *-compactification of $(\mathcal{C}_\cap(X),\mathcal{U}_H)$.*

Proof. First note that every point symmetric T_0 quasi-uniform space is T_1.

(1) implies (2). If $(S_D(X),\mathcal{U}_D)$ is compact, then it is precompact and hence $(X,\mathcal{U})$ is precompact by Remark 3. By Corollary 2, $(X,\mathcal{U}^{-1})$ is hereditarily precompact. By [16, Lemma 2.2], $(X,\mathcal{U})$ is T_1 *-compactifiable.

(2) implies (3). Since $(X,\mathcal{U})$ is point symmetric and T_0 *-compactifiable, then it is T_1 *-compactifiable. By [16, Lemma 2.2], $(X,\mathcal{U})$ is precompact. By Corollary 1 and Remark 2, $(S_D(X),\mathcal{U}_D)$ is compact.

On the other hand, since $(X,\mathcal{U})$ is T_1 *-compactifiable, then $(G(X),\widetilde{\mathcal{U}})$ is compact and T_1 and hence point symmetric, so $(G(X),\widetilde{\mathcal{U}}^{-1})$ is hereditarily precompact by Corollary 3.

(3) implies (1). By Corollary 4 and Remark 2.

(3) implies (4). By Theorem 1.

(4) implies (3). By Proposition 2 and Remark 1 (note that $(\widetilde{X})_m = G(X)$).

(1) equivalent to (5). By Remark 2.

(3) implies (6). By Corollary 4.

(6) implies (5). Obvious. □

The next result is a characterization of the compactness of the stability space, similar to the characterization of the compactness of the hyperspace (see Remark 1).

Corollary 6. *Let $(X,\mathcal{U})$ be a compact T_0 quasi-uniform space. Then the stability space $(S_D(X),\mathcal{U}_D)$ is compact if and only if $(X_m,\mathcal{U}^{-1})$ is hereditarily precompact.*

Proof. It is a consequence of Corollary 3 on the one hand, and Theorem 1 and Remark 2 on the other hand. □

Example 1. Let X be the set of non-negative integers and let $\mathcal{U}$ be the quasi-uniformity generated by the usual order $\leqslant$ on X. Clearly $(X,\mathcal{U})$ is precompact since $X = U(0)$ for $U = \leqslant$. Since $(X,\mathcal{U})$ is bicomplete, its set of closed points is $G(X) = \{0\}$ and hence $(G(X),\widetilde{\mathcal{U}}^{-1})$ is hereditarily precompact. Since $(X,\mathcal{U})$ is compact, it is T_0 *-compactifiable and then $(\mathcal{C}_\cap(X),\mathcal{U}_H)$ is T_0 *-compactifiable by Corollary 5. Note also that $(X,\mathcal{U}^{-1})$ is not precompact and hence it is not hereditarily precompact. Therefore, the fact that $(\mathcal{C}_\cap(X),\mathcal{U}_H)$ is T_0 *-compactifiable does not imply that $(X,\mathcal{U}^{-1})$ is hereditarily precompact.

The following open question arises in a natural way in the light of the above example.

Question 1. If $(\mathcal{C}_\cap(X),\mathcal{U}_H)$ is T_0 *-compactifiable, is either $(X_m,\mathcal{U}^{-1})$ or $(G(X),\widetilde{\mathcal{U}}^{-1})$ hereditarily precompact?

Concerning Question 1, note that if the conjecture "$(\mathcal{C}_\cap(X),\mathcal{U}_H)$ T_0 *-compactifiable implies $(X_m,\mathcal{U}^{-1})$ hereditarily precompact" is true, then it is also true "$(\mathcal{C}_\cap(X),\mathcal{U}_H)$ T_0 *-compactifiable implies $(G(X),\widetilde{\mathcal{U}}^{-1})$ hereditarily precompact".

Indeed, if $(\mathcal{C}_\cap(X),\mathcal{U}_H)$ is T_0 *-compactifiable then its bicompletion is compact, so by Proposition 3, the bicompletion of $(\mathcal{C}_\cap(\widetilde{X}),\widetilde{X}_H)$ is also compact and hence $(\mathcal{C}_\cap(\widetilde{X}),\widetilde{\mathcal{U}}_H)$ is T_0 *-compactifiable. By applying the conjecture to $(\mathcal{C}_\cap(\widetilde{X}),\widetilde{\mathcal{U}}_H)$ it follows that $((\widetilde{X})_m,\widetilde{\mathcal{U}}^{-1})$ is hereditarily precompact. Since $(\widetilde{X})_m = G(X)$, we conclude that $(G(X),\widetilde{\mathcal{U}}^{-1})$ is hereditarily precompact.

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