The Oscillation of Higher-Order Nonlinear Difference Equations

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Abstract—The authors discuss the relation of the oscillation of the following two difference equations,

\[ \Delta^m x_{n-1} + f(n, x_n) = 0, \]
\[ \Delta^m x_{n-1} + f(n, x_{\tau(n)}) = 0, \]

where \( m \geq 2 \), \( \tau : \mathbb{N} \to \mathbb{N} \), \( \mathbb{N} \) is the set of integers, \( |n - \tau(n)| \leq M \) for \( n \in \mathbb{N}_0 \), \( M \) is a positive integer, \( \lim_{n \to \infty} \tau(n) = \infty \), \( f : \mathbb{N} \times \mathbb{R} \to \mathbb{R} \), \( f \) is nondecreasing in \( x \) and \( xf(n, x) > 0 \), as \( x \to 0 \). We will show some relations of the oscillation of the above two equations. Especially, for \( m \) to be even, we establish the equivalence of the oscillation of the above two difference equations. © 2004 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

We consider the higher-order nonlinear difference equation

\[ \Delta^m x_{n-1} + f(n, x_{\tau(n)}) = 0, \quad (1.1) \]

where we assume the following, Condition (H), holds.

(H) \( m \geq 2 \), \( \tau : \mathbb{N} \to \mathbb{N} \), \( \mathbb{N} \) is the set of integers, \( \lim_{n \to \infty} \tau(n) = \infty \), and \( |n - \tau(n)| \leq M \), for \( n \in \mathbb{N}_0 = \{0, 1, 2, \cdots \} \), \( M \) is a positive integer; \( f : \mathbb{N} \times \mathbb{R} \to \mathbb{R} \), \( f \) is nondecreasing in \( x \) and \( xf(n, x) > 0 \), as \( x \neq 0 \).

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A special case of (1.1) is
\[ \Delta^m x_{n-1} + f(n, x_n) = 0. \] (1.2)

A solution \( \{x_n\} \) of (1.1) is said to be eventually positive, if \( x_n > 0 \), for all large \( n \), and eventually negative, if \( x_n < 0 \), for all large \( n \). It is said to be oscillatory, if it is neither eventually positive nor eventually negative. An equation is said to be oscillatory, if every solution of this equation is oscillatory.

We will establish some comparison theorems for the oscillation of (1.1) and (1.2). Especially, for \( m \) to be even, we will prove that every solution of (1.1) oscillates, if and only if, every solution of (1.2) oscillates.

2. RESULTS

First, we need the following lemma.

**Lemma 2.1.** Assume that (H) holds. If
\[ \Delta x_{n-1} - f(n, x_{\tau(n)}) \geq 0 \] (2.1)
has a positive solution on \( n \geq N_1 > 0 \), so does the equation,
\[ \Delta x_{n-1} - f(n, x_{\tau(n)}) = 0. \] (2.2)

**Proof.** Let \( \{x_n\} \) be a positive solution of (2.1) on \( n \geq N_1 \). Choose \( N_2 > N_1 \), so that \( \tau(n) \geq N_1 \), for \( n \geq N_2 \). Summing (2.1) from \( N_2 \) to \( n \), we obtain
\[ x_n \geq x_{N_2-1} + \sum_{i=N_2}^{n} f(i, x_{\tau(i)}). \] (2.3)

Let \( y^{(1)}_n = x_n \), for \( n \geq N_1 \). Define
\[ y^{(k)}_n = \begin{cases} x_{N_2-1} + \sum_{i=N_2}^{n} f(i, y^{(k-1)}_{\tau(i)}), & n \geq N_2, \\ x_n, & N_1 \leq n < N_2, \end{cases} \] (2.4)
and \( k = 2, 3, \ldots \).

In view of (2.3), we have
\[ x_n = y^{(1)}_n \geq y^{(2)}_n \geq \cdots \geq x_{N_2-1}, \quad \text{for } n \geq N_2. \]

Hence, \( \lim_{k \to \infty} y^{(k)}_n = y_n \) exists, \( n \geq N_2 \). From (2.4), we have
\[ y_n = x_{N_2-1} + \sum_{i=N_2}^{n} f(i, y_{\tau(i)}), \]
which implies that \( \{y_n\} \) is a positive solution of (2.2). The proof is complete.

We consider the equation,
\[ \Delta^m x_{n-1} + f^*(n, x_{\tau(n)}) = 0, \] (2.5)
where
\[ f^*(n, x_{\tau(n)}) = \begin{cases} f(n, x_{\tau(n)}), & \text{if } x_{\tau(n)} \leq 0, \\ -f(n, -x_{\tau(n)}), & \text{if } x_{\tau(n)} > 0. \end{cases} \]

Clearly, \( f^*(n, -x_{\tau(n)}) = -f^*(n, x_{\tau(n)}) \) and \( x_{\tau(n)} f^*(n, x_{\tau(n)}) > 0 \), if \( x_{\tau(n)} \neq 0 \). Also, if \( \{x_n\} \) is a solution of (2.5), so is \( \{-x_n\} \); furthermore, \( y_n < 0, n \geq N_1 \) is a solution of (2.5), if and only if, \( \{y_n\} \) is a solution of (1.1).
Theorem 2.1. Assume $\tau(n)$ is strictly increasing and $\tau(n) \leq n$, $n \geq N_1$. If, for $m$ even, (1.1) has a nonoscillatory solution, then, the equation,

$$\Delta^m y_{s-1} + f(\tau^{-1}(s), y_s) = 0,$$

has a nonoscillatory solution. If, for $m$ odd, (1.1) has an unbounded nonoscillatory solution, so does equation (2.6).

Proof. Let $\{x_n\}$ be a nonoscillatory solution of (1.1). First of all, we assume that $x_n > 0$, for $n \geq N_1 \geq 0$. Then, $x_n \Delta^m x_n < 0$, $n \geq N_2$. By a known result [1, Theorem 1.8.11], there exist $N_3 \geq N_1$ and an integer $l$, $0 \leq l \leq m - 1$, $m + l$ is odd, such that $\Delta^k x_n > 0$, $k = 0, 1, \ldots, l$; $(-1)^{m+k-1}\Delta^k x_n > 0$, $k = l + 1, \ldots, m - 1, n \geq N_3$. Thus, if $m$ is even, or $m$ is odd and $\{x_n\}$ is unbounded, then, $l \geq 1$. Summing (1.1) from $s_1$ to $s_2 - 1$, $s_2 > s_1 \geq N_3$, we obtain

$$\Delta^m x_{s_2} - \Delta^m x_{s_1} + \sum_{i=s_1}^{s_2-1} f(i, x_{\tau(i)}) = 0,$$

and, hence,

$$\Delta^m x_{s_1} \geq \sum_{i=s_1}^{s_2-1} f(i, x_{\tau(i)}), \quad s_1 \geq N_3. \quad (2.7)$$

Let $v = \tau(i)$, $i = \tau^{-1}(v)$, then,

$$\Delta^m x_{s_1} \geq \sum_{v=\tau(s_1)}^{ \infty} f(\tau^{-1}(v), x_v).$$

Since $\tau(s_1) \leq s_1$, we have

$$\Delta^m x_{s_1} \geq \sum_{v=s}^{ \infty} f(\tau^{-1}(v), x_v). \quad (2.8)$$

Define

$$F_i x_n = \sum_{v=n}^{ \infty} f(\tau^{-1}(v), x_v)$$

and

$$F_j x_n = \sum_{i=n}^{ \infty} F_{j-1} x_i, \quad j = 2, 3, \ldots, m - l.$$ 

Then, it follows from (2.8) that the operators $F_i$, $i = 1, 2, \ldots, m - l$ are well defined and that $F_j x_n \geq F_j y_n$, $j = 1, 2, \ldots, m - l$, whenever $x_n \geq y_n$, $n \geq N_3$. Furthermore, $F_j x_n > 0$ and

$$\Delta F_j x_n = -F_{j-1} x_n, \quad n \geq N_3, \quad j = 2, 3, \ldots, m - l. \quad (2.9)$$

Thus, (2.8) becomes

$$\Delta^m x_{s-1} \geq F_1 x_s, \quad s \geq N_3.$$ 

By successive summations of the above inequality from $s_1$ to $s_2$, $s_2 \geq s_1 \geq N_3$, discarding positive terms, we obtain

$$(-1)^{i+1} \Delta^{m-i} x_{s-1} \geq F_i x_s, \quad s \geq N_3, \quad i = 1, 2, \ldots, m - l.$$ 

In particular,

$$\Delta^l x_{s-1} \geq F_{m-l} x_s, \quad s \geq N_3. \quad (2.10)$$

Define

$$T_0 x_n = F_{m-l} x_n$$
and
\[ T_j x_n = \sum_{i=0}^{n-1} T_{j-1} x_i, \quad n \geq N_3, \quad j = 1, 2, \ldots, l. \]

Then, \( T_j x_n \geq T_j y_n \), \( j = 1, 2, \ldots, l \), whenever \( x_n \geq y_n, n \geq N_3 \). Furthermore, \( T_j x_n > 0 \), and
\[ \Delta T_j x_n = T_{j-1} x_n, \quad n \geq N_3, \quad j = 1, 2, \ldots, l. \] (2.11)

By successive summations of (2.10), we obtain
\[ \Delta^{i-1} x_{s-1} \geq T_i x_s, \quad s \geq N_3, \quad i = 0, 1, \ldots, l - 1. \]

In particular,
\[ \Delta x_{s-1} \geq T_{l-1} x_s, \quad s \geq N_3. \]

By Lemma 2.1, equation \( \Delta x_{s-1} = T_{l-1} x_s \) has a positive solution \( \{y_n\} \) and \( \Delta y_n > 0 \), for \( n \geq N_3 \).

By successive differences of equation \( \Delta y_{s-1} = T_{l-1} y_s \) and noting (2.11), we have
\[ \Delta^{t} y_{s-1} = T_0 y_s = F_{m-t} y_s. \] (2.12)

Furthermore, noting (2.9), from (2.12), we obtain
\[ \Delta^{m-1} y_{s-1} = F_1 y_s = \sum_{i=s}^{\infty} f \left( \tau^{-1}(v), y_v \right). \]

Hence, \( \{y_n\} \) satisfies (2.6), i.e.,
\[ \Delta^{m} y_{s-1} = -f \left( \tau^{-1}(s), y_s \right). \]

If \( m \) is odd, then, \( \Delta^{2} y_s > 0 \) and, hence, \( y_s \) is unbounded.

Next, we assume \( \{x_n\} \) is a negative solution of (1.1), say \( x_n < 0, n \geq N_2 \). Let \( u_n = -x_n > 0 \), then, \( \{u_n\} \) is a positive solution of (2.5), which is unbounded, if \( m \) is odd and \( \{x_n\} \) is unbounded.

By the above conclusion, the equation,
\[ \Delta^{m} y_{s-1} + f^* \left( \tau^{-1}(s), y_s \right) = 0, \] (2.13)

has a positive solution \( \{u_n\} \). Let \( y_n = -u_n \), then, \( \{y_n\} \) is a solution of (2.13) and that of (2.6) and which is unbounded when \( m \) is odd. The proof is complete.

If \( \tau(n) > n, \) from (2.7) and the fact that \( x_n \) is increasing, we can obtain
\[ \Delta^{m-1} x_{s-1} \geq \sum_{i=s}^{\infty} f \left( i, x_{\tau(i)} \right) \geq \sum_{i=s}^{\infty} f \left( i, x_i \right). \]

By the same argument of Theorem 2.1, we have the following proposition.

**COROLLARY 2.1.** Assume \( \tau(n) \) is increasing and \( \tau(n) > n \). If, for \( m \) even, (1.1) has a nonoscillatory solution, then, equation (1.2) has a nonoscillatory solution. If, for \( m \) odd, (1.1) has an unbounded nonoscillatory solution, so does equation (1.2).

As in the proof of Theorem 2.1, we can obtain the following proposition.

**COROLLARY 2.2.** Assume \( (H) \) holds and \( m \) is even. Then, equation (1.1) is oscillatory, if and only if, the inequality,
\[ \Delta^{m} x_{n-1} + f \left( n, x_{\tau(n)} \right) \leq 0, \] (2.14)

has no eventually positive solutions.

Now, we consider (1.1) together with
\[ \Delta^{m} x_{n-1} + f \left( n, x_{\sigma(n)} \right) = 0, \] (2.15)

where \( \sigma: N \rightarrow N, \lim_{n \rightarrow \infty} \sigma(n) = \infty \).
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THEOREM 2.2. Assume that \( \tau(n) \geq \sigma(n) \), \( n \geq 0 \). If, for \( m \) even, equation (1.1) has a nonoscillatory solution, so does (2.15). If, for \( m \) odd, equation (1.1) has an unbounded nonoscillatory solution, so does (2.15).

PROOF. Let \( \{x_n\} \) be a positive solution of (1.1), i.e., \( x_n > 0 \), for \( n \geq N_1 \). As in the proof of Theorem 2.1, there exists \( N_3 \), such that

\[
\Delta^{m-1} x_{s-1} \geq \sum_{i=s}^{\infty} f(i, x_{\tau(i)}), \quad s \geq N_3.
\]

Define the sequence of operators \( F_i \) and \( T_i \), respectively, by

\[
F_j^{(x)} x_n = \sum_{i=n}^{\infty} f(i, x_{\tau(i)}),
\]

\[
F_j^{(x)} x_n = \sum_{i=n}^{\infty} F_{j-1}^{(x)} x_i, \quad j = 2, 3, \ldots, m - l,
\]

and

\[
T_j^{(x)} x_n = F_{m-1}^{(x)} x_n,
\]

\[
T_j^{(x)} x_n = \sum_{i=N_3}^{n-1} T_{j-1}^{(x)} x_i, \quad j = 1, 2, \ldots, l,
\]

which satisfy (2.9) and (2.11). Hence, by successive summations of (2.16), we have

\[
\Delta x_{n-1} \geq T_{m-1}^{(x)} x_n, \quad n \geq N_3.
\]

Since \( \tau(n) \geq \sigma(n) \) and \( f \) is nondecreasing in \( x \), we have

\[
\Delta x_{n-1} \geq T_{m-1}^{(x)} x_n,
\]

and, hence, by Lemma 2.1, the equation,

\[
\Delta x_{n-1} = T_{m-1}^{(x)} x_n,
\]

has a nonoscillatory solution \( \{y_n\} \), which is unbounded when \( m \) is odd. By successive differences of equation \( \Delta y_{n-1} = T_{m-1}^{(x)} y_n \), we conclude that \( \{y_n\} \) satisfies (2.15).

Now, if we assume \( x_n < 0, n \geq N_1 \) is a solution of (1.1), then, the equation,

\[
\Delta^{m} x_{n-1} + f^* (n, x_{\sigma(n)}) = 0,
\]

(2.17)

has a positive solution \( \{v_n\} \), which is unbounded when \( m \) is odd. Hence, \( y_n = -v_n \) is a solution of (2.17), which satisfies (2.15). The proof is complete.

THEOREM 2.3. Suppose that \( \tau(n) \geq \sigma(n) \), \( \sigma(n) \) is increasing, \( \sigma(n) \leq n, n \geq N_1 \). If, for \( m \) even, the equation,

\[
\Delta^{m} y_{s-1} + f (\sigma^{-1} (s), y_{s}) = 0,
\]

(2.18)

is oscillatory, so is (1.1). If, for \( m \) odd, equation (2.18) has no unbounded nonoscillatory solution, neither does (1.1).

PROOF. It follows from Theorems 2.1 and 2.2.

REMARK 2.1. If \( \sigma(n) > n \), (2.18) is replaced by (1.2). The conclusions of Theorem 2.3 are also true.

Since \( n - \tau(n) \) is bounded, we have the following conclusions. For \( m \) odd, if \( n - M \leq \tau(n) \leq n \) holds, then, we can get the following result.
THEOREM 2.4. For m odd, if \( n - M \leq \tau(n) \leq n \), then, every nonoscillatory solution of (1.1) is bounded, if and only if, every nonoscillatory solution of (1.2) is bounded.

PROOF. Since \( n - M \leq \tau(n) \leq n \), \( M \) is a positive integer.

Set \( \sigma(n) = n - M \), \( s = \sigma(n) \), and \( y_s = x_n \); then, we have
\[
\begin{align*}
  n &= \sigma^{-1}(s), \\
  y_s &= x_n = x_{\sigma^{-1}(s)}, \\
  \tau(n) &= \sigma(n), \\
  \Delta y_s &= y_{s+1} - y_s = x_{\sigma^{-1}(s+1)} - x_{\sigma^{-1}(s)} = \Delta x_n, \\
  \Delta^m y_s &= \Delta^m x_n.
\end{align*}
\]

(2.19)

Sufficiency, i.e., if every nonoscillatory solution of (1.2) is bounded, so does (1.1). From the above argument, every nonoscillatory solution of (1.2) is bounded implies that of (2.18), then, the result follows from Theorem 2.3 directly.

Necessity, i.e., if every nonoscillatory solution of (1.1) is bounded, so does (1.2). If we suppose that (1.2) has an unbounded nonoscillatory solution, then, from Theorem 2.2 and \( n \geq \tau(n) \), (1.1) has an unbounded nonoscillatory solution, which is a contradiction.

The proof is complete.

For \( m \) even, we are going to prove the equivalence of the oscillation of (1.1) and (1.2).

THEOREM 2.5. For \( m \) even, assume (H) holds, then, (1.1) is oscillatory, if and only if, (1.2) oscillates.

PROOF. Since \( |n - \tau(n)| \leq M \), \( M \) is a positive integer. Set \( \sigma(n) = n - M \), \( s = \sigma(n) \), as before, we can obtain (2.19).

Sufficiency, i.e., the oscillation of equation (1.2) implies that of (1.1). For the case \( \tau(n) > n \), the conclusion follows from Corollary 2.1. For the case \( \tau(n) \leq n \), the result follows from Theorem 2.3 directly.

Necessity, i.e., the oscillation of equation (1.1) implies that of (1.2). For \( n \geq \tau(n) \), the result follows from Theorem 2.2. For \( \tau(n) > n \), without loss of generality, we will give the proof for \( m = 4 \). Suppose that the equation,
\[
\Delta^4 x_{n-1} + f(n, x_n) = 0,
\]
has an eventually positive solution \( \{x_n\} \) on \( n \geq N_1 \), i.e., there exists \( N_1 \), such that \( x_n > 0 \), for \( n \geq N_1 \), then, \( \Delta^4 x_{n-1} \leq 0 \). By [1, Theorem 1.8.11], there are two possible cases,

(i) \( \Delta x_n > 0, \Delta^2 x_n < 0, \Delta^3 x_n > 0, n \geq N_2 > N_1 \), and

(ii) \( \Delta x_n > 0, \Delta^2 x_n > 0, \Delta^3 x_n > 0, n \geq N_2 > N_1 \).

For Case (i), \( \lim_{n \to \infty} \Delta x_n = k \geq 0 \) exists.

(a) If \( k > 0 \), then, there exists \( N_3 > N_2 \), such that \( \Delta x_n \leq k + 1 \), for all \( n \geq N_3 \). Since \( n < \tau(n) \leq n + M \), then,
\[
\sum_{i=n}^{\tau(n)-1} \Delta x_i = x_{\tau(n)} - x_n \leq \sum_{i=n}^{n+M-1} \Delta x_i \leq M(k + 1).
\]

Hence, \( x_n \geq x_{\tau(n)} - M(k + 1) \). Let \( z_n = x_n - M(k+1) \), then, \( z_n > 0 \), \( z_{\tau(n)} \leq x_n \), \( \Delta z_n = \Delta x_n \) eventually.

From (1.1), we have
\[
\Delta^4 z_{n-1} + f(n, z_{\tau(n)}) \leq \Delta^4 x_{n-1} + f(n, x_n) = 0.
\]

Hence, if (1.2) has a positive solution, then, (1.1) has a positive solution, which is a contradiction.

(b) If \( k = 0 \), since \( \{x_n\} \) is increasing and \( x_n > 0 \), then, there exist \( \varepsilon_0 > 0 \) and \( N_2 \geq N_1 \), such that \( x_n > M\varepsilon_0 \), for \( n \geq N_2 \). Corresponding to this \( \varepsilon_0 \), there exists \( N_3 \geq N_1 \), such that \( \Delta x_n \leq \varepsilon_0 \), for all \( n \geq N_3 \). Let \( N^* = \max\{N_2, N_3\} \), if \( n \geq N^* \), we obtain
\[
x_{\tau(n)} - x_n \leq x_{n+M} - x_n = \sum_{i=n}^{n+M-1} \Delta x_i \leq M\varepsilon_0.
\]

Hence, \( x_n \geq x_{\tau(n)} - M\varepsilon_0, n \geq N^* \). Similar to Case (a), we also get a contradiction.
For Case (ii), \( \lim_{n \to \infty} \Delta^3 x_n = k \geq 0 \) exists.

Set
\[
H_n = \begin{cases} 
\Delta x_{n-1}, & n \geq N, \\
0, & n < N.
\end{cases}
\]

Hence, \( H_n \geq 0 \). Define \( z_n = \sum_{i=0}^{\infty} H_{n-i} \). Clearly, \( z_n - z_{n-1} = H_n, z_{N-1} = 0 \), and \( z_n > 0, \Delta z_n > 0 \), for \( n \geq N \). For \( n \geq N \), we have
\[
\begin{align*}
z_n - z_{n-1} &= \Delta x_{n-1}, \\
z_N &= \Delta x_{N-1}, \\
z_{N+1} - z_N &= \Delta x_N, \\
z_{N+1} &= \Delta x_{N-1} + \Delta x_N, \\
z_{N+2} - z_{N+1} &= \Delta x_{N+1}, \\
z_{N+2} &= \Delta x_{N-1} + \Delta x_N + \Delta x_{N+1}.
\end{align*}
\]

By induction, for \( n \geq N \), we have
\[
\begin{align*}
z_n &= \sum_{i=N-1}^{n-1} \Delta x_i = x_n - x_{N-1} \leq x_n.
\end{align*}
\]

Hence,
\[
z_{n+M} \leq x_{n+M} - x_n + x_n \leq x_n + M \max_{i=n}^{n+M-1} \{|\Delta x_i|, \ i = n, \ldots, n + M - 1\}. \tag{2.21}
\]

We discuss three possible cases.

(1) If \( k > 0 \), then,
\[
\begin{align*}
\Delta^2 x_n &= k \Delta x_n + o(n) , \\
\Delta x_n &= \frac{k n^2}{2} + o(n^2) , \\
x_n &= \frac{k n^3}{6} + o(n^3) , \\
z_n &= \frac{k n^3}{6} + o(n^3).
\end{align*}
\]

From (2.21), we have \( z_{n+M} \leq x_n + kMn^2 \), for \( n \geq N \). Set \( \tilde{z}_n = z_n - k M (n - M)^2 > 0, n \geq N \). Then, we obtain
\[
\begin{align*}
\tilde{z}_{n+M} &= z_{n+M} - kMn^2 \leq x_n, \\
\tilde{z}_n - \tilde{z}_{n-1} &= z_n - z_{n-1} - k M (n - M)^2 + k M (n - 1 - M)^2, \\
\Delta^4 \tilde{z}_{n-1} &= \Delta^3 (\tilde{z}_n - \tilde{z}_{n-1}) = \Delta^3 (z_n - z_{n-1}) = \Delta^4 x_{n-1}.
\end{align*}
\]

From the above notation, we obtain \( \tilde{z}_{\tau(n)} \leq x_n \), then, \( f(n, \tilde{z}_{\tau(n)}) \leq f(n, x_n) \). Hence,
\[
\Delta^4 \tilde{z}_{n-1} + f(n, \tilde{z}_{\tau(n)}) \leq \Delta^4 x_{n-1} + f(n, x_n) = 0,
\]
which implies (1.1) has a positive solution. It leads to a contradiction.

(2) If \( k = 0 \) and \( \lim_{n \to \infty} \Delta^2 x_n = l \), then,
\[
\begin{align*}
\Delta x_n &= \ln + o(n) , \\
x_n &= \frac{\ln^2}{2} + o(n^2) , \\
z_n &= \frac{\ln^2}{2} + o(n^2).
\end{align*}
\]

Hence, from (2.21), we have \( z_{n+M} \leq x_n + 2 M l n, n \geq N \). Set \( \tilde{z}_n = z_n - 2 M l (n - M) > 0, n \geq N \). Then,
\[
\begin{align*}
\tilde{z}_{n+M} &= z_{n+M} - 2 M l n \leq x_n, \\
\Delta^4 \tilde{z}_{n-1} &= \Delta^4 x_{n-1}.
\end{align*}
\]
Similar to Case 1, we get
\[ \Delta^4 \overline{z}_{n-1} + f(n, \overline{z}_{\tau(n)}) \leq \Delta^4 x_{n-1} + f(n, x_n) = 0, \]
which is a contradiction.

(3) If \( k = 0 \) and \( \lim_{n \to \infty} \Delta^2 x_n = \infty \), then,
\[
\Delta^2 x_n = o(n), \quad \Delta x_n = o(n^2), \quad n = o(\Delta x_n), \quad x_n = o(n^3), \\
n^2 = o(x_n), \quad z_n = o(n^3), \quad n^2 = o(z_n).
\]

From (2.21), we have \( z_{n+M} \leq x_n + M n^2 \), for \( n \geq N \). Define \( \overline{z}_n = z_n - M(n - M)^2 \), then, \( \overline{z}_n > 0 \), and \( x_n \geq \overline{z}_{n+M} \). Similar to Cases 1 and 2, we can obtain
\[ \Delta^4 \overline{z}_{n-1} + f(n, \overline{z}_{\tau(n)}) \leq 0, \]
which implies (1.1) has a positive solution. As before, it leads to a contradiction. The proof is complete.

**Remark 2.2.** Theorem 2.2 and Theorem 2.5 generalize the related results in [2,3].

**REFERENCES**