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Multivector Calculus

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INTRODUCTION

The object of this paper is to show how differential and integral calculus in many dimensions can be greatly simplified by using Clifford algebra. Here the necessary notations, definitions, and fundamental theorems are developed to make the calculus ready to be used. Those features of Clifford algebra which are needed for this task are described without proof.[1]

The discussion of differentiation and integration omits without comment many important problems in analysis, because they are in no way affected by the special features of the approach advanced here. The object throughout is to show how Clifford algebra can be used to advantage.

1. ALGEBRA

The notion of a vector as a directed number can be made precise by introducing rules for addition and multiplication of vectors which have a geometric interpretation. The rules governing vector addition and scalar[2] multiplication are too familiar to require comment. By these operations an n-dimensional linear space $\mathcal{O}\!\ell_n$, here called *arithmetic n-space*, can be generated from n linearly independent vectors. Appropriate rules governing multiplication of vectors can be arrived at by requiring that the product of any nonzero vector with itself be a positive scalar. The "square" of a vector a is written

$$a^2 = |a|^2 \geqslant 0, \tag{1.1}$$

where $|a|$ is a positive scalar called the *modulus* (or *magnitude*) of a, and equality holds if and only if $a = 0$. With the exception of the commutative rule for multiplication, all the rules of scalar algebra can be applied to vectors without contradicting (1.1). Then, by multiplication and addition, a Clifford

[1] Further discussion of Clifford algebra together with applications to physics can be found in my book "Space-Time Algebra," Gordon and Breach, New York, 1966.

[2] In this paper "scalar" always means "real number."

Algebra $\mathcal{M}_n$, here called the *multivector algebra* of $\mathcal{O}_n$, can be generated from the vectors in $\mathcal{O}_n$. To emphasize their relation to vectors, the elements of $\mathcal{M}_n$ are called *multivectors.*

Far from being a defect as some might be inclined to think, the absence of universal commutativity for vector products is a great advantage. For the "degree of commutativity" in a product is a *measure of the relative directions* of directed numbers. This is easily seen by decomposing the product of vectors a and b into a sum of commutative and anticommutative parts.

$$ab = a \cdot b + a \wedge b, \tag{1.2}$$

where $a \cdot b$ and $a \wedge b$ can be regarded as new kinds of multiplication, respectively called *inner* and *outer products*, and defined by the equations

$$a \cdot b = \tfrac{1}{2}(ab + ba) = b \cdot a \tag{1.3}$$

$$a \wedge b = \tfrac{1}{2}(ab - ba) = -b \wedge a. \tag{1.4}$$

By virtue of (1.1), $a \cdot b$ is a scalar and may be interpreted as the usual "Euclidean" scalar product of vectors in $\mathcal{O}_n$. Subject to this interpretation, (1.2) shows that two vectors are *collinear* if and only if they commute, and they are *orthogonal* if and only if they anticommute. The products $a \cdot b$ and $a \wedge b$ were invented and given a geometrical interpretation by H. Grassmann more than one hundred years ago. Through (1.2) they imbue the noncommutative product ab and all of $\mathcal{M}_n$ with geometrical significance.

A multivector which can be factored into a product of k orthogonal vectors is called a *simple k-vector*. Since k orthogonal vectors also span a k-dimensional subspace of $\mathcal{O}_n$, it is apparent that to every simple k-vector there corresponds a unique k-dimensional subspace of $\mathcal{O}_n$. In fact, every simple k-vector can be interpreted geometrically as an oriented volume of some k-dimensional subspace of $\mathcal{O}_n$.

Any linear combination of simple k-vectors is called simply a *k-vector*. The terms "1-*vectors*" and "0-*vector*" are synonyms for "vector" and "scalar," respectively. An n-vector of $\mathcal{M}_n$ is often called a *pseudoscalar*.

The product aA_k of a vector a with a k-vector A_k consists of a $(k-1)$-vector plus a $(k+1)$-vector, denoted by $a \cdot A_k$ and $a \wedge A_k$ respectively, so

$$aA_k = a \cdot A_k + a \wedge A_k. \tag{1.5}$$

This is a straightforward generalization of (1.2), to which it reduces if $k = 1$. In general, the product of a simpler r-vector A_r with a simple s-vector B_s is more complicated than (1.5), but the $(r + s)$-vector part of the product A_rB_s is important enough to be given a symbol: $A_r \wedge B_s$, and a name: *outer product* of A_r with B_s.

Any multivector A can be expressed as the sum of k-vectors A_k, where $k = 0, 1, 2, \ldots, n$. Thus,

$$A = \sum_{k=0}^{n} A_k . \tag{1.6}$$

Equation (1.5) can now be generalized by introducing the definitions

$$a \cdot A \equiv \sum_k a \cdot A_k , \qquad a \wedge A \equiv \sum_k a \wedge A_k \tag{1.7}$$

so that

$$aA = a \cdot A + a \wedge A. \tag{1.8}$$

The *reverse* (or *adjoint*) of A, denoted by $A^\dagger$, can be obtained by expressing the simple k-vector components as products of vectors and reversing the order of multiplication. It follows that

$$A^\dagger = \sum_{k=0}^{n} (-1)^{\frac{1}{2}k(k-1)} A_k . \tag{1.9}$$

The scalar part of the product $A^\dagger B$ is called the *scalar product* of multivectors A and B, and is written $(A^\dagger B)_0$, the subscript zero denoting 0-vector (or scalar) part. This scalar product is symmetric and positive definite, the latter property being due to (1.1).

$$(A^\dagger B)_0 = \sum_{k=0}^{n} (A_k^\dagger B_k)_0 = \sum_{k=0}^{n} (B_k^\dagger A_k)_0 = (B^\dagger A)_0 \tag{1.10}$$

$$(A^\dagger A)_0 = \sum_{k=0}^{n} (A_k^\dagger A_k)_0 = \sum_{k=0}^{n} A_k^\dagger A_k \geqslant 0. \tag{1.11}$$

The *modulus* (or norm) of A is defined by the equation

$$| A | = [(A^\dagger A)_0]^{\frac{1}{2}} = \left[\sum_{k=0}^{n} | A_k |^2\right]^{\frac{1}{2}} \geqslant 0. \tag{1.12}$$

We have $| A | = 0$ if and only if $A = 0$. A multivector with unit modulus is said to be *unitary*. Any multivector A can be expressed as a scalar multiple of a unitary multivector $\hat{A}$.

$$A = | A | \hat{A}. \tag{1.13}$$

The scalar product has many important properties. For instance, every scalar determinant of rank k can be expressed as the scalar product of two k-vectors, and all the properties of determinants follow automatically from properties of multivectors algebra.

There are two unitary pseudoscalars in $\mathscr{M}_n$ corresponding to the two possible orientations of an n-dimensional unit volume in $\mathscr{O}_n$. Denote by i the pseudoscalar representing the unit volume with positive orientation. The product iA is called the *dual* of A. In $\mathscr{M}_3$, the "cross product" $a \times b$ introduced by J. Willard Gibbs is the dual of the bivector $a \wedge b$. When proper account is taken of the usual sign conventions, this duality is expressed by the equation

$$a \wedge b = i(a \times b). \tag{1.14}$$

With this definition, the entire vector algebra of Gibbs is seen as a subalgebra of $\mathscr{M}_3$.

2. Geometry

Intuitive geometrical notions such as "continuous," "straight," "distance," and "dimension" require a special language for precise expression. To meet this need, multivector algebra is cultivated here.

The points of *Euclidean n-space* $\mathscr{E}_n$ can be put into one-to-one correspondence with the vectors of $\mathscr{O}_n$. So it is convenient to use a vector x as a name for the point to which it corresponds. A vector used as a name for a point is called the *coordinate* of the point.

The correspondence between $\mathscr{E}_n$ and $\mathscr{O}_n$ gives much more than names. Points in $\mathscr{E}_n$ are named for the purpose of describing the properties of geometric objects (point sets) in $\mathscr{E}_n$. The multivector algebra $\mathscr{M}_n$ provides a grammar and vocabulary designed to simplify such descriptions. For instance, the distance between two points x_1 and x_2 in $\mathscr{E}_n$ is simply the modulus $| x_2 - x_1 |$, and trigonometrical relations for discrete point sets in $\mathscr{E}_n$ are readily computed with multivector algebra. It is worth remarking that $\mathscr{M}_n$ can also be used to describe non-Euclidean geometries if only an appropriate change is made in the definition of scalar product.

This paper is concerned with continuous surfaces in $\mathscr{E}_n$. Let $\mathscr{S}$ be a smooth k-dimensional surface in $\mathscr{E}_n$. A *multivector function* (or *multivector field*) *on* $\mathscr{S}$ is a mapping of $\mathscr{S}$ into $\mathscr{M}_n$. Let $x_1, x_2, x_3, \ldots$ be a sequence of points in $\mathscr{S}$ converging a point x in $\mathscr{S}$. The unit vector

$$n(x) = \lim_{i \to \infty} \frac{x_i - x}{| x_i - x |} \tag{2.1}$$

is said to be *tangent to $\mathscr{S}$ at x*. The surface $\mathscr{S}$ is k-dimensional if and only if there are k linearly independent vectors tangent to $\mathscr{S}$ at each point of $\mathscr{S}$. By multiplication and addition, these vectors generate a multivector algebra $\mathscr{M}_k(x)$ which, of course, is a subalgebra of $\mathscr{M}_n$. A multivector function on $\mathscr{S}$ with values in $\mathscr{M}_k(x)$ at x is said to be *tangent to $\mathscr{S}$ at x*. A multivector function is said to be *tangent to $\mathscr{S}$* if it is tangent at every point of $\mathscr{S}$.[3]

A surface can be characterized by the multivector functions tangent to it. If a smooth k-dimensional surface $\mathscr{S}$ is orientable, there are exactly two unitary k-vector functions tangent to $\mathscr{S}$, each corresponding to one of the two orientations which can be given to $\mathscr{S}$. So tangent to each point x of an oriented k-dimensional surface there is a unique unitary k-vector $v(x)$ characterizing the orientation of $\mathscr{S}$ at x. Call $v(x)$ *the tangent* of $\mathscr{S}$ at x. Call the dual of $v(x)$ *the normal* of $\mathscr{S}$ at x.

3. Integration

Let f be a multivector function defined on a smooth r-dimensional surface $\mathscr{V}$. Define the *directed integral* of f over $\mathscr{V}$ by the formula

$$\int_{\mathscr{V}} dvf = \int_{\mathscr{V}} dv(x)f(x) \equiv \lim_{n\to\infty} \sum_{i=1}^{n} \Delta v_i(x)f(x_i). \tag{3.1}$$

This differs from the usual definition of a Riemann integral only in one important respect. Both dv and $\Delta v(x_i)$ are directed volume elements. The magnitudes $| dv |$ and $| \Delta v |$ are to be understood as the usual *Riemann measures* of volume. The direction of a volume element at x is characterized by the unitary simple r-vector $v(x)$ tangent to $\mathscr{V}$ at the point x. This can be expressed by writing

$$\Delta v(x_i) = | \Delta v(x_i) | \, v(x_i) \tag{3.2a}$$

$$dv(x) = | dv(x) | \, v(x). \tag{3.2b}$$

So it is clear that the *directed integral* of f is equivalent to the *Riemann integral of vf*.

$$\lim_{n\to\infty} \sum_{i=1}^{n} | \Delta v(x_i) | \, v(x_i)f(x_i) = \int_{\mathscr{V}} | dv | \, vf = \int_{\mathscr{V}} dvf. \tag{3.3}$$

Therefore, the details to the limiting process in (3.1) can be handled by the techniques of Riemann integration theory.

[3] For brevity, the effects of discontinuities in a surface are not discussed here.

The directed integral (3.1) is a nontrivial generalization of the Riemann integral which makes essential use of multivector algebra. The significance of this generalization can be seen in complex variable theory, for the integral with respect to a complex variable is a 1-dimensional directed integral, and it is this feature which makes the theory so powerful. In another paper it will be shown that complex variable theory can be regarded as a special case of the multivector calculus developed here.

At a deeper level, "directed integration" is founded on a generalization of measure theory which uses "directed measure" instead of "scalar measure." A "directed measure" associates a direction and a dimension as well as a magnitude to a set. Thus, it may be said that the directed integral (3.1) makes use of "directed Riemann measure" rather than the usual "scalar Riemann measure."

The volume $|\mathscr{V}|$ of the surface $\mathscr{V}$ is

$$|\mathscr{V}| = \int_{\mathscr{V}} dv v^{\dagger} = \int_{\mathscr{V}} |dv|. \tag{3.4}$$

One can also associate a "directed volume" with the surface $\mathscr{V}$ given by the integral $\int_{\mathscr{V}} dv$. From the definition (3.1), it follows that the directed volume of any closed surface vanishes. This is expressed by the equation

$$\oint dv = 0, \tag{3.5}$$

where $\oint$ indicates that the integral is over a closed surface. Clearly, (3.5) obtains because on a closed surface "directed volume elements" occur in pairs with opposite orientations which cancel when added.

Because multivector multiplication is noncommutative, (3.1) is not the most general form for a directed integral. The appropriate generalization is

$$\int_{\mathscr{V}} g\, dvf = \lim_{n\to\infty} \sum_{i=1}^{n} g(x_i)\, \Delta v(x_i) f(x_i), \tag{3.6}$$

where, of course, f and g are multivector functions defined on $\mathscr{V}$.

4. Differentiation

Let A be a nonzero simple r-vector in $\mathscr{M}_n$, and f some multivector function defined on $\mathscr{E}_n$. The "(left) *derivative* of f with respect to A (evaluated) at x" is denoted by $\nabla_A f(x)$ and defined as follows:

$$\nabla_A f(x) \equiv \lim_{|\mathscr{V}|\to 0} \frac{\hat{A}^{\dagger}}{|\mathscr{V}|} \int_{\partial\mathscr{V}} da f, \tag{4.1}$$

where

(1) $\hat{A} = | A |^{-1} A$.

(2) A smooth open r-dimensional surface $\mathscr{V}$ has been chosen which passes through x "tangent to A," i.e., so that $\hat{A} = v(x)$, the tangent of $\mathscr{V}$ at x.

(3) The integral of f is taken over the boundary, $\partial\mathscr{V}$, of $\mathscr{V}$. The orientation of the $(r - 1)$-vector $da = | da | \, a$ describing a volume element of $\partial\mathscr{V}$ is chosen so that the vector $n(x') = v^{\dagger}(x') \, a(x')$ is the outward normal at a point x' of $\partial\mathscr{V}$.

(4) The limit is taken by shrinking $\mathscr{V}$ and its volume $| \mathscr{V} |$ to zero at the point x.

A discussion of the extent to which the choice of $\mathscr{V}$ and the process of shrinking is arbitrary is too involved to give here.

To get at the significance of the operator ∇_A let us look at some special cases and then ascertain its general properties.

On the basis of (4.1) no meaning can be attached to the derivative with respect to a scalar, so the simplest example is the derivative with respect to some vector n. In this case, $\mathscr{V}$ is an oriented curve with arc length $s = | \mathscr{V} |$ passing through the point x with tangent proportional to n. The boundary of $\mathscr{V}$ consists of the end points, x_1 and x_2, of the curve. To evaluate the integral over $\partial\mathscr{V}$, appeal must be made to the definition of the integral (3.1), which shows that limiting process is unnecessary since the surface in question consists of only two points. A point is a 0-dimensional surface, so its volume element is a 0-vector. Use of Riemann measure requires that the volume element at x_2 have unit wieght, i.e., $\Delta a(x_2) = 1$. The volume element at x_1 must have opposite orientation to be consistent with (3.5), so $\Delta a(x_1) = -1$. Thus

$$\begin{aligned} \int_{\partial\mathscr{V}} da f &= \Delta a(x_2) f(x_2) + \Delta a(x_1) f(x_1) \\ &= f(x_2) - f(x_1), \end{aligned} \tag{4.2}$$

and (4.1) can be written

$$\nabla_n f(x) = \hat{n} \lim_{s \to 0} \frac{1}{s} \left(f(x_2) - f(x_1) \right). \tag{4.3}$$

The right side of (4.3) is recognized as the average of left and right derivatives with respect to arc length.

It is desirable to eschew such expressions as df/ds for "derivative with respect to arc length" and $\partial f / \partial x$ for "partial derivatives with respect to the (scalar) coordinate x," because they contain irrelevances. Different as they appear, they refer to one and the same limiting process. The derivative at x

depends only on the direction at x along which the limit is taken and not on any particular curves passing through the point. The essentials are expressed by the notation $\nabla_n f$.

The operator ∇_n may sometimes be awkward to use because the $\hat{n}$ on the right of (4.3) does not commute with other multivectors. In such cases the "scalar differential operator" $\partial_n = \hat{n}\nabla_n$ may be recommended. Nevertheless, ∇_n is more fundamental than ∂_n because of its generalization by (4.1). A comparably simple generalization of ∂_n does not exist.

The derivative with respect to a pseudoscalar is expecially important. The same result is obtained for all pseudoscalars, so it is convenient to drop the subscript and write ∇.

Call ∇ the *gradient* operator, to agree with common parlance when ∇ operates on a scalar. The gradient is a "vector differential operator," so, by virtue of (1.8),

$$\nabla f = \nabla \cdot f + \nabla \wedge f. \tag{4.4}$$

Call $\nabla \cdot f$ the *divergence* of f and $\nabla \wedge f$ the *curl* of f, to agree with the terminology of vector and tensor analysis. For the special case of a vector field on $\mathscr{E}_3$, (1.4) can be used to get

$$\nabla \wedge f = i\nabla \times f. \tag{4.5}$$

Another familiar differential operator easily obtained from ∇ is the *laplacian* ∇^2. In fact, every differential operator on $\mathscr{E}_n$ can be expressed as some operator function of ∇.

One can think of ∇_A as the gradient operator for the subspace of $\mathscr{E}_n$ determined by A. However, if A is a function of x, the subspace will depend on the point at which the derivative is evaluated. Let it be understood that, unless otherwise specified, $\nabla_A f$ has the value $\nabla_{A(x)}f(x)$ at x, that is, the derivative of f at x is taken with respect to the value of A at x.

The general properties of the operator ∇_A follow from the definition (4.1).

$$\nabla_{-A} = \nabla_A \tag{4.6}$$

$$\nabla_{\lambda A} = \nabla_A \qquad \text{for positive scalar } \lambda \tag{4.7}$$

$$\nabla_{AB} = \nabla_A + \nabla_B \qquad \text{if} \qquad AB = A \wedge B \tag{4.8}$$

$$\nabla_A\nabla_B = -\nabla_B\nabla_A \qquad \text{if} \qquad AB = A \wedge B \qquad \text{and} \qquad \nabla_A B = \nabla_B A = 0 \tag{4.9}$$

$$\nabla_A(f + g) = \nabla_A f + \nabla_A g \tag{4.10}$$

$$\nabla_A(fg) = (\nabla_A f)g \qquad \text{if} \qquad g \text{ constant.} \tag{4.11}$$

The operator equations (4.6) and (4.7) express the fact that ∇_A depends only on the direction of A and not on the orientation or magnitude of A. Equations (4.8) and (4.9) show how "gradient operators for orthogonal subspaces of $\mathscr{E}_n$" are related, and they determine how "laplacians" for orthogonal subspaces "combine":

$$(\nabla_{AB})^2 = \nabla_A{}^2 + \nabla_B{}^2. \tag{4.12}$$

Equations (4.10) and (4.11) hardly need comment.

The convention that ∇_A differentiates only to the right can be awkward because of the noncommutivity of multiplication. If the convention is retained, it is convenient to have a mark which indicates differentiation both to the left and right when desired. Accordingly, the definition

$$g\,\bar{\bar{\nabla}}_A f \equiv \lim_{|\mathscr{V}|\to 0} \int_{\partial\mathscr{V}} g\,\frac{\hat{A}^\dagger\, da}{|\mathscr{V}|}\, f. \tag{4.13}$$

This definition admits a simple form for the "Leibnitz rule" for differentiating a product:

$$g\,\bar{\bar{\nabla}}_A f = (g\,\bar{\nabla}_A) f + g(\bar{\nabla}_A f). \tag{4.14}$$

On the right, only the function inside the parenthesis is to be differentiated. The proof of (4.14) uses the identity

$$\begin{aligned}\int_{\partial\mathscr{V}} g'\,\frac{\hat{A}^\dagger\, da'}{|\mathscr{V}|}\, f' = \Big[\int_{\partial\mathscr{V}} g'\,\frac{\hat{A}^\dagger\, da'}{|\mathscr{V}|}\Big] f + g\,\Big[\int_{\partial\mathscr{V}} \frac{\hat{A}^\dagger\, da'}{|\mathscr{V}|}\, f'\Big] \\ + \int_{\partial\mathscr{V}} (g' - g)\,\frac{A^\dagger\, da'}{|\mathscr{V}|}\,(f' - f) \\ - g\,\frac{A^\dagger}{|\mathscr{V}|}\int_{\partial\mathscr{V}} da' f,\end{aligned} \tag{4.15}$$

where $f = f(x)$ is the value f at the point where the derivative is to be taken, and $f' = f(x')$ is the value of f at a point x' on $\partial\mathscr{V}$; likewise for the other quantities. The last term on the right of (4.15) is identically zero because of (3.5). In the limit, the next to the last term on the right of (4.15) vanishes and the remaining terms give (4.14).

The relation of ∇_A to the gradient is shown by the following:

$$\nabla = A^{-1}A\nabla = A^{-1}(A \cdot \nabla + A \wedge \nabla)$$

$$\nabla = \nabla_A + \nabla_{iA} \tag{4.16}$$

$$\nabla_A = A^{-1}A \cdot \nabla \tag{4.17}$$

$$\nabla_{iA} = A^{-1}A \wedge \nabla. \tag{4.18}$$

5. The Fundamental Theorem of Calculus

Let f be a multivector function defined on an oriented r-dimensional surface $\mathscr{V}$ in $\mathscr{E}_n$ with tangent v. Call $\nabla_v f$ the *tangential derivative* of f on $\mathscr{V}$. These things being understood, the fundamental theorem can be stated as follows:

The integral of the tangential derivative of f over $\mathscr{V}$ is equal to the integral of f over the boundary of $\mathscr{V}$. As an equation,

$$\int_{\mathscr{V}} dv\, \nabla_v f = \int_{\partial\mathscr{V}} da f. \tag{5.1}$$

Note that this formula is independent of the dimension of $\mathscr{V}$ and of the space in which $\mathscr{V}$ is imbedded. A major motivation for the formulation of integration and differentiation in this paper has been to achieve as simple and general a statement of the fundamental theorem as possible. For instance, the "1-vector property" of ∇_v is appropriate because it relates integrals over $\mathscr{V}$ and $\partial\mathscr{V}$-surfaces which differ by one dimension. Furthermore, the definition of the derivative (4.1) has been made as similar to (5.1) as possible.

Various special cases of the fundamental theorem are called Green's theorem, Gauss' theorem, Stoke's theorem, etc. But the general theorem is so basic that it deserves a name which describes its scope.

A proof of the fundamental theorem is obtained by establishing the following sequence of equations

$$\begin{aligned}\int_{\mathscr{V}} dv\, \nabla_v f &= \lim_{n\to\infty} \sum_{i=1}^{n} \Delta v_i \left\{\frac{1}{\Delta v_i} \int_{\partial\mathscr{V}_i} da f\right\} \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \int_{\partial\mathscr{V}_i} da f = \int_{\partial\mathscr{V}} da f. \end{aligned} \tag{5.2}$$

The analytical details of the proof do not depend on the dimension of $\mathscr{V}$; they differ in no essential way from details in the proofs of special cases of the theorem. Such proofs have been given on many occasions, though seldom with the utmost generality, so no further comment is needed here.[4]

To illustrate the felicity and generality of (5.1), the integration theorems of Gibbs' "vector calculus" in $\mathscr{E}_3$ can easily be derived.

If $\mathscr{V}_3$ is a 3-dimensional region in $\mathscr{E}_3$, then one can write

$$dv = |\,dv\,|\, i, \qquad \nabla_v = \nabla, \qquad da = in\,|\,da\,|\,,$$

[4] For a careful discussion of the problems involved, see M. R. Hestenes, *Duke Math. J.*, **8**, 300 (1941); A. B. Carson in "Contributions to the Calculus of Variations," p. 457. Univ. of Chicago Press, Chicago, Ill., 1938–1941.

where n is the outward normal to $\mathscr{V}_3$. Since the pseudoscalar i is constant, it can be "factored out," and (5.1) becomes

$$\int_{\mathscr{V}_3} |\,dv\,|\,\nabla f = \oint |\,da\,|\,nf. \tag{5.3}$$

If $f = \varphi$ is a scalar function, then (5.3) becomes

$$\int_{\mathscr{V}_3} |\,dv\,|\,\nabla\varphi = \oint |\,da\,|\,n\varphi.$$

If $f = E$ is a vector field, then (1.2), (1.14), (4.4), (4.5) can be used, and 0-vector and 2-vector parts on each side of the equation can be equated separately to get

$$\int_{\mathscr{V}_3} |\,dv\,|\,\nabla \cdot E = \oint |\,da\,|\,n \cdot E$$

$$\int_{\mathscr{V}_3} |\,dv\,|\,\nabla \times E = \oint |\,da\,|\,n \times E.$$

If $\mathscr{V}_2$ is a 2-dimensional surface, then one can write

$$dv = -in\,|\,da\,|, \qquad da = dx$$

$$\nabla_v = (in)^{-1}\,(in) \cdot \nabla = -ini(n \wedge \nabla) = in(n \times \nabla),$$

where dx is the differential of the coordinate x of a point on $\partial\mathscr{V}_2$, and n is the "right-handed normal" to the surface $\mathscr{V}_2$. So (5.1) becomes

$$\int_{\mathscr{V}_2} |\,da\,|\,n \times \nabla f = \oint dxf. \tag{5.4}$$

If $f = \varphi$ is a scalar, then

$$\int_{\mathscr{V}_2} |\,da\,|\,n \times \nabla\varphi = \oint dx\varphi.$$

If $f = E$ is a vector, then, as before, 0-vector and 2-vector parts in (5.4) can be separately equated to get

$$\int_{\mathscr{V}_2} |\,da\,|\,n \cdot (\nabla \times E) = \oint dx \cdot E$$

$$\int_{\mathscr{V}_2} |\,da\,|\,(n \times \nabla) \times E = \oint dx \times E.$$

If $\mathscr{V}_1$ is a curve in $\mathscr{E}_3$ with endpoints a and b, then one can write

$$dv = dx, \qquad dv\nabla_v = dx \cdot \nabla.$$

So (5.1) becomes the familiar formula

$$\int_{\mathscr{V}_1} dx \cdot \nabla f = \int_a^b df = f(b) - f(a). \tag{5.5}$$

In spite of the ease with which the formulas of vector analysis can be derived, it is even easier to use (5.1) as it is or sometimes the special forms (5.3), (5.4), or (5.5).

For each choice of a particular function f, (5.1) yields a formula relating integrals over $\mathscr{V}$ to integrals over $\partial\mathscr{V}$. For instance, if v is a k-vector and x is the coordinate of a point in $\mathscr{E}_n$, then

$$\nabla_v x = k. \tag{5.6}$$

So, if $\mathscr{V}$ is an k-dimensional surface, then

$$\int_{\mathscr{V}} dv = \frac{1}{k} \oint dax. \tag{5.7}$$

Because of (3.5), this integral is independent of the choice of origin. If $\mathscr{V}$ is a flat surface, then its tangent v is constant, so

$$|\mathscr{V}| = \frac{1}{kv} \oint dax. \tag{5.8}$$

If $\partial\mathscr{V}$ is an $(r-1)$-dimensional sphere with radius R and area $|\partial\mathscr{V}|$, then (5.8) reduces to

$$|\mathscr{V}| = \frac{R}{k} |\partial\mathscr{V}|. \tag{5.9}$$

Many other useful consequences of (5.7) can be easily found.

Actually, because of the noncommutivity of multiplication, (5.1) is not the most general form of the fundamental theorem. The necessary generalization can be written

$$\int_{\mathscr{V}} g\, dv\, \bar{\nabla}_v f = \int_{\partial\mathscr{V}} g\, daf, \tag{5.10}$$

where it is understood that dv is not differentiated by $\bar{\nabla}_v$. More explicitly, since

$$\bar{\nabla}_v v = (-1)^{r-1} v\, \bar{\nabla}_v, \tag{5.11}$$

(5.10) can be written

$$\int_{\mathscr{V}} g\, dv\, \nabla_v f + (-1)^{r-1} \int_{\mathscr{V}} (g\, \bar{\nabla}_v)\, dvf = \int_{\partial\mathscr{V}} g\, daf. \tag{5.12}$$

The fundamentals have been set down. A complete geometric calculus of multivector functions is now waiting to be worked out along lines similar to the calculus of real and complex functions.