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Maximal arcs in Steiner systems $S(2,4,v)$

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Abstract

A *maximal arc* in a Steiner system $S(2,4,v)$ is a set of elements which intersects every block in either two or zero elements. It is well known that $v \equiv 4 \pmod{12}$ is a necessary condition for an $S(2,4,v)$ to possess a maximal arc. We describe methods of constructing an $S(2,4,v)$ with a maximal arc, and settle the longstanding sufficiency question in a strong way. We show that for any $v \equiv 4 \pmod{12}$, we can construct a resolvable $S(2,4,v)$ containing a triple of maximal arcs, all mutually intersecting in a common point. An application to the motivating colouring problem is presented.

Keywords: Maximal arc; Steiner system; Colouring

1. Introduction

A *maximal arc* in a Steiner system $S(2,4,v)$ is a set of $(v+2)/3$ elements, with no three contained in a block. Alternatively, a maximal arc is a set of elements which intersects every block in either two or zero elements.

It is well known that $v \equiv 4 \pmod{12}$ is a necessary condition for an $S(2,4,v)$ to possess a maximal arc (cf. [15]). It is also well known that the projective spaces $PG(d,3)$ cannot contain a maximal arc [9].

On the other hand, it appears that $S(2,4,v)$'s with a maximal arc were known to exist only for orders of the form $v = 12.2^h + 4$, $h \geqslant 0$, as a consequence of Denniston's construction of maximal arcs in finite projective planes of even order (see [16]; cf. also [6]). Also, in [13], an $S(2,4,40)$ with a maximal arc was constructed.

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In this article, we describe methods to construct $S(2,4,v)$'s with a maximal arc. In particular, we show that for any $v \equiv 4 \pmod{12}$, we can construct an $S(2,4,v)$ containing a maximal arc. Actually, we show something rather stronger, namely that for any $v \equiv 4 \pmod{12}$, we can construct a resolvable $S(2,4,v)$ containing a triple of maximal arcs, with each pair intersecting in the same common point.

2. Denniston's construction

Although the basic consequence of Denniston's construction of maximal arcs in finite projective planes of even order is well known, if we actually examine his construction [4], we can get a stronger, more detailed result.

Theorem 1. *If k is a power of 2, then the resolvable $S(2,k,k(k-1)2^h+k)$ given by Denniston's maximal arc consists of $k-1$ maximal arcs of $k2^h+2$ points, all mutually intersecting in a common point.*

Proof. Working in $PG(2,k2^h)$, Denniston considers an irreducible second-order curve, $Q(x,y)=ax^2+bxy+cy^2$ over $GF(k2^h)$ in the non-homogeneous coordinates (x,y) of the plane. Let G be any additive subgroup of order k of the additive group $GF(k2^h)$. Then the set of $k(k-1)2^h+k$ points satisfying $Q(x,y) \subset G$ forms Denniston's arc. This set of points intersects every line of the plane in either k or 0 points. Taking one of the external lines, we can use its points to indicate the parallel classes of the arc. Next note that if $a \in G$, then $\{0,a\}$ forms an additive subgroup of G as well as $GF(k2^h)$, hence the points of the arc it generates are a subset of the points generated by G as well as being a Denniston arc in its own right, with intersections of 2 or 0 with the lines of the plane. Finally, the trivial additive subgroup $\{0\}$ generates a single point arc, and since this trivial subgroup was the intersection of all the order 2 subgroups, this generated point is the intersection point of the arcs. □

Remark 2. Let $q=k2^h$. If we take the additive subgroup, G, of $GF(q)$ as $GF(q)$ itself, then as an extreme case of Denniston's construction we get $AG(2,q)$ as Denniston's arc. Now Theorem 1 says that there will be a set of $q-1$ maximal arcs, all intersecting in a common point, which cover $AG(2,q)$. This phenomenon has already been noted for $q=64$ in [7], where one may find further information on the automorphism groups acting on this configuration.

3. Construction of maximal arcs

In this section, we describe a general approach to constructing Steiner systems $S(2,4,v)$ with a maximal arc of size $(v+2)/3$. We then show that our method is always successful whenever $n=(v-1)/3$ is a prime power, even though this condition is not necessary.

Let $n=4m+1$. Define a class of graphs $\mathcal{G}_n$ as follows. Let V_1, V_2 be two sets, $|V_i|=n$, $V_1\cap V_2=\emptyset$, and let $H_1=(V_1,E_1)$, $H_2=(V_2,E_2)$ be two regular graphs, each of degree $2m$. Let $G=(V,E)$ be the graph with $V=V_1\cup V_2$, $E=E_1\cup E_2\cup M$ where M is a perfect matching given by $M=\{[x,\alpha x]\colon x\in V_1\}$; here $\alpha: V_1\to V_2$ is any bijection between V_1 and V_2. Then $G\in\mathcal{G}_n$.

Thus any $G\in\mathcal{G}_n$ is a regular graph of degree $2m+1=(n+1)/2$. For any $G\in\mathcal{G}_n$, the complement $\bar{G}$ of G is a regular graph of degree $6m=\frac{3}{2}(n-1)$.

For a regular graph H, a 1-factorization $\mathcal{F}$ of the graph $2H$ (every edge of H is "doubled") is *self-orthogonal* if any two distinct 1-factors of $\mathcal{F}$ have at most one edge in common (cf. [11]).

Remark 3. The number of K_4's in the decomposition of $\bar{G}$ is $m(4m+1)=mn$. Observe also that if $G\in\mathcal{G}_n$ then in any self-orthogonal 1-factorization of $2G$, any two 1-factors must have *exactly* one edge in common.

Theorem 4. *Suppose for some $G\in\mathcal{G}_n$ there exists a self-orthogonal 1-factorization of $2G$, and suppose there exists a decomposition of the complement $\bar{G}$ of G into copies of K_4. Then there exists an $S(2,4,12m+4)$ with a maximal arc (of size $4m+2$).*

Proof. Let $\mathcal{F}=\{F_1,F_2,\ldots,F_{4m+2}\}$ be a self-orthogonal 1-factorization of $2G$ ($G\in\mathcal{G}_n$). Let X be a set, $|X|=4m+2$, $X\cap V(G)=\emptyset$, and let $\beta: X\to\mathcal{F}$ be a bijection. Let now B_{xy} be the block of size 4 containing x, y ($x,y\in X$) and the two elements of the edge common to the two 1-factors $\beta x, \beta y$ associated with x and y, respectively. It is readily seen that the set of blocks $\mathcal{B}=\{B_{xy}\colon x,y\in X\}$, together with the blocks given by the decomposition of the complement $\bar{G}$ of G into K_4's, is the set of blocks of an $S(2,4,12m+4)$ on $X\cup V(G)$, and X is a maximal arc. □

Theorem 5. *Let $n=(v-1)/3$ be a prime power. Then there exists an $S(2,4,v)$ with a maximal arc.*

Proof. We describe a particular graph G and a specific self-orthogonal 1-factorization of $2G$, together with a decomposition of $\bar{G}$ into copies of K_4, so that Theorem 4 will apply.

Let x be a primitive root of $GF(n)$ where $n=(v-1)/3=4m+1$. Set $V_i=GF(n)\times\{i\}$, $i=1,2$, and define a "base" 1-factor F_0 by

$\{[0_1,0_2]\}\cup\{[x_1,(\varepsilon x)_1],[(-x)_1,(-\varepsilon x)_1],[x_2,(-\varepsilon x)_2],[(-x)_2,(\varepsilon x)_2]\colon x\in X\}$,

where ε is a fourth root of unity and X is a system of representatives of the cosets of $\langle\varepsilon\rangle$ in the multiplicative group of $GF(n)$. 1-factors $F_1,\ldots,F_{n-1}$ are obtained from F_0 by developing over $GF(n)$. Also, let F^* be the 1-factor given by

$$F^*=\{[i_1,i_2]\colon i\in GF(n)\}.$$

We note that the "pure" differences on level 1 in F_0 comprise the set $\{\pm(x(\varepsilon-1)\colon x\in X\}$ while the pure differences on level 2 comprise the set $\{\pm x(\varepsilon+1)\colon x\in X\}$ (each twice). Also, the edges in F^* are all of mixed difference 0. It is not hard to verify that we do have a self-orthogonal 1-factorization.

Consider now the following set $\mathscr{C}$ of blocks of size 4 (decomposition of the complement $\bar{G}$ into K_4's), where $\mathscr{C}=\{C_x\colon x\in X\}$ and:

$$C_x=\{0_1,(x(\varepsilon+1))_1,x_2,(\varepsilon x)_2\}.$$

Since $x^j+x^{j+2m}-x^{j+m}=x^{j+3m}$, we see that C_j comprises the mixed differences $\pm x,\pm\varepsilon x$, the pure difference $\pm(x(\varepsilon+1))$ on level 1, and the pure difference $\pm(x(\varepsilon-1))$ on level 2, and thus the differences in $\mathscr{C}$ comprise precisely the differences not present in $\{F_0,F_1,\dots,F_{n-1},F^*\}$. Thus, all conditions of Theorem 4 are satisfied. □

Actually, Theorem 5 provides examples of systems $S(2,4,v)$ having not just one but *three* maximal arcs, all intersecting in one common point, namely the point (say) ∞ of X that is associated with the 1-factor F^*; the other two maximal arcs besides X are then $(\mathrm{GF}(n)\times\{i\})\cup\{\infty\}$, $i=1,2$.

The next example of an $S(2,4,64)$ with a maximal arc of size 22 shows that the prime power restriction as given in Theorem 5 is not necessary.

Example 6. $V(G)=Z_{21}\times\{1,2\}$; $E(G)$ is obtained as a union of the following 22 1-factors (brackets enclosing edges are omitted):

$F_0=\{0_10_2,1_17_1,2_19_1,3_111_1,4_113_1,5_115_1,6_116_1,8_117_1,10_118_1,12_119_1,14_120_1,1_22_2, 3_220_2,4_218_2,5_211_2,6_213_2,7_210_2,8_214_2,9_212_2,15_219_2,16_217_2\} \pmod{21}$,

$F^*=\{0_10_2 \pmod{21}\}$, $\mathscr{F}=\{F_0,F_1,\dots,F_{20},F^*\}$.

Blocks of size 4 decomposing the complement $\bar{G}$ of G:

$\{0_11_110_212_2\},\{0_15_18_218_2\},\{0_14_16_219_2\},\{0_13_14_220_2\},\{0_12_17_216_2\} \pmod{21}$.

Theorem 4 can now be applied, yielding the further base blocks:

$\{7_020_00_16_1\},\{9_019_00_17_1\},\{11_018_00_18_1\},\{13_017_00_19_1\},\{15_016_00_110_1\}$,

$\{5_020_00_21_2\},\{12_014_00_23_2\},\{1_06_00_24_2\},\{13_016_00_26_2\},\{3_015_00_27_2\} \pmod{21}$.

together with the "short" base block $\{\infty 0_01_02_0\}$ (here we denoted for convenience $X=(Z_{21}\times\{0\})\cup\{\infty\}$).

4. A difference family construction

Theorem 7 is essentially Moore's construction [14], although he was not concerned with the embedded arcs that we will need.

Theorem 7. *If $n\equiv 1 \pmod 4$ is a prime power, then a resolvable $S(2,4,v)$ exists, with $v=3n+1$, which contains three maximal arcs of $n+1$ points, all mutually intersecting in a common point.*

Proof. We construct a 1-rotational difference family over $(Z_3 \times \mathrm{GF}(n)) \cup \{\infty\}$. Let x be a primitive element of $\mathrm{GF}(n)$, with $n = 4m + 1$, and let $\varepsilon = x^m$ be a fourth root of unity. Let:

$$B_0 = \{(0,1),(0,-1),(1,\varepsilon),(1,-\varepsilon)\},$$

$$B_\infty = \{\infty,(0,0),(1,0),(2,0)\},$$

$$\mathscr{B} = B_\infty \cup \{(1,x^i) * B_0\colon\ i = 0,1,\ldots,m-1\}.$$

Then $\mathscr{B}$ is the required difference family. The block B_∞ has a short orbit, and developing $\mathscr{B}$ over Z_3 only gives a parallel set of blocks that span all points, and developing these blocks over $\mathrm{GF}(n)$ gives the design. The sets $(\{i\} \times \mathrm{GF}(n)) \cup \{\infty\}$ form the maximal arcs. □

Remark 8. Most of the designs of the form $S(2,k,v)$ with $v \equiv k-1 \pmod{k^2-k}$ with k even that are constructed in [5] also have the properties of being resolvable, and consist of $k-1$ maximal arcs intersecting in a common point. The existence of a maximal arc there has been exploited in constructing PBDs by Ling et al. (see their [12, Lemma 4.10]).

Our next theorem is in the spirit of several similar constructions [1–3]; however, our interest is in the embedded maximal arcs, rather than other embedded subdesigns, such as whist teams.

Theorem 9. *If q is a product of prime powers, all equivalent to* $1 \pmod 4$, *then there exists a resolvable* $S(2,4,3q+1)$ *which contains three maximal arcs of $q+1$ points, all mutually intersecting in a common point.*

Proof. We shall proceed by induction. Suppose we have a 1-rotational difference family over $(Z_3 \times G_1) \cup \{\infty\}$, with $|G_1| = 4m+1$, and with all full orbit base blocks of the form $\{(0,a),(0,-a),(1,b),(1,-b)\}$, and a similar 1-rotational difference family over $(Z_3 \times G_2) \cup \{\infty\}$, with $|G_2| = 4n+1$, with all full orbit base blocks of the form $\{(0,c),(0,-c),(1,d),(1,-d)\}$, then we shall construct a 1-rotational difference family over $(Z_3 \times G_1 \times G_2) \cup \{\infty\}$, as follows:

For each pair of base blocks we construct the following four base blocks:

$$\{(0,a,c),(0,-a,-c),(1,b,d),(1,-b,-d)\},$$

$$\{(0,a,-c),(0,-a,c),(1,b,-d),(1,-b,d)\},$$

$$\{(0,a,d),(0,-a,-d),(1,b,-c),(1,-b,c)\},$$

$$\{(0,a,-d),(0,-a,d),(1,b,c),(1,-b,-c)\}.$$

This generates $4mn$ base blocks. Now for every base block in the first family, we construct:

$$\{(0,a,0),(0,-a,0),(1,b,0),(1,-b,0)\}$$

and for every base block in the second family, we construct:

$$\{(0,0,c),(0,0,-c),(1,0,d),(1,0,-d)\}.$$

Finally, we have the short orbit base block

$$\{\infty,(0,0,0),(1,0,0),(2,0,0)\}.$$

This set of base blocks is the required difference family. The resolvability and the maximal arcs are as in Theorem 7. □

Remark 10. It is worth pointing out that we only used the special prime form of q to ensure the existence of the two difference families, and that if we had difference families of the correct form, the product result would follow. The same applies to the resolvability, whilst the maximal arc property is clear from the form of the product difference family.

Remark 11. A referee pointed out to us that we could have proved Theorem 9 whilst proving Theorem 7 by working in $\mathrm{GR}(q)$, the Galois ring of order q, and noting there is a group, $\langle\varepsilon\rangle$, of units of $\mathrm{GR}(q)$ of order 4 acting semiregularly on $\mathrm{GR}(q)\setminus\{0\}$. Use that ε and redefine B_0 in the proof of Theorem 7; similarly, redefine $\mathscr{B}$ as with $\mathscr{B}=B_\infty\cup\{(1,s)*B_0\colon s\in S\}$, where S is a system of representatives for the $\langle\varepsilon\rangle$ orbits on $\mathrm{GR}(q)\setminus\{0\}$. It now follows as in [3], that $\mathscr{B}$ is a 1-rotational difference family over $(Z_3\times \mathrm{GR}(q))\cup\{\infty\}$ for a resolvable $\mathrm{S}(2,4,3q+1)$ which contains three maximal arcs intersecting in $\{\infty\}$.

Whilst Remark 11 is certainly more elegant than our exposition, we have decided to retain Theorem 9 separate from Theorem 7. As we point out in Remark 10, the proof of Theorem 9 has the potential of wider application than we used in the statement of Theorem 9.

Actually, for Theorem 14, our main result, we only need Theorem 7, and then only for the prime powers 5, 9, 13, 17 and 29.

5. A PBD construction

In this section, we will employ a variant of Wilson's fundamental construction (WFC) [17] to construct an $\mathrm{S}(2,4,3n+1)$ with three maximal arcs intersecting in a common point.

Theorem 12. *Suppose an* $\mathrm{S}(2,P_{1(4)},n)$ *PBD exists, where* $P_{1(4)}$ *denotes the prime powers of the form* $4t+1$; *then a resolvable* $\mathrm{S}(2,4,3n+1)$ *exists, which contains three maximal arcs of* $n+1$ *points, all mutually intersecting in a common point.*

Proof. Start with the PBD, and assign a weight of 3 to each point in WFC. Now the standard version of WFC will yield a $\{4\}$-frame of type 3^n, noting that the deletion of the lines through the infinite point in Theorem 7 actually yields frames. For our variant, we will assign a labelled weight to each point of the PBD, with the labels 0, 1 and 2. Now when we take a block of size k in the PBD, and construct our frame of type 3^k on its weighted points, we can do so in such a way that every block contains 2 points from each of a pair of labels. Now filling in the groups of the resulting frame with the aid of an infinite point gives the resolvable $S(2,4,3n+1)$, and this infinite point also is the common intersection point of the three maximal arcs. □

Next, we need a result of [10].

Lemma 13. *Let $P_{1(4)}$ denote the prime powers of the form $4m+1$. If $n \equiv 1 \pmod 4$, and $n \neq 33$, then an $S(2, P_{1(4)}, n)$ PBD exists.*

We can now state our main theorem.

Theorem 14. *If $v \equiv 4 \pmod{12}$, then a resolvable $S(2,4,v)$ exists which contains three maximal arcs of size $(v+2)/3$ mutually intersecting in a common point.*

Proof. Except for $v=100$, this follows from applying Theorem 12 to the PBD's of Lemma 13. The case $v=100$ is treated by Theorem 1 with $k=4$ and $h=3$. □

6. Discussion

We have presented two ostensibly different approaches to the problem of constructing $S(2,4,v)$'s with maximal arcs: the self-orthogonal 1-factorization approach of Theorem 4, and the difference family approach of Section 4. The question which obviously arises is to what extent are these approaches reconcilable? We are unable to give a complete answer, but we do have some insights.

Let us first consider the $S(2,4,64)$ example we gave earlier in Section 3. For comparison, we note that we constructed an $S(2,4,64)$ in Theorem 14. Since the component PBD was actually $PG(2,4)$, which has a difference set representation of $\{3,6,7,12,14\}$ over Z_{21}, and the ingredient design was a frame of type 3^5, i.e., the punctured $AG(2,4)$, and this also has a difference set representation of $\{1_1 4_1 2_2 3_2\}$ over $Z_5 \times Z_3$ (with the Z_3 element as a subscript), we might expect that Theorem 12 yields a design with a difference family representation, as is the case:

$\{6_1 14_1 7_2 12_2\}, \{7_1 3_1 12_2 14_2\}, \{12_1 6_1 14_2 3_2\}, \{14_1 7_1 3_2 6_2\}, \{3_1 12_1 6_2 7_2\}$.

This set, together with the "short" block $\{\infty 0_0 1_0 2_0\}$, gives the difference family over $Z_{21} \times Z_3$. The shifts of 18, 15, 14, 9, 7, respectively, exhibit the parallel class.

Now we will assume the above difference family came from a Theorem 4 construction, and by exploiting the cyclic automorphism of order 21, we present a "cook-book" approach to finding the underlying self-orthogonal 1-factorization. We will take our X for Theorem 4 as the elements with the zero subscript, and the blocks not containing any element of X as the decomposition of $\bar{G}$. First we consider the block $\{6_0 14_0 7_1 12_1\}$, and subtract the first element from the last two, then the second, and claim $1_1 6_1$ and $14_1 19_1$ are edges in the factorization; we repeat this for all full base blocks. Next we consider the block $\{6_2 14_2 7_0 12_0\}$, and subtract the third element from the first two, then the fourth, and claim $20_2 7_2$ and $15_2 2_2$ are edges in the factorization; we repeat this for all full base blocks. With $V(G) = Z_{21} \times \{1,2\}$, $E(G)$ is obtained as a union of the following 22 1-factors (brackets enclosing edges are omitted):

$F_0 = \{0_1 0_2, 1_1 6_1, 14_1 19_1, 5_1 7_1, 9_1 11_1, 2_1 12_1, 8_1 18_1, 10_1 13_1, 17_1 20_1, 3_1 4_1, 15_1 16_1, 20_2 7_2, 15_2 2_2, 16_2 12_2, 14_2 10_2, 19_2 13_2, 9_2 3_2, 11_2 4_2, 8_2 1_2, 18_2 6_2, 17_2 5_2\} \pmod{21}$,

$F^* = \{0_1 0_2 \pmod{21}\}$, $\mathscr{F} = \{F_0, F_1, \ldots, F_{20}, F^*\}$.

The conclusions we drew from this demonstration were, that provided the underlying 1-factorization was developed over some Abelian additive group of order $|H| = (v-1)/3$, and the decomposition of $\bar{G}$ was given by a difference family over H, then we could give a difference family construction over H, but not necessarily a group of order $3|H| = v - 1$; also given a difference family over H for a triple-arc design, we could reverse engineer the difference family construction to present it as a consequence of Theorem 4.

Another aspect we considered was the resolvability, but we could see no obvious way to decide whether the S(2, 4, 64) example in Section 3 was resolvable. Further, although we chose to use frames in Theorem 12, we could have picked non-frame GDDs of type 3^n in their place, and again there would be no obvious way to decide whether the resulting S(2, 4, v) was resolvable. Thus, we concluded that the resolvability, although clearly worth commenting on, was not an inherent property of our constructions.

Finally, we looked more closely at the constructions of Theorems 5 and 7, and found that they gave essentially identical difference families. We say "essentially" as we may multiply any subset of base blocks in Theorem 7 by 1_{-1} (i.e., multiply the subscripts by -1) and still have a difference family; also each block is invariant under multiplication by ε_{-1}, so we get essentially the same difference family by multiplying B_0 by any sequence of m successive powers of x.

The first step is to establish that the blocks other than $\mathscr{C}$ in Theorem 5 can be composed into a difference family where we treat the blocks in $\mathscr{C}$ as base blocks over $GF(n) \times Z_3$, rather than just over $GF(n)$. We will not exhibit this mechanical step. Now examining C_0 of Theorem 5, we see we have

$$C_0 = \{0_1, (1+\varepsilon)_1, 1_2, \varepsilon_2\}.$$

Let $\alpha = (1-\varepsilon)^{-1}$. If we add $(-\alpha)_0$ to each element, after simplification, noting that $(1-\varepsilon)(1+\varepsilon) = 2$, we get

$$C_0 = \{(-\alpha)_1, \alpha_1, (-\varepsilon\alpha)_2, (\varepsilon\alpha)_2\}$$

from which the essential identification of the two difference families follows.

7. An application to colourings

A specialized colouring of S(2, 4, v) of type AC, as defined in [13] is a colouring of elements such that each block is either monochromatic or has the block colour pattern 2 + 2. Provided all colours are used, apart from the trivial colouring with one colour (which is always possible for any $S(2,4,v)$, the only other possibility is for a 2-colouring to exist (i.e. a colouring using two colours), and $v \equiv 4 \pmod{12}$ is then necessary [13]. It was conjectured in [13] that an $S(2,4,v)$ with a 2-colouring of type AC exists for all $v \equiv 4 \pmod{12}$. Since an $S(2,4,v)$ with a maximal arc admits a 2-colouring of type AC [13], we can extend the results of [13] as follows.

Theorem 15. *An* $S(2,4,v)$ *with a* 2-*colouring of type* AC *exists whenever* $v \equiv 4 \pmod{12}$.

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