Oscillation Theorems for Certain Second Order Perturbed Nonlinear Differential Equations

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1. Introduction

In this paper we discuss, using weighted integrals, the oscillations of solutions of the perturbed differential equations

\[(A) \quad (a(t)x')' + m(t)x + Q(t, x) = P(t, x, x'), \quad \left( \frac{d}{dt} \right) \]

and

\[(B) \quad (a(t)x')' + Q(t, x) = P(t, x, x'). \]

Problems of these types for less general equations have been studied by Onose [5], Kamanev [3] and Wong [7] for the equation

\[\ddot{x} + q(t)f(x) = 0,\]

and Coles [1] for the equation

\[\ddot{x} + Q(t, x) = 0.\]

We are concerned with only continuable solutions of (A) and (B), which exist on some half line \([t_0, \infty].\) A solution \(x(t)\) of (A) (resp. (B)) will be called nonoscillatory if there exists a \(t_1 \geq t_0\) such that \(x(t) \neq 0\) for \(t \geq t_1,\) and \(x(t)\) will be called oscillatory if it has no last zero.

Our main results are given in Section 2 (Theorems 2.1–2.3), which generalize some of the results established by Coles [1], Graef, Rankin and Spikes [2], and Kamanev [3]. In Section 3 we give some illustrative examples.
2. Oscillation Criteria

In the sequel we assume that

(i) \(a: [t_0, \infty) \to (0, \infty), m: [t_0, \infty) \to R\), are continuous, where \(R = (-\infty, \infty)\):

(ii) \(Q: [t_0, \infty) \times R \to R, P: [t_0, \infty) \times R \to R\), are continuous functions such that

\[Q(t, x) \geq q(t)f'(x) \quad \text{and} \quad P(t, x, \dot{x}) \leq p(t)g(x)f_2(x),\]

for \(x \neq 0\), where

\[q: [t_0, \infty) \to R, p: [t_0, \infty) \to [0, \infty) \quad \text{and} \quad g, f_1: R \to R\]

are continuous functions;

(iii) \(x f_i(x) > 0\) for \(x \neq 0, i = 1, 2\);

(iv) \(f_i(x) \leq f_i(x)\) for all \(x \neq 0\);

(v) \(0 < g(\dot{x}) \leq c\) for some constant \(c\).

We prove the following theorem for the case when \(m(t) \equiv 0\).

**Theorem 2.1.** If, in addition to conditions (i)-(v), we assume that

(vi) \(f_i'(x) \geq 0\) for \(x \neq 0, (\prime \equiv d/dx)\);

(vii) \(\int_{t_0}^{\infty} (1/a(s)) \, ds = \infty\);

(viii) \(\int_{t_0}^{\infty} (q(s) - cp(s)) \, ds = \infty\), where \(c\) is the constant appearing in (v),

then all the continuable solutions of (A) are oscillatory.

**Proof.** Suppose that \(x(t)\) is a continuable nonoscillatory solution of (A), and that \(x(t) \neq 0\) for \(t \geq t_i \geq t_0\). Then

\[
\left(\frac{a(t)x}{f_i(x)}\right)' = -\frac{Q(t, x)}{f_i(x)} + \frac{P(t, x, \dot{x})}{f_i(x)} - \frac{a(t)f_i'(x)x^2}{f_i^2(x)} \\
\leq -q(t) + \frac{p(t)g(x)f_2(x)}{f_i(x)} \\
\leq -(q(t) - cp(t)).
\]

Integrating the above inequality from \(t_i\) to \(t(>t_i)\) we obtain

\[
\frac{a(t)\dot{x}(t)}{f_i(x(t))} \leq \frac{a(t_1)\dot{x}(t_1)}{f_i(x(t_1))} - \int_{t_i}^{t} (q(s) - cp(s)) \, ds.
\]
We assume that \( x(t) > 0 \) for \( t \geq t_1 \); the proof for the case \( x(t) < 0 \) for \( t \geq t_1 \) is similar and will be omitted. In view of condition (viii), it follows from the last inequality that there exists a \( T_0 \geq t_1 \) such that \( \dot{x}(t) < 0 \) for \( t \geq T_0 \). It also follows from (viii) that there exists a \( T \geq T_0 \) such that

\[
\int_{T_0}^{T} (q(s) - cp(s)) \, ds = 0 \quad \text{and} \quad \int_{T}^{T} (q(s) - cp(s)) \, ds \geq 0 \quad \text{for} \quad t \geq T.
\]

Integrating (A) by parts, we have

\[
\begin{align*}
a(t) \dot{x}(t) & \leq a(t) \dot{x}(t) - \int_{T}^{t} q(s) f_1(x(s)) \, ds + \int_{T}^{t} p(s) g(\dot{x}(s)) f_2(x(s)) \, ds \\
& \leq a(T) \dot{x}(T) - \int_{T}^{t} (q(s) - cp(s)) f_1(x(s)) \, ds \\
& \leq a(T) \dot{x}(T) - f_1(x(t)) \int_{T}^{t} (q(s) - cp(s)) \, ds \\
& \quad + \int_{T}^{t} f_1(x(s)) \dot{x}(s) \int_{T}^{s} (q(u) - cp(u)) \, du \, ds \\
& \leq a(T) \dot{x}(T).
\end{align*}
\]

Thus

\[
x(t) \leq x(T) + a(T) \dot{x}(T) \int_{T}^{t} \frac{1}{a(s)} \, ds.
\]

By virtue of (vii), and the fact that \( \dot{x}(T) < 0 \), it follows that \( x(t) \to -\infty \) as \( t \to \infty \), which contradicts the assumption that \( x(t) > 0 \) for \( t \geq t_1 \). This completes the proof.

Remark. Theorem 2.1 includes Theorem 1 in [2] and Theorem 6(i) in [6].

Theorem 2.2. If, in addition to conditions (i)–(vi), we assume that

(ix) \( \int_{0}^{T} du/f_1(u) < \infty \) and \( \int_{0}^{T} du/f_1(u) < \infty \) for every \( \epsilon > 0 \);

(x) there exists a function \( \rho \) which is twice continuously differentiable on \((t_0, \infty), \rho(t) > 0 \) for \( t \geq t_0 \), and such that

\[
\int_{t_0}^{\infty} \frac{1}{\rho(s)} \int_{t_0}^{s} \rho(\tau)(q(\tau) - cp(\tau)) \, d\tau \, ds = \int_{\rho}^{\infty} (s)(q(s) - cp(s)) \, ds = \infty;
\]
then each of the following conditions ensures the oscillation of each of the continuable solution of (A):

(C) \( a(t), m(t) \) are twice continuously differentiable on \( (t_0, \infty) \) and \( \gamma(t) \leq 0, \, \dot{\gamma}(t) \geq 0 \) for \( t > t_0 \), where \( \gamma(t) = a(t) \dot{\rho}(t) - m(t) \rho(t) \);

(D) \( a(t), m(t) \) are continuously differentiable on \( (t_0, \infty) \)

\[
\int_{t_0}^{\infty} \frac{du}{f_1(u)} < \infty, \quad \int_{t_0}^{\infty} |\dot{\gamma}(s)| \, ds < \infty \quad \text{for} \quad \epsilon > 0;
\]

(E) \( f_1'(x) \geq k > 0 \) for \( x \neq 0, \rho(t) > 0, \rho(t) \) is continuously differentiable on \( (t_0, \infty) \) and

\[
\int_{t_0}^{\infty} \frac{\gamma^2(s)}{a(s) \rho(s)} \, ds < \infty;
\]

(F) \( f_1'(x) > 0 \) for \( x \neq 0, a(t), m(t) \) are twice continuously differentiable on \( (t_0, \infty) \), \( \gamma(t) = a(t) \dot{\rho}(t) - m(t) \rho(t) > 0 \), and there exists a constant \( K_1 > 0 \) such that

\[
\frac{G''(x) G(x)}{G^2(x)} \leq -\frac{1}{k_1}
\]

and

\[
\frac{\dot{\gamma}(t)}{\gamma^2(t)} a(t) \rho(t) \leq -k_1 \quad \text{for} \quad t \geq t_0,
\]

where \( G(v) = \int_0^v du/f_1(u) \).

Proof: Our proof is an adaptation of the argument developed by Coles [1] and Kamenev [3]. Let \( x(t) \) be a continuable nonoscillatory solution of (A). Without loss of generality we may assume that \( x(t) \neq 0 \) on \( [t_0, \infty) \). Suppose \( x(t) > 0 \) for \( t \geq t_1 \geq t_0 \). The case \( x(t) < 0 \) is handled similarly.

We multiply (A) by \( \rho(t)/f_1(x(t)) \) and integrate to obtain

\[
\begin{align*}
& a(t) \rho(t) \frac{\dot{x}(t)}{f_1(x(t))} + \int_{t_1}^{t} a(s) \rho(s) f_1'(x(s)) \frac{\dot{x}^2(s)}{f_1^2(x(s))} \, ds \\
& - \int_{t_1}^{t} (a(s) \dot{\rho}(s) - m(s) \rho(s)) \frac{\dot{x}(s)}{f_1(x(s))} \, ds \\
& + \int_{t_1}^{t} \rho(s) \left[ q(s) - \rho(s) \frac{g(\dot{x}(s)) f_2(x(s))}{f_1(x(s))} \right] \, ds \\
& \leq \frac{a(t_1) \rho_1(t_1) \dot{x}(t_1)}{f_1(x(t_1))}.
\end{align*}
\]
Let \( u(t) = \dot{x}(t)/f_i(x(t)) \) and \( c_1 = a(t_1) \rho(t_1) \dot{x}(t_1)/f_i(x(t_1)) \); then the above inequality reduces to

\[
a(t) \rho(t) u(t) \leq c_1 - \int_{t_1}^{t} \rho(s)[q(s) - p(s)] \, ds + \int_{t_1}^{t} \gamma(s) u(s) \, ds
- \int_{t_1}^{t} a(s) \rho(s) f_i'(x(s)) u^2(s) \, ds. \tag{2.1}
\]

We consider four cases:

**Case 1.** Let (C) hold, then (2.1) becomes

\[
a(t) \rho(t) u(t) \leq c_1 - \int_{t_1}^{t} \rho(s)[q(s) - c p(s)] \, ds + \gamma(t) G(x(t))
- \gamma(t_1) G(x(t_1)) - \int_{t_1}^{t} \gamma(s) G(x(s)) \, ds,
\]

which implies that

\[
a(t) \rho(t) u(t) \leq c_2 - \int_{t_1}^{t} \rho(s)[q(s) - c p(s)] \, ds,
\]

where \( c_2 = c_1 - \gamma(t_1) G(x(t_1)). \)

**Case 2.** Let (D) hold, and consider

\[
\int_{t_1}^{t} \gamma(s) u(s) \, ds = \gamma(t) G(x(t)) - \gamma(t_1) G(x(t_1))
- \int_{t_1}^{t} \gamma(s) G(x(s)) \, ds. \tag{2.2}
\]

From the conditions, it follows that \( 0 \leq G(x(t)) \leq M \) and \( |\gamma(t)| \leq N \) for some suitable constants \( M \) and \( N \), and hence (2.2) implies that

\[
\int_{t_1}^{t} \gamma(s) u(s) \, ds \leq -\gamma(t_1) G(x(t_1)) + MN + M \int_{t_1}^{t} |\gamma(s)| \, ds
\leq k_1, \quad \text{for some } k_1,
\]

and hence (2.1) becomes

\[
a(t) \rho(t) u(t) \leq c_3 - \int_{t_1}^{t} \rho(s)[q(s) - c p(s)] \, ds,
\]

where \( c_3 = c_1 + k_1. \).
Case 3. Let (E) hold. From (2.1) we get

\[
\begin{align*}
\int_{t_1}^{t} \left[ a(s) \rho(s) - m(s) \rho(s) \right]^2 \, ds &= \int_{t_1}^{t} \left[ \frac{a(s) \rho(s) - m(s) \rho(s)}{4a(s) \rho(s) f'(x(s))} \right] ds \\
- \int_{t_1}^{t} \left[ \frac{a(s) \rho(s) - m(s) \rho(s)}{2a(s) \rho(s) f'(x(s))} \right] \left[ q(s) - cp(s) \right] ds \\
&= \left[ \int_{t_1}^{t} \left( a(s) \rho(s) - m(s) \rho(s) \right)^2 \, ds \right]^{1/2} \left[ \int_{t_1}^{t} \left( a(s) \rho(s) - m(s) \rho(s) \right) \left[ q(s) - cp(s) \right] ds \right]^{1/2}
\end{align*}
\]

for some constant \( c_3 \).

Case 4. Let (F) hold, and let \( B_1 + B_2 = 1 \). Then

\[
\begin{align*}
\int_{t_1}^{t} \gamma(s) \, u(s) \, ds &= B_1 \int_{t_1}^{t} \gamma(s) \, u(s) \, ds + B_2 \int_{t_1}^{t} \gamma(s) \, u(s) \, ds.
\end{align*}
\]

Integrate \( B_2 \int_{t_1}^{t} \gamma(s) \, u(s) \, ds \) by parts and define a function \( h \) such that

\[
2a(t) \rho(t) f'(x(t)) \, h(t) = B_1 \gamma(t).
\]

From (2.1) we obtain

\[
\begin{align*}
\int_{t_1}^{t} \left( a(s) \rho(s) - m(s) \rho(s) \right) \, ds &= \int_{t_1}^{t} \left( a(s) \rho(s) - m(s) \rho(s) \right)^2 \, ds \\
- \int_{t_1}^{t} \left( a(s) \rho(s) - m(s) \rho(s) \right) \left[ q(s) - cp(s) \right] ds \\
&= \int_{t_1}^{t} \left( a(s) \rho(s) - m(s) \rho(s) \right) \left[ q(s) - cp(s) \right] ds
\end{align*}
\]

which leads to

\[
a(t) \rho(t) \, u(t) \leq c_5 \int_{t_1}^{t} \rho(s) \left[ q(s) - cp(s) \right] ds,
\]

where \( c_5 = c_1 - B_2 \gamma(t_1) \, G(x(t_1)) \), provided

\[
1 - B_1 \gamma(t) \leq 0,
\]
and
\[
\frac{B_1^2 \dot{y}^2(t)}{4a(t) \rho(t) f_1'(x(t))} \leq (1 - B_1) \dot{y}(t) G(x(t)).
\] (2.4)

We consider only the case when \( B_1 > 1 \), in which case (2.4) reduces to
\[
\frac{B_1^2}{4(1 - B_1)} \geq \frac{\ddot{y}(t)}{\gamma^2(t)} a(t) \rho(t) G(x(t)) f_1'(x(t)).
\] (2.5)

The maximum of the expression in the left-hand side of (2.5) is \(-1\) (for \( B_1 = 2 \)). So (2.5) is implied for each \( B_1 > 1 \) by
\[
-1 \geq \frac{\ddot{y}(t)}{\gamma^2(t)} a(t) \rho(t) G(x(t)) f_1'(x(t)).
\]

Since
\[
G(x(t)) = \int_0^{x(t)} \frac{du}{f_1(u)}, \quad G'(x(t)) = \frac{1}{f_1'(x(t))}
\]
and
\[
G''(x(t)) = -\frac{f_1'(x(t))}{f_1''(x(t))},
\]
the hypotheses in (F) imply that
\[
G(x) f_1'(x) \geq \frac{1}{k_1} > 0.
\]

Thus (2.4) is verified.

From all the four cases we have
\[
a(t) \rho(t) u(t) \leq L - \int_{t_1}^{t} \rho(s)[q(s) - cp(s)] \, ds,
\] (2.6)

for some constant \( L \).

Proceeding now from (2.6), if
\[
\int_{t_1}^{\infty} \rho(s)[q(s) - cp(s)] \, ds = \infty,
\]
then for sufficiently large \( t \), say \( t \geq t_2 \geq t_1 \), we get
\[
\int_{t_1}^{t} \rho(t)[q(s) - cp(s)] \, ds \geq 2L.
\]
So (2.6) implies that
\[ a(t) \rho(t) u(t) \leq -\frac{1}{2} \int_{t_i}^{t} \rho(s) [q(s) - c \rho(s)] \, ds \]
or
\[ u(t) \leq -\frac{1}{2} \frac{1}{a(t) \rho(t)} \int_{t_i}^{t} \rho(s) [q(s) - c \rho(s)] \, ds, \]
which implies that
\[ G(x(t)) \leq G(x(t_i)) - \frac{1}{2} \int_{t_i}^{t} \frac{1}{a(s) \rho(s)} \int_{t_i}^{s} \rho(\tau) [q(\tau) - c \rho(\tau)] \, d\tau \, ds. \]
Consequently \( G(x(t)) \to -\infty \) as \( t \to \infty \), contradicting the fact that \( G(x) \geq 0 \). This completes the proof of Theorem 2.2.

The following corollary is an immediate consequence of Theorem 2.2, and it includes Theorem 8 in [2] as a special case.

**Corollary.** Let condition (x) in Theorem 2.2 be replaced by
\[ (x)' \int_{-\infty}^{\infty} \frac{M}{a(s) \rho(s)} \, ds - \int_{-\infty}^{\infty} \frac{1}{a(s) \rho(s)} \int_{t_i}^{s} \rho(\tau) [q(\tau) - c \rho(\tau)] \, d\tau \, ds = -\infty, \]
then the conclusion of Theorem 2.2 holds

**Theorem 2.3.** Let condition (ix) in Theorem 2.2 be replaced by
\[ (ix)' \int_{0}^{\infty} \frac{\psi(u)}{f_1(u)} \, du < \infty \quad \text{and} \quad \int_{0}^{\infty} \frac{\psi(u)}{f_1(u)} \, du < \infty, \]
for every \( \varepsilon > 0 \), where \( \psi(u) \) is a continuous bounded and a positive function on \( R \). Then the conclusion of Theorem 2.2 holds for \( (B) \).

**Proof.** The proof of Theorem 2.3 can be modelled on that of Theorem 2.2 and hence is omitted.

3. **Illustrative Examples**

**Example 1.** Consider the differential equation
\[ (tx)' + tf(x) = 0, \]
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where
\[ f(x) = x, \quad \text{for } x \in [0, \infty), \]
\[ = 2 \tan^{-1} x - x, \quad \text{for } x \in (-\infty, 0]. \]

We rewrite the above equation as
\[ (t\ddot{x} + 2tf'(x) = tf_2(x), \quad t \geq 1, \]
where
\[ f_1(x) = x, \quad x \in [0, \infty), \]
\[ = \tan^{-1} x, \quad x \in (-\infty, 0], \]
and
\[ f_2(x) = x. \]

It is easy to check that the hypotheses of Theorem 2.1 are satisfied. Hence all the continuable solutions of the equation are oscillatory. This conclusion does not appear to be deducible from other known oscillation criteria.

**Example 2.** Consider the equation
\[ \ddot{x} + \dot{x} + x^{1/3} = 0. \]
If we take \( p(t) = e^t \), then all the hypotheses of Theorem 2.2(E) are satisfied. We may add that the fact that this equation is oscillatory is also deducible from Lemma 3.1 in [4].

**Example 3.** Consider the equation
\[ (2t\ddot{x})' - \dot{x} + x^{1/3} = 0. \]
We may choose \( p(t) = 1/t \) defined on \([1, \infty)\). This equation is oscillatory in view of Theorem 2.2(A). However, we cannot apply results of Kartsatos and Toro in [4], since \( m(t) \) is a negative functions and \( a(t) \neq 1 \).

**Example 4.** Consider the equation
\[ \ddot{x} + f(x) = 0, \]
where
\[ f(x) = x^{1/3}, \quad x \in [-1, \infty), \]
\[ = 2x^{1/3} - \frac{x - 2}{3}, \quad x \in (-\infty, -1]. \]
We will show that this equation is oscillatory, this fact, we believe, is not deducible from other known criteria. We rewrite the above equation as

\[ x + 2f'(x) = f(x), \]

where

\[
\begin{align*}
f_1(x) &= x^{1/3}, \\
f_2(x) &= x^{1/3}, \quad x \in [-1, \infty), \\
&= \frac{x - 2}{3}, \quad x \in (-\infty, -1].
\end{align*}
\]

We may choose \( \rho(t) = 1/t \), defined on \([1, \infty)\). It is easy to check that Theorem 2.2 can be applied in order to prove that the above equation is oscillatory.

Remarks. (1) Theorem 6 and Corollary 7 in [2] are included in Theorem 2.2 for the case that \( a(t) \equiv 1, m(t) = 0, \) and \( \rho(t) = t \).

(2) Theorem 2.2 generalizes some of the results in [4].

(3) Interesting consequences of Theorem 2.2 are obtained by letting \( \rho(t) \) to be one of the following:

\[ t, \frac{1}{a(t)}, \quad \frac{t}{a(t)}, \quad 1 + \int_{t_0}^{t} \frac{ds}{a(s)}, \quad \text{for } m = 0 \]

and

\[ \exp \left( \int_{t}^{t_0} \frac{m(s)}{a(s)} ds \right) \quad \text{for } m \neq 0. \]

REFERENCES