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Tilting complexes, perpendicular categories and recollements of derived module categories of rings

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Abstract

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A necessary and sufficient criterion is given for the existence of recollements of unbounded derived module categories of rings. The criterion is applied to several previously investigated situations.

1. Introduction

Recently, derived module categories of rings, in particular of finite-dimensional algebras, have attracted interest in connection with the following two topics:

Firstly, Happel, who introduced the concept of derived module categories of algebras into representation theory, gave a description of the process of tilting in terms of derived equivalences [7]. Rickard [14] generalized this result by showing that derived equivalences correspond to what he calls tilting complexes. (The problem to develop a Morita theory for derived module categories was suggested by Cline, Parshall and Scott in their paper [2] where they already gave a partial converse to Happel's work.) Thus, derived equivalences can be viewed as very general forms of the well-known Morita-equivalences.

Secondly, Cline, Parshall and Scott (cf. [13]) used the definition of recollement of triangulated categories, given by Beilinson, Bernstein and Deligne in their work on perverse sheaves [1] as a generalization of a well-known situation in topology, to obtain what they call stratification of certain derived module

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categories, in particular for the so-called quasi-hereditary algebras which play an important role in the Cline–Prashall–Scott program [13] dealing with the Kazhdan–Lusztig conjecture and its modular analogue.

The aim of this note is to provide a link between the two concepts of tilting complexes and recollement of derived module categories of rings. Using Rickard's theorem we prove the following main result which gives a necessary and sufficient criterion for the existence of recollement situations (for notation and definitions we refer to Section 2; all rings we are dealing with are supposed to have an identity 1):

Theorem. *Let A be any ring. Then the derived module category* $D^-(\text{Mod-}A)$ *admits recollement relative to the derived module categories of two rings B and C,*

$$D^-(\text{Mod-}B) \leftrightarrows D^-(\text{Mod-}A) \leftrightarrows D^-(\text{Mod-}C)\,,$$

if and only if there exist two partial tilting complexes $\mathfrak{B}$ *in* $K^b(\text{Proj-}A)$ *and* $\mathfrak{C}$ *in* $K^b(P_A)$ *which satisfy*:

(I) $\text{End}_A(\mathfrak{B}) \cong B$,
(II) $\text{End}_A(\mathfrak{C}) \cong C$,
(III) $\text{Hom}^\bullet_A(\mathfrak{C}, \mathfrak{B}) = 0$,
(IV) $\mathfrak{B}^\perp \cap \mathfrak{C}^\perp = \{0\}$.

The proof of this theorem will be given in Section 3; the main ingredients are an application of Rickard's results and a description of subcategories of $D^-(\text{Mod-}A)$, which appear as images respectively kernels of functors in the recollement situation, as perpendicular categories to the partial tilting complexes $\mathfrak{B}$ and $\mathfrak{C}$.

For rings of finite global dimension the bounded derived module category is of much more interest than the unbounded one. In this case we show in Section 4 that the analogue of the above theorem also holds for these categories. However, without assumptions on the finiteness of global dimensions it is not always true that recollements of the unbounded derived categories restrict to recollements of the bounded derived categories (cf. Example 8 in Section 4). In Section 4 we also give some results on restrictions of recollements from the level of the unbounded derived categories to the bounded ones in the case of arbitrary rings without assumptions on the global dimension. Moreover, we prove a criterion for a recollement situation to possess a symmetric recollement situation (where the rings B and C have changed sides).

From the theorem it is now obvious that quasi-hereditary algebras [13, 16] and quasi-hereditary orders [11] admit stratifications of derived module categories. But also the more general results of Cline, Parshall and Scott [3, 4, 12] on recollements involving three derived module categories immediately follow from our theorem as will be shown in Section 5. Note that Cline, Parshall and Scott did not use Rickard's theorem; in the proof of each of their theorems on recollements of derived module categories they gave explicit constructions of functors; this method first led to more general recollement situations where only two of the involved triangulated categories could be identified as derived module categories of certain rings; additional assumptions then were necessary in order to make it possible to identify also the third category as a derived module category. It is inherent in our method that we avoid these difficulties. However, we note that many of the technical arguments used in the proof of our theorem have already appeared in the work of Cline, Parshall and Scott, in particular, in the proofs of Theorem 2.1 in [3] and Theorem 1.1 in [4]. In Section 5 we also collect several other situations where recollements arise in a natural way.

2. Definitions and needed results

In the following we assume the reader to be familiar with the definition of triangulated categories and the interpretation of derived module categories as triangulated categories [8, 10, 13, 17].

First, we want to cite Rickard's fundamental theorem on derived equivalences. If R is any ring (always with 1), then Mod-R denotes the unitary right R-modules (not necessarily finitely generated).

Two rings A and B are called derived equivalent if and only if $D^{\mathrm{b}}(\text{Mod-}A)$ and $D^{\mathrm{b}}(\text{Mod-}B)$ are equivalent (as triangulated categories).

Proj-R denotes the category of projective R-modules, $K^{\mathrm{b}}(\text{Proj-}R)$ the homotopy category of bounded complexes over Proj-R (i.e. the complexes over Proj-R with almost all entries being 0); P_R denotes the category of finitely generated projective R-modules and $K^{\mathrm{b}}(P_R)$ the homotopy category of bounded complexes over P_R.

Theorem (Rickard [14]). *Let A and B be two rings. Then A and B are derived equivalent if and only if the unbounded derived categories $D^-(\text{Mod-}A)$ and $D^-(\text{Mod-}B)$ are equivalent (as triangulated categories) which happens if and only if B is isomorphic to $\text{End}_{K^{\mathrm{b}}(P_A)}(T)$, where T is an object of $K^{\mathrm{b}}(P_A)$ and satisfies:*

(I) $\text{Hom}(T, T[i]) = \{0\}$ *for* $i \neq 0$,

(II) $\text{add}(T)$ *generates* $K^{\mathrm{b}}(P_A)$ *as a triangulated category* □

(Here $\text{add}(T)$ denotes the full subcategory having as objects all direct summands of finite direct sums of copies of T, $[i]$ means translation by i in the triangulated category $D^{\mathrm{b}}(\text{Mod-}A)$.)

In the following we will call an object like the T in the theorem a tilting complex. We will need the following generalization of the notion of tilting complex:

Definition. A *partial tilting complex* over a ring R is a complex T which is in $K^{\mathrm{b}}(\text{Proj-}R)$ and satisfies:

(I) $\mathrm{Hom}(T, T[i]) = \{0\}$ for $i \neq 0$ and
(II) for all indexed families $\{T_i\}_{i\in I}$ of copies of T holds:

$$\bigoplus_{i\in I} \mathrm{Hom}(T, T_i) \overset{\mathrm{nat}}{\cong} \mathrm{Hom}\left(T, \bigoplus_{i\in I} T_i\right).$$

Note that—in contrast to the situation in Rickard's theorem—we do not require T to be a complex over $K^{\mathrm{b}}(P_R)$. In fact, later on we will give an example of a recollement situation where a tilting complex is involved which is not a complex over finitely generated projective modules (Example 9 in Section 4).

Rickard's proof of his theorem consists of two parts. In one part he shows that equivalences between the unbounded derived module categories restrict to equivalences between various other derived categories as $D^{\mathrm{b}}(\text{Mod-}A)$, $K^{\mathrm{b}}(P_A)$ etc. In the other part he gives explicit constructions of functors between the unbounded homotopy categories (which are equivalent to the unbounded derived module categories). In our context it is important that Rickard has constructed a functor F from the unbounded homotopy category $K^{-}(\text{Proj-}B)$ of the endomorphism ring B of T to the unbounded homotopy category $K^{-}(\text{Proj-}A)$ of A and a right adjoint G to F even for a partial tilting complex T. (Note that formally our partial tilting complexes do not satisfy Rickard's assumption, since they need not be complexes of finitely generated projective modules; however, his construction for the embedding uses only the two properties (I) and (II), whereas the construction of the right adjoint depends only on the boundedness of T.) Since the unbounded derived module category $D^{-}(\text{Mod-}A)$ is equivalent to the unbounded homotopy category $K^{-}(\text{Proj-}A)$ we may use Rickard's construction also for this category. (In the following we will not distinguish between $K^{-}(\text{Proj-}A)$ and $D^{-}(\text{Mod-}A)$).

Now we turn to the second topic we are interested in and recall the definition of recollement, given by Beilinson, Bernstein and Deligne in their work on perverse sheaves.

Definition (Beilinson, Bernstein and Deligne [1]). Let $\mathfrak{D}$, $\mathfrak{D}'$ and $\mathfrak{D}''$ be triangulated categories. Then a *recollement of* $\mathfrak{D}$ *relative to* $\mathfrak{D}'$ *and* $\mathfrak{D}''$, diagrammatically expressed by

$$\mathfrak{D}' \leftrightarrows \mathfrak{D} \leftrightarrows \mathfrak{D}'',$$

is given by six exact functors

$$i_* = i_! : \mathfrak{D}' \to \mathfrak{D}, \quad j^* = j^! : \mathfrak{D} \to \mathfrak{D}'', \quad i^*, i^! : \mathfrak{D} \to \mathfrak{D}', \quad j_!, j_* : \mathfrak{D}'' \to \mathfrak{D},$$

which satisfy the following four conditions:

(R1) $(i^*, i_* = i_!, i^!)$ and $(j_!, j^* = j^!, j_*)$ are adjoint triples, i.e., i^* is left adjoint to i_* which is left adjoint to $i^!$ etc., (R2) $i^! j_* = 0$ (and thus $j^* i_* = 0$ and $i^* j_! = 0$),

(R3) i_*, $j_!$ and j_* are full embeddings (and thus $i^* i_* \cong i^! i_* \cong \mathrm{id}(\mathfrak{D}')$ and $j^* j_* \cong j^* j_! \cong \mathrm{id}(\mathfrak{D}'')$),

(R4) any object X in $\mathfrak{D}$ determines distinguished triangles

$$i_! i^! X \to X \to j_* j^* X \to \quad \text{and} \quad j_! j^! X \to X \to i_* i^* X \to$$

(where the morphisms $i_! i^! X \to X$ etc. are the adjunction morphisms).

We will need a weaker form of recollement (cf. [12]): A *right recollement* is said to hold if the lower two rows of a recollement (as defined above) exist and the functors appearing in these two rows (i.e. i_*, $i^!$, $j^!$ and j_*) satisfy all the conditions in the definition above which involve only these functors. Similarly, a *left recollement* is defined via the upper two rows.

In the following we will only be interested in recollements where all the triangulated categories involved are derived module categories of rings, i.e. in recollements of the following type:

$$D^-(\text{Mod-}B) \leftrightarrows D^-(\text{Mod-}A) \leftrightarrows D^-(\text{Mod-}C)$$

or

$$D^b(\text{Mod-}B) \leftrightarrows D^b(\text{Mod-}A) \leftrightarrows D^b(\text{Mod-}C)\,.$$

We note that in the latter situation, for example, the Grothendieck group of A is isomorphic to the direct sum of the Grothendieck groups of B and C (here one uses a result of Grothendieck [6] saying that the Grothendieck group of A is isomorphic to the Grothendieck group of the triangulated category $D^b(\text{Mod-}A)$). Furthermore, A is of finite global dimension if and only if B and C are.

It should be noted that many finite-dimensional algebras are known to admit recollements on the derived level (cf. Section 5) but there also are examples of (not necessarily local) algebras which do not admit recollements (for an example with infinite global dimension cf. [18], for examples of finite global dimension cf. [9]).

For later use in Section 5 we now cite results of Geigle and Lenzing [5].

Definition (Geigle and Lenzing [5]). Let R and S be two rings and $\varphi : R \to S$ a homomorphism. Then φ is called a *homological epimorphism* if and only if the following conditions are satisfied:

(a) the multiplication map $S \otimes_R S \to S$ is an isomorphism;

(b) for all $i \geq 1$ $\mathrm{Tor}_i^R(S_R, {}_RS) = 0$.

(Note that condition (a) just states that φ is an epimorphism of rings.)

The connection between homological epimorphisms and the context we are interested in is given by the following result of Geigle and Lenzing:

Theorem (Geigle and Lenzing [5, 4.4]). *Let R and S be two rings and $\varphi : R \to S$ a homomorphism. Then the following conditions are equivalent*:

(1) *φ is a homological epimorphism*,

(2) *for all right S-modules M and all left S-modules N and for all $i \geq 0$ the natural map $\mathrm{Tor}_i^R(M, N) \to \mathrm{Tor}_i^S(M, N)$ is an isomorphism*,

(3) *for all left S-modules M and N and for all $i \geq 0$ the natural map $\mathrm{Ext}_S^i(M, N) \to \mathrm{Ext}_R^i(M, N)$ is an isomorphism*,

(4) *the induced functor of derived module categories*

$$D^b(\varphi_*) : D^b(\text{Mod-}S) \to D^b(\text{Mod-}R)$$

is a full embedding. □

Also, Geigle and Lenzing proved several other characterizations of homological epimorphisms in terms of conditions involving Hom and Ext (respectively ⊗ and Tor) of certain modules.

In the following we usually will abbreviate $D^b(\text{Mod-}A)$ by $D^b(A)$, $D^-(\text{Mod-}A)$ by $D^-(A)$, etc. The term triangle always means distinguished triangle. Since for each ring A there are full embeddings between the various derived categories and homotopy categories associated to A, we may denote homomorphisms in any of these categories just by $\mathrm{Hom}_A(-, -)$.

We need some more notation: $\mathrm{Hom}_A^\bullet(-, -)$ denotes the collection of all sets of translated homomorphisms in the derived module category, i.e. $\mathrm{Hom}_A^\bullet(X, Y)$ is the collection having as nth element the set $\mathrm{Hom}_A(X, Y[n])$, where $[n]$ denotes as above translation by n; $\mathrm{Hom}_A^\bullet(X, Y) = 0$ means $\mathrm{Hom}_A(X, Y[n]) = 0$ for all $n \in \mathbb{Z}$ (including $n = 0$). By a *right perpendicular category* $X^\perp$ to an object $X \in D^-(\text{Mod-}A)$ we mean the full triangulated subcategory of $D^-(\text{Mod-}A)$ generated by all objects Y with $\mathrm{Hom}_A^\bullet(X, Y) = 0$, similarly left perpendicular categories are defined. The right (respectively left) perpendicular category of a subcategory is the intersection of the right (respectively left) perpendicular categories of all its objects.

We note that similarly defined perpendicular categories within module categories have been defined and studied by Geigle and Lenzing [5] and independently by Schofield.

The kernel $\ker(F)$ of a functor F between triangulated categories denotes the full triangulated subcategory of the domain of F which is generated by the objects X with $F(X) = 0$. Similarly the image $\mathrm{im}(F)$ is generated by all $F(X)$ where X runs through the domain of F.

3. The main theorem and its proof

Theorem 1. *Let A be any ring. Then the unbounded derived module category $D^-(\text{Mod-}A)$ admits recollement relative to the unbounded derived module* *categories of two rings B and C,*

$$D^-(\text{Mod-}B) \leftrightarrows D^-(\text{Mod-}A) \leftrightarrows D^-(\text{Mod-}C),$$

if and only if there exist two partial tilting complexes $\mathfrak{B}$ *in* $K^b(\text{Proj-}A)$ *and* $\mathfrak{C}$ *in* $K^b(P_A)$ *which satisfy:*

(I) $\text{End}_A(\mathfrak{B}) \cong B$,

(II) $\text{End}_A(\mathfrak{C}) \cong C$,

(III) $\text{Hom}^\bullet_A(\mathfrak{C}, \mathfrak{B}) = 0$,

(IV) $\mathfrak{B}^\perp \cap \mathfrak{C}^\perp = \{0\}$.

Moreover, if recollement holds with these data then we have:

$\mathfrak{B}^\perp = \ker(i^!) = \text{im}(j_*)$,

$\mathfrak{C}^\perp = \ker(j^*) = \text{im}(i_*)$,

${}^\perp(\mathfrak{C}^\perp) = \ker(i^*) = \text{im}(j_!)$,

$\mathfrak{B}^\perp = (\mathfrak{C}^\perp)^\perp$,

$\mathfrak{C}^\perp = {}^\perp(\mathfrak{B}^\perp)$.

Proof. ($\Rightarrow$) Assume recollement holds with notation as above. Denote by $\mathfrak{B}$ respectively $\mathfrak{C}$ the images of B respectively C under the full embeddings i_* respectively $j_!$. First we have to show that $\mathfrak{B}$ and $\mathfrak{C}$ lie already in $K^b(\text{Proj-}A)$. Here we use the following criterion which is analogous to a result of Rickard [14, proof of Proposition 8.1]: An object $X \in D^-(A)$ lies in $K^b(\text{Proj-}A)$ (bounded complexes over Proj-A, the projective A-modules) if and only if for each object $Y \in D^-(A)$ there exists a natural number N (depending on Y) such that for all $n \geq N$: $\text{Hom}_A(X, Y[n]) = 0$. (To prove this criterion one considers for a given X the direct sum of objects $X^i[i]$, where X^i is the same as X on places with index smaller than i and 0 elsewhere.) In our special situation we have

$$\text{Hom}_A(\mathfrak{B}, Y[n]) = \text{Hom}_B(B, i^! Y[n])$$

(note $i^!$ commutes by definition with translation), hence N clearly exists. The same argument shows that $\mathfrak{C}$ is in $K^b(\text{Proj-}A)$. Our assertion is that $\mathfrak{C}$ moreover is a complex over the finitely generated A-modules. By a result of Rickard [14, proof of 6.3] we have to show that $\text{Hom}_A(\mathfrak{C}, -)$ preserves arbitrary direct sums. Since $j_!$ has a right adjoint $j^* = j^!$ and j^* has a right adjoint j_* and functors which have a right adjoint commute with direct sums, we have:

$$\begin{aligned}\text{Hom}_A(\mathfrak{C}, \textstyle\bigoplus Y_i) &\cong \text{Hom}_A(\mathfrak{C}, j_! j^! \textstyle\bigoplus Y_i) \cong \text{Hom}_A(\mathfrak{C}, j_! \textstyle\bigoplus j^! Y_i) \\ &\cong \text{Hom}_C(C, \textstyle\bigoplus j^! Y_i) \cong \textstyle\bigoplus \text{Hom}_C(C, j^! Y_i) \\ &\cong \textstyle\bigoplus \text{Hom}_A(\mathfrak{C}, j_! j^! Y_i) \cong \textstyle\bigoplus \text{Hom}_A(\mathfrak{C}, Y_i).\end{aligned}$$

From the definition of $\mathfrak{B}$ and $\mathfrak{C}$ and the fact that i_* and $j_!$ are full embeddings it follows that $\mathfrak{B}$ and $\mathfrak{C}$ satisfy condition (I) in the definition of partial tilting complex, since their translated endomorphisms are via the full embeddings just the translated endomorphisms of B (respectively C). Moreover, it follows that $\mathrm{End}_A(\mathfrak{B}) \cong B$ and $\mathrm{End}_A(\mathfrak{C}) \cong C$.

Next we have to consider condition (II) for partial tilting complexes. As above we use that a functor which has a right adjoint commutes with direct sums; hence we have for each set $\{\mathfrak{C}_\lambda = (j_!C)_\lambda\}$ of copies of $\mathfrak{C}$ that

$$\mathrm{Hom}_A(\mathfrak{C}, \bigoplus \mathfrak{C}_\lambda) \cong \mathrm{Hom}_A(j_!C, j_!(\bigoplus C_\lambda)) \cong \mathrm{Hom}_C(C, \bigoplus C_\lambda)$$
$$\cong \bigoplus \mathrm{Hom}_C(C, C_\lambda) \cong \bigoplus \mathrm{Hom}_A(\mathfrak{C}, \mathfrak{C}_\lambda);$$

similarly for $\mathfrak{B}$.

Thus we have shown that $\mathfrak{B}$ and $\mathfrak{C}$ are partial tilting complexes. (We note here that the complex j_*C is not necessarily a partial tilting complex, cf. Example 8 in Section 4.)

Also $\mathrm{Hom}^\bullet_A(\mathfrak{C}, \mathfrak{B}) = \mathrm{Hom}^\bullet_A(j_!C, i_*B) = \mathrm{Hom}^\bullet_C(C, j^*i_*B) = 0$.

Finally, assume X to lie in $\mathfrak{B}^\perp \cap \mathfrak{C}^\perp$. From the standard triangle $i_!i^!X \to X \to j_*j^*X \to$ and a long exact homology sequence it follows (since $\mathrm{Hom}^\bullet_A(\mathfrak{B}, j_*j^*X) \cong \mathrm{Hom}^\bullet_B(B, i^!j_*j^*X) = 0$) that $\mathrm{Hom}^\bullet_A(\mathfrak{B}, i_!i^!X) = 0$, but this means $0 = \mathrm{Hom}^\bullet_B(B, i^!i_!i^!X) \cong \mathrm{Hom}^\bullet_B(B, i^!X)$, hence $i^!X = 0$ and thus $X \cong j_*j^*X$. Using similar arguments with $\mathfrak{B}$ replaced by $\mathfrak{C}$ we infer $X = 0$.

($\Leftarrow$) Assume the existence of partial tilting complexes $\mathfrak{B}$ and $\mathfrak{C}$ with the properties (I) to (IV).

(1) By Rickard's results there exists a full embedding $i_* = i_! : D^-(B) \to D^-(A)$ which sends B to $\mathfrak{B}$ and has a right adjoint $i^! : D^-(A) \to D^-(B)$; also there exists a full embedding $j_! : D^-(C) \to D^-(A)$ which sends C to $\mathfrak{C}$ and has a right adjoint $j^! = j^* : D^-(A) \to D^-(C)$.

Our main task is to construct the two missing functors and to show they satisfy the required adjunction conditions. If we already knew that there is a recollement then these functors would be described by the (unique) standard triangles appearing in the axiom (R4), hence we first try to find such triangles; then we will define i^* and j_* via these triangles.

(2) *Statement*: $\mathfrak{B}^\perp = \mathrm{kernel}(i^!)$. *Proof*: By adjointness of i_* and $i^!$ we have for any X in $D^-(A)$ that $\mathrm{Hom}^\bullet_A(\mathfrak{B}, X) \cong \mathrm{Hom}^\bullet_B(B, i^!X)$, hence $i^!X$ is acyclic (which means 0 in the derived category) if and only if X lies in $\mathfrak{B}^\perp$.

In the same way we get the following:

Statement: $\mathfrak{C}^\perp = \mathrm{kernel}(j^*)$.

(3) Choose an object X in $D^-(A)$ and complete the adjunction morphism $i_*i^!X \to X$ to a triangle $i_*i^!X \to X \to Y \to$. This triangle will be seen to be the desired one.

Statement: Y is an object of $\mathfrak{B}^\perp$. *Proof*: Since i_* is a full embedding, adjointness yields

$$\mathrm{Hom}^\bullet_A(\mathfrak{P}, i_* i^! X) \cong \mathrm{Hom}^\bullet_B(B, i^! X) \cong \mathrm{Hom}^\bullet_A(\mathfrak{P}, X)\,.$$

By a long exact homology sequence follows $\mathrm{Hom}^\bullet_A(\mathfrak{P}, Y) = 0$.

Similarly we have a triangle $j_! j^! X \to X \to Z \to$, where Z is an object of $\mathfrak{C}^\perp$.

(4) *Statement*: image$(i_*) = \mathfrak{C}^\perp$. *Proof*: By assumption (III) we know that $\mathfrak{P}$ (and of course each direct summand of $\mathfrak{P}$) lies in $\mathfrak{C}^\perp$. $\mathfrak{C}^\perp$ is a triangulated category, hence closed under taking direct sums of finitely many objects; since $\mathfrak{C}$ is in $K^b(P_A)$, $\mathfrak{C}^\perp$ is even closed under taking arbitrary direct sums (cf. [14, 2.1]), hence it contains also Add $\mathfrak{P}$ (i.e. all direct summands of direct sums of $\mathfrak{P}$). Now $\mathfrak{C}^\perp$ is a full triangulated subcategory, thus it contains the full triangulated subcategory generated by Add $\mathfrak{P}$, hence it contains $i_*(K^b(\text{Proj-}B))$, since i_* is exact.

Let X be an object in $i_*(D^-(B))$, we have to show that X lies in $\mathfrak{C}^\perp$. Assume the contrary; since $\mathfrak{C}^\perp$ is closed under translation, we may assume that there is a nontrivial map f from $\mathfrak{C}$ to X. But $\mathfrak{C}$ is a finite complex, hence f induces a nontrivial map from $\mathfrak{C}$ to a (large enough) bounded complex which is a truncation of X. Now, by Rickard's construction of i_*, this truncated complex is also the truncated complex of the i_*-image of a bounded complex Y. Hence f induces a nontrivial map from $\mathfrak{C}$ to $i_*(Y)$, so we have reached a contradiction.

To see the other inclusion we choose an object X in $\mathfrak{C}^\perp$ and look at the triangle constructed in (3): $i_* i^! X \to X \to Y \to$ with Y in $\mathfrak{P}^\perp$. We already know that the first term and by assumption of X also the second term of this triangle lies in $\mathfrak{C}^\perp$, hence by a long exact homology sequence it follows that also Y lies in $\mathfrak{C}^\perp$. But now by assumption (IV) Y has to be 0, hence X is isomorphic to $i_* i^! X$. (Note that an analogous statement for $j_!$ does not hold, but cf. the statement on im(j_*) in (7).)

(5) Let X be an object in $D^-(A)$. In (3) we constructed a triangle $j_! j^! X \to X \to Z \to$ which starts with the adjunction morphism and satisfies $Z \in$ obj$(\mathfrak{C}^\perp)$. From the statement in (4) and the property of i_* to be a full embedding it follows that we can write Z uniquely as $i_*(U)$ for some U in $D^-(B)$. U is unique up to isomorphism, thus an application of the axiom of choice on relations on classes gives us a function which defines i^* on objects: $i^*(X)$ is an element of the isomorphism class of U. By an axiom of triangulated categories we can define i^* on morphisms, too, by using diagrams like the following:

$$\begin{array}{ccc} j_! j^! X \to & X \to & i_*(i^* X) \to \\ \downarrow & \downarrow & \\ j_! j^! Y \to & Y \to & i_*(i^* Y) \to \end{array}$$

However, in order to see that the function i^* is a functor, we need to know more, viz. that there is no choice in the definition of i^* on morphisms.

Statement: For any V in $D^-(B)$, $\mathrm{Hom}^\bullet_A(j_! j^! X, i_* V) = 0$. *Proof*: Adjointness of $j_!$ and $j^!$ and im$(i_*) = \mathfrak{C}^\perp = $ kernel$(j^!)$.

Now it follows from a long exact homology sequence that

$$\mathrm{Hom}_A^\bullet(X, i_*i^*Y) \cong \mathrm{Hom}_A^\bullet(i_*i^*X, i_*i^*Y),$$

thus the relation used to define i^* on morphisms is already a function, hence i^* is indeed functorial.

Statement: i^* is left adjoint to i_*. *Proof*: This follows from the previous statement by a long exact homology sequence:

$$\mathrm{Hom}_B^\bullet(i^*M, N) \cong \mathrm{Hom}_A^\bullet(i_*i^*M, i_*N) \cong \mathrm{Hom}_A^\bullet(M, i_*N)$$

for all objects M in $D^-(A)$ and N in $D^-(B)$.

(6) We have to show that all functors involved are exact, i.e. commute with the translation functor and carry triangles to triangles. Since we will use exactness of j^* in the proof of the existence of j_*, it is convenient now to give a proof for the functors already constructed; we note however that the same argument also works for the functor j_* which yet has to be constructed. Rickard has shown that the embeddings i_* and $j_!$ which he constructed are exact. It is clear, that all the other functors commute with the respective translation functors, too. Obviously functors which are quasi-inverse (i.e. inverse up to a natural equivalence) to an equivalence (which is by definition exact) are exact, too. Hence certain restrictions of the other functors are exact as well. It is now well known how to show the exactness of, for example, i^*. (We copy the following argument from [3, proof of theorem 2.1].) Choose a triangle $X \to Y \to Z \to$ in $D^-(A)$. Using the exactness of i_* and all its quasi-inverses we have to show that the sextuple $i_*i^*X \to i_*i^*Y \to i_*i^*Z \to$ is a triangle. Now consider the following commutative square:

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ i_*i^*X & \to & i_*i^*Y \end{array}$$

where all morphisms are those we considered above.

By the 9-lemma of [1, 1.1.11], this square can be embedded in the following diagram where all rows and columns are triangles:

$$\begin{array}{ccccccc} X' & \longrightarrow & Y' & \longrightarrow & Z' & \to \\ \downarrow & & \downarrow & & \downarrow & \\ X & \longrightarrow & Y & \longrightarrow & Z'' & \to \\ \downarrow & & \downarrow & & \downarrow & \\ i_*i^*X & \to & i_*i^*Y & \to & Z''' & \to \\ \downarrow & & \downarrow & & \downarrow & \end{array}$$

What we have to do now is to apply several times the 5-lemma and use the uniqueness of certain maps which has been proved above in order to see that the triangles in this diagram are up to isomorphism the ones we want to have. In particular, the last row is up to a unique isomorphism the sextuple from above. (The same argument works for j^*, since its left adjoint $j_!$ is a full embedding.)

(7) The proof of the existence of j_* is similar to (5), but a little bit more difficult, since j_* goes into the opposite direction. Again we start with a triangle constructed in (3): $i_!i^!X \to X \to Y \to$ with Y lying in $\mathfrak{B}^{\perp}$.

Statement: $\mathrm{Hom}^{\bullet}_A(i_!i^!X, V) = 0$ for all $V \in \mathfrak{B}^{\perp}$. *Proof*: adjointness.

Hence we may define a functor $F : D^-(A) \to D^-(A)$ with $F(X) = Y$ by proceeding as above. Our aim is to show that F factorizes via j^*.

Statement: $\mathrm{im}(F) = \mathfrak{B}^{\perp}$. *Proof*: triangle in the definition of F.

Statement: $\ker(F) = \ker(j^*)$. *Proof*: With the notation above we have to show that $j^*X = 0$ is equivalent to $Y = 0$. But $j^*X = 0$ means by (2) X lies in $\mathfrak{C}^{\perp}$ and this means by (4) X lies in the image of i_*, and this holds if and only if X is by the adjunction morphism isomorphic to $i_!i^!X$, thus $Y = 0$.

Statement: $j^*i_! = 0$. *Proof*: contained in the proof of the previous statement.

Statement: $F(X) \cong F(j_!j^*X)$. *Proof*: Consider the following diagram:

$$\begin{array}{ccccc}
i_!i^!j_!j^*X & \longrightarrow & j_!j^*X & \to & F(j_!j^*X) \to \\
\downarrow & & \downarrow & & \downarrow \\
i_!i^!X & \longrightarrow & X & \longrightarrow & F(X) \to \\
 & & \downarrow & & \downarrow \\
 & & i_*i^*X & \longrightarrow & Q
\end{array}$$

where the maps are induced from the adjunction maps and Q completes the last column to a triangle. From a long exact homology sequence it follows that Q lies in $\mathfrak{B}^{\perp}$. The maps in the diagram induce the following isomorphisms:

$$\begin{aligned}\mathrm{Hom}^{\bullet}_A(\mathfrak{C}, F(j_!j^*X)) &\cong \mathrm{Hom}^{\bullet}_A(\mathfrak{C}, j_!j^*X) \\ &\cong \mathrm{Hom}^{\bullet}_A(\mathfrak{C}, X) \cong \mathrm{Hom}^{\bullet}_A(\mathfrak{C}, F(X)) .\end{aligned}$$

Hence Q lies in $\mathfrak{C}^{\perp}$, too, thus $Q = 0$.

Statement: F factorizes via j^*, i.e. $F = j_*j^*$, where j_* is a functor from $D^-(C)$ to $D^-(A)$. *Proof*: obvious.

Statement: $i^*j_! = 0$. *Proof*: For X in $D^-(C)$ we have:

$$\begin{aligned}\mathrm{Hom}^{\bullet}_B(i^*j_!X, i^*j_!X) &\cong \mathrm{Hom}^{\bullet}_A(j_!X, i_*i^*j_!X) \\ &\cong \mathrm{Hom}^{\bullet}_C(X, j^*i_*i^*j_!X) \cong 0 ,\end{aligned}$$

since $j^*i_* = 0$.

Statement: For X in $D^-(C)$, $X \cong j^*j_!X$. *Proof*: Consider the triangle $j_!j^*j_!X \to j_!X \to i_*i^*j_!X = 0 \to$ and note that $j_!$ is a full embedding.

Statement: For X in $D^-(C)$, $X \cong j^*j_*X$. *Proof*: Consider the triangle $i_!i^!j_!X \to j_!X \to j_*j^*j_!X \to$, apply the exact functor j^* and use that by (5) $j^*i_! = 0$.

Statement: j_* is a full embedding. *Proof*: Let X be an object of $D^-(C)$. Considering the long exact homology sequence to the triangle $j_!j^*j_*X \to j_*X \to i_*i^*j_*X \to$ and using $i^!j_* = 0$ (since $\mathrm{im}(j_*) = \mathrm{im}(F) = \ker(i^!)$) we get

$$\mathrm{Hom}^\bullet_A(j_*X, j_*X) \cong \mathrm{Hom}^\bullet_A(j_!j^*j_*X, j_*X) \cong \mathrm{Hom}^\bullet_C(j^*j_*X, j^*j_*X) \cong \mathrm{Hom}^\bullet_C(X, X).$$

Statement: j_* satisfies $\mathrm{im}(j_*) = \mathfrak{B}^\perp$, and is a right adjoint for j^*. *Proof*: The first part follows from the corresponding statement for F. Adjointness follows from

$$\mathrm{Hom}^\bullet_C(j^*M, N) \cong \mathrm{Hom}^\bullet_A(j_*j^*M, j_*N) \cong \mathrm{Hom}^\bullet_A(M, j_*N)$$

for all objects M in $D^-(A)$ and N in $D^-(C)$.

(8) The remaining statements in the axioms of recollement and the equalities between kernels and images follow now easily. As an example we prove the statements $\mathfrak{B}^\perp = (\mathfrak{C}^\perp)^\perp$ and $\mathfrak{C}^\perp = {}^\perp(\mathfrak{B}^\perp)$: Assume X lies in $\mathfrak{B}^\perp$, hence $i^!X = 0$ and thus $X \cong j_*j^*X$. Now for each Y in $\mathfrak{C}^\perp$ we have $\mathrm{Hom}_A(Y, X) \cong \mathrm{Hom}_C(j^*Y, j^*X)$, but $j^*Y = 0$. The argument can be reserved, and the other statement follows in the same way. □

4. Applications to other types of recollements

First, we observe that for more general recollements involving triangulated categories which are not necessarily derived module categories, our construction goes through if one already knows that there are two full embeddings with right adjoints satisfying the equality kernel(j^*) = image(i_*).

Of course, all the mentioned equalities of kernels and images hold also in the more general recollement situation.

However, the main purpose of this section is to apply Theorem 1 to recollements involving derived module categories which are not of type $D^-(A)$. In particular, we want to show that for rings of finite global dimension there holds an analogous theorem for recollements of the bounded derived module category D^b (which is of great interest in this case, cf. [7, 8]). However, for rings of infinite global dimension we show by example that an analogous statement for the bounded derived categories does not hold. (The reason is that for such rings the categories D^b and K^b are not isomorphic whereas D^- and K^- always are isomorphic.)

Hence we first have to consider restrictions of functors from larger derived module categories to smaller ones.

Lemma 2. *Let A and B rings and $F: D^-(A) \to D^-(B)$ an exact functor of triangulated categories which has a left adjoint G. Then F sends complexes of bounded homology to complexes of bounded homology, hence F restricts to a functor $D^b(A) \to D^b(B)$.*

Proof. We use the following criterion of Rickard [14, proof of 6.1]: An object X in $K^- = D^-$ has bounded homology if and only if for any Y in D^- there exists some N such that $\mathrm{Hom}_{K^-}(Y, X[n]) = 0$ for all $n < N$. Let now X be an object in $D^b(A)$; then for any Y in $D^-(B)$ we have:

$$\mathrm{Hom}_{D^-(B)}(Y, FX[n]) \cong \mathrm{Hom}_{D^-(A)}(GY, X[n]),$$

thus by Rickard's criterion it follows that FX has bounded homology. □

Lemma 3. *Let A and B be rings and $F: D^-(A) \to D^-(B)$ or $F: D^b(A) \to D^b(B)$ an exact functor between triangulated categories which has a right adjoint. Then F sends bounded complexes to bounded complexes, hence it restricts to a functor between $K^b(A)$ and $K^b(B)$.*

Proof. For the D^--case we use again the criterion of Rickard [14, 8.1] which was mentioned at the beginning of the proof of Theorem 1. For D^b there is an analogous criterion of Rickard [14, proof of Proposition 6.2]. □

Proposition 4. *Let A, B and C be rings and assume there is a recollement situation of the following type:*

$$D^-(\text{Mod-}B) \leftrightarrows D^-(\text{Mod-}A) \leftrightarrows D^-(\text{Mod-}C).$$

Then this recollement restricts to a right recollement of the type

$$D^b(\text{Mod-}B) \rightleftarrows D^b(\text{Mod-}A) \rightleftarrows D^b(\text{Mod-}C)$$

and a left recollement of the type

$$K^b(\text{Proj-}B) \leftrightarrows K^b(\text{Proj-}A) \leftrightarrows K^b(\text{Proj-}C).$$

If at least one of the rings A and C has finite global dimension, then the given recollement restricts to the following recollement:

$$D^b(\text{Mod-}B) \leftrightarrows D^b(\text{Mod-}A) \leftrightarrows D^b(\text{Mod-}C).$$

If at least one of the rings A and B has finite global dimension, then the given recollement restricts to the following recollement:

$$K^{b}(\text{Proj-}B) \leftrightarrows K^{b}(\text{Proj-}A) \leftrightarrows K^{b}(\text{Proj-}C).$$

Proof. The first part follows from the lemmas.

Assume C to have finite global dimension. Then $K^{b}(C)$ is equivalent to $D^{b}(C)$ hence $j_!$ restricts to D^{b}. Let X be an object in $D^{-}(A)$ which has bounded homology. In the standard triangle $j_!j^!X \to X \to i_*i^*X \to$ the two objects X and $j_!j^!X$ have bounded homology, therefore also the third one, i_*i^*X, has bounded homology. If i^*X would have unbounded homology then Rickard's criterion (see the proof of Lemma 2 above) would lead to a contradiction, since i_* is a full embedding; hence i^* restricts to D^{b}.

Similarly, if B has finite global dimension, then $i^!$ sends bounded complexes to complexes of bounded homology which are by assumption isomorphic to bounded complexes. Hence the assertion follows again from a standard triangle.

The remaining assertions follow if we can show that B and C have finite global dimension if A has. But this follows from the fact that the restrictions of i_* and j_* to the bounded derived categories are full embeddings. □

For recollement situations of the type

$$D^{b}(\text{Mod-}B) \leftrightarrows D^{b}(\text{Mod-}A) \leftrightarrows D^{b}(\text{Mod-}C)$$

it is well known (cf. [18]) that A is of finite global dimension if and only if B and C are so. Thus from the proposition and its proof we get the following:

Corollary 5. *Let A, B and C be rings and assume there is a recollement of the following type*:

$$D^{-}(\text{Mod-}B) \leftrightarrows D^{-}(\text{Mod-}A) \leftrightarrows D^{-}(\text{Mod-}C).$$

Then A is of finite global dimension if and only if B and C so are. □

For recollement situations on the level of the bounded derived categories one can copy the $(\Rightarrow)$-part of the proof of Theorem 1. (To show that $\mathfrak{B}$ and $\mathfrak{C}$ are bounded complexes one has to replace the mentioned criterion of Rickard by another criterion of Rickard [14, Section 6], which was already used in the proof of Lemma 2 above.) Thus we get the following:

Corollary 6. *Let A be any ring. Assume the derived module category* $D^{b}(\text{Mod-}A)$ *admits recollement relative to the derived module categories of two rings B and C*

$$D^{b}(\text{Mod-}B) \leftrightarrows D^{b}(\text{Mod-}A) \leftrightarrows D^{b}(\text{Mod-}C).$$

Then there is also recollement of the following type:

$$D^-(\text{Mod-}B) \mathrel{\substack{\leftarrow\\ \rightarrow\\ \leftarrow}} D^-(\text{Mod-}A) \mathrel{\substack{\leftarrow\\ \rightarrow\\ \leftarrow}} D^-(\text{Mod-}C)\,. \qquad \square$$

If we already know that one of the rings A and C has finite global dimension, we can summarize:

Theorem 7. *Let A, B and C be rings. Assume at least one of the rings A and C has finite global dimension. Then the following conditions are equivalent:*

(I) *The unbounded derived module category $D^-(\text{Mod-}A)$ admits recollement relative to the unbounded derived module categories of B and C,*

$$D^-(\text{Mod-}B) \mathrel{\substack{\leftarrow\\ \rightarrow\\ \leftarrow}} D^-(\text{Mod-}A) \mathrel{\substack{\leftarrow\\ \rightarrow\\ \leftarrow}} D^-(\text{Mod-}C)\,.$$

(II) *The bounded derived module category $D^b(\text{Mod-}A)$ admits recollement relative to the bounded derived module categories of B and C,*

$$D^b(\text{Mod-}B) \mathrel{\substack{\leftarrow\\ \rightarrow\\ \leftarrow}} D^b(\text{Mod-}A) \mathrel{\substack{\leftarrow\\ \rightarrow\\ \leftarrow}} D^b(\text{Mod-}C)\,.$$

(III) *There exist two partial tilting complexes $\mathfrak{B}$ in $K^b(\text{Proj-}A)$ and $\mathfrak{C}$ in $K^b(P_A)$ which satisfy:*

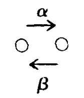
(i) $\text{End}_A(\mathfrak{B}) \cong B$,
(ii) $\text{End}_A(\mathfrak{C}) \cong C$,
(iii) $\text{Hom}^\bullet_A(\mathfrak{C}, \mathfrak{B}) = 0$,
(iv) $\mathfrak{B}^\perp \cap \mathfrak{C}^\perp = \{0\}$.

Moreover, if recollement holds with these data then we have in both recollement situations for the respective functors:

$\mathfrak{B}^\perp = \ker(i^!) = \text{im}(j_*)$,
$\mathfrak{C}^\perp = \ker(j^*) = \text{im}(i_*)$,
${}^\perp(\mathfrak{C}^\perp) = \ker(i^*) = \text{im}(j_!)$,
$\mathfrak{B}^\perp = (\mathfrak{C}^\perp)^\perp$,
$\mathfrak{C}^\perp = {}^\perp(\mathfrak{B}^\perp)$. $\square$

It remains to show by example that for rings of infinite global dimension there are recollements on the unbounded level which do not restrict to the bounded level. We also have to show by example that the assumption on $\mathfrak{B}$ cannot be strengthened, in particular $\mathfrak{B}$ need not be a complex of finitely generated projective A-modules.

Example 8. (This example is based on an example of Rickard [15].) Let k be a field and A the finite-dimensional k-algebra which is defined by the following quiver:



with the relation $\alpha\beta\alpha = 0$. It has the two indecomposable projectives:

$$P(a) = \begin{bmatrix} a \\ b \\ a \\ b \end{bmatrix} \quad \text{and} \quad P(b) = \begin{bmatrix} b \\ a \\ b \end{bmatrix}.$$

The global dimension of A is infinite.

$\mathfrak{B} := S(a)$ (the simple top of $P(a)$) and $\mathfrak{C} := P(b)$ are partial tilting complexes (as a complex $\mathfrak{B}$ is equal to $0 \to P(b) \to P(a) \to 0$). The endomorphism rings B and C are local k-algebras of k-dimension 1 respectively 2. Obviously there are no translated homomorphisms from $\mathfrak{C}$ to $\mathfrak{B}$; moreover, there is an exact sequence of the form $0 \to P(b) \oplus P(b) \to A \to S(a) \to 0$, which shows that only 0 can be right perpendicular to both $\mathfrak{B}$ and $\mathfrak{C}$. Thus from Theorem 1 we get a recollement on the unbounded level:

$$D^{-}(\text{Mod-}B) \leftrightarrows D^{-}(\text{Mod-}A) \leftrightarrows D^{-}(\text{Mod-}C).$$

Note that C has infinite global dimension.

Let S be the simple C-module. We show that there is no embedding $j_! : D^{-}(\text{Mod-}C) \to D^{-}(\text{Mod-}A)$ which sends C to $\mathfrak{C}$ and S to a complex of bounded homology, hence in any recollement situation as above the functor $j_!$ does not restrict to the bounded derived category. (The following short argument is due to the referee.)

Assume $j_!$ exists; then it sends the triangle $S \to C \to S \to$ to a triangle $j_!S \to j_!C \to j_!S \to$. From the long exact sequence of homology for this triangle follows $\text{Hom}^i(j_!S, -) \cong \text{Hom}^{i+1}(j_!S, -)$ for $i < -1$ or $i \geq 1$, thus $j_!S$—which is by assumption a complex of bounded homology—can have homology only in degree 0. So we have a module $X \cong j_!S$ and a short exact sequence $0 \to X \to P(b) \to X \to 0$, which is clearly impossible.

We note however, that this algebra A admits recollement situations on the unbounded derived level which do restrict to the bounded derived level (choose for example $\mathfrak{B} := P(a)$ and $\mathfrak{C} := S(a)$).

The example shows that for rings of infinite global dimension one cannot use Rickard's construction to find recollement situations on the level of the bounded derived categories. Since any recollement situation—on any level of the derived categories—provides a lot of information, it seems that for rings of infinite global dimension the unbounded derived category is more interesting than the bounded one.

Example 9. In [14] Rickard showed that the tilting complexes which belong to derived equivalences between two rings are always in the category of bounded complexes over finitely generated projective modules. In this example we show that in recollement situations there may occur partial tilting complexes which are not in this category since some projectives occur which are not finitely generated. Moreover, we show that the complex j_*C in a recollement situation need not be a bounded complex (while it always has bounded homology by Lemma 2), thus in the theorem it is essential to work with the complex $j_!C$. Therefore, in some recollement situations there is a remarkable lack of symmetry. However, after this example we will give a criterion for a recollement to admit a 'symmetry' in the sense defined below.

Let k be any field and V a k-vectorspace of infinite dimension.

A is the infinite-dimensional k-algebra with two projectives, one of them being simple, the other having simple top and a socle which is as a vectorspace isomorphic to V:

$$A = \begin{bmatrix} k & V \\ 0 & k \end{bmatrix}.$$

Choose $\mathfrak{C}$ the simple projective A-module and $\mathfrak{B}$ the other simple A-module. Then it is easy to see that $\mathfrak{B}$ and $\mathfrak{C}$ satisfy the assumptions of Theorem 1, hence the derived module category $D^-(\text{Mod-}A)$ admits recollement where both sides are isomorphic to the derived category of k-vectorspaces. However, $\mathfrak{B}$ is not in the category of complexes over finitely generated projective A-modules (while it is finitely generated as an A-module). Now assume there would be a recollement situation with $\mathfrak{B}$ and j_*C playing the roles of $\mathfrak{C}$ respectively $\mathfrak{B}$. To get a contradiction we just have to remember that in each recollement situation the partial tilting complex $\mathfrak{C}$ is in $K^b(P_A)$.

We note that for this algebra A there is another recollement (cf. the example after Corollary 15) with $\mathfrak{B}$ and $\mathfrak{C}$ satisfying the assumptions of the following criterion for the existence of a 'symmetric' recollement situation.

Here we call two recollement situations (left- respectively right-) symmetric if the complexes $\mathfrak{B}' = j_*C$ and $\mathfrak{C}' = \mathfrak{B}$ in the first situation play the role of $\mathfrak{B}$ and $\mathfrak{C}$ in the second situation.

Theorem 10. *Assume there is a recollement situation as in Theorem* 1. *Then* $\mathfrak{B}$ *and* j_*C *play the role of* $\mathfrak{C}$ *and* $\mathfrak{B}$ *in a right-symmetric recollement situation if and only if* $\mathfrak{B}$ *is in* $K^b(P_A)$ *and* j_*C *is a bounded complex.*

Proof. From Theorem 1 and its proof it follows that the conditions are necessary. Conversely, assume the conditions are satisfied. Then most of the needed properties of $\mathfrak{B}$ and j_*C follow from the assumed recollement situation by arguments similar to those in the proof of Theorem 1. What remains to be shown is that j_*C satisfies condition (II) in the definition of a partial tilting complex. Denote j_*C by $\mathfrak{T}$ and choose a family of copies $\{\mathfrak{T}_i\}$ of $\mathfrak{T}$. Since $\text{Im}(j_*) = \mathfrak{B}^\perp$ and $\mathfrak{B}^\perp$ is closed under taking direct sums (since $\mathfrak{B}$ is in $K^b(P_A)$) the direct sum

$\bigoplus \mathfrak{T}_i$ can be written as $j_* X$ for a certain X. Thus we get

$$\mathrm{Hom}_A(\mathfrak{T}, \bigoplus \mathfrak{T}_i) \cong \mathrm{Hom}_C(j^*\mathfrak{T}, j^*j_*X) \cong \mathrm{Hom}_C(j^*\mathfrak{T}, \bigoplus j^*\mathfrak{T}_i)$$

(because j^*—which has a right adjoint—commutes with direct sums). Since j^*j_* is an autoequivalence, $j^*\mathfrak{T}$ is a bounded complex (by [14, 6.1 and 6.2]), hence by [14, 2.1] we get

$$\mathrm{Hom}_C(j^*\mathfrak{T}, \bigoplus j^*\mathfrak{T}_i) \cong \bigoplus \mathrm{Hom}_C(j^*\mathfrak{T}, j^*\mathfrak{T}_i) \cong \bigoplus \mathrm{Hom}_A(\mathfrak{T}, \mathfrak{T}_i)$$

(again since j^*j_* is an equivalence). □

We note that the conditions in Theorem 10 are in particular satisfied if A is of finite global dimension and $\mathfrak{B}$ is a complex of finitely generated projective modules. This is the case in many examples (see the following section).

5. Examples

In this section we apply our theorems to some special situations in which recollement can be shown to hold. Thereby we prove anew results of [3–5, 11–13].

To begin with, let A be any ring and assume J is a finitely generated two-sided ideal of A, B is the quotient ring A/J. Define $\mathfrak{B} := B$ and $\mathfrak{C} := J$. What we have to do now, is to translate the conditions of Theorem 1 for $\mathfrak{B}$ and $\mathfrak{C}$ to induce recollement into this special context. There are two long exact homology sequences:

$$\begin{aligned} 0 &\to \mathrm{Hom}_A(B, J) \to \mathrm{Hom}_A(A, J) \to \mathrm{Hom}_A(J, J) \\ &\to \mathrm{Ext}^1_A(B, J) \to \mathrm{Ext}^1_A(A, J) \to \mathrm{Ext}^1_A(J, J) \to \cdots \end{aligned}$$

and

$$\begin{aligned} 0 &\to \mathrm{Hom}_A(B, B) \to \mathrm{Hom}_A(A, B) \to \mathrm{Hom}_A(J, B) \\ &\to \mathrm{Ext}^1_A(B, B) \to \mathrm{Ext}^1_A(A, B) \to \mathrm{Ext}^1_A(J, B) \to \cdots, \end{aligned}$$

hence several conditions are equivalent; for example, $\mathrm{Ext}^1_A(B, B) = 0$ implies $\mathrm{Hom}_A(J, B) = 0$ and vice versa. Moreover, we have a ring homomorphism $\varphi : A \to B$ which is an epimorphism in the category of rings. On the derived level φ induces a functor

$$D^b(\varphi_*) : D^b(\text{Mod-}B) \to D^b(\text{Mod-}A)$$

which sends B to $\mathfrak{B}$. If $\mathfrak{B}$ and $\mathfrak{C}$ are partial tilting complexes inducing recollement in the sense of our Theorem 1, then $D^b(\varphi_*)$ is a full embedding, hence φ is a homological epimorphism as defined by Geigle and Lenzing (cf. Section 2). (Note that the full embedding i_* in a recollement situation on D^--level always restricts to a full embedding on D^b-level by Proposition 4.) Vice versa, if φ is a homological epimorphism, then $\mathfrak{B}$ is a partial tilting complex satisfying all the conditions in the theorem of Geigle and Lenzing cited above.

Corollary 11. *Let A be a ring, J a two-sided ideal of A, which has (considered as a right A-module) a finite resolution by finitely generated projective A-modules, and B the quotient ring A/J. Then the following conditions are equivalent:*
(1) $\mathfrak{B} = B$ *and* $\mathfrak{C} = J$ *satisfy the assumptions of Theorem* 1;
(2) B *is a partial tilting module and* $\operatorname{Ext}^i_A(B, J) = 0$ *for all* $i \geq 2$;
(3) J *is a partial tilting module and* $\operatorname{Ext}^i_A(J, B) = 0$ *for all* $i \geq 0$;
(4) $\varphi : A \to B$ *is a homological epimorphism and J is a partial tilting module*;
(5) $\operatorname{Tor}^A_i(B_A, {}_AB) = 0$ *for all* $i \geq 1$ *and J is a partial tilting module.* □

An important case is that of J a projective right A-module:

Corollary 12. *Let A be any ring, J a two-sided ideal in A which is projective and finitely generated as a right A-module, define B to be the quotient A/J and assume* $\operatorname{Hom}_A(J, B) = 0$. *Then the derived module category of A admits recollement relative to B and* $\operatorname{End}_A(J)$:

$$D^-(\text{Mod-}B) \leftrightarrows D^-(\text{Mod-}A) \leftrightarrows D^-(\text{Mod-End}_A(J)) . \qquad \square$$

The structure of quasi-hereditary algebras [16, 13] and quasi-hereditary orders [11] now tells us that we can apply Corollary 12 several times in order to get what is called a stratification of derived module categories (mod-A denotes the category of finitely generated right A-modules):

Corollary 13. *Let A be either a quasi-hereditary algebra or a quasi-hereditary order. Then A is of finite global dimension and the derived module category* $D^b(\text{Mod-}A)$ *(and similarly* $D^b(\text{mod-}A)$*) admits a stratification, i.e. there is a finite sequence of recollement situations starting with a recollement for* $D^b(\text{Mod-}A)$ *such that at each stage the derived module category on one side of the recollement (and in the last step on both sides) is the derived category of vectorspaces over a skewfield or the derived module category of a local maximal order (this last possibility only occurs if A is a quasi-hereditary order).*

Proof. First consider the corresponding stratification on D^--level which is given by several applications of Corollary 12. Then it follows that A is of finite global dimension and Theorem 7 gives the desired stratification on the D^b-level. □

With regard to applications, a more special situation turns out to be very fruitful. Recollements of derived module categories first arose in the context of quasi-hereditary algebras [16], investigated by Cline, Parshall and Scott. In this case the two-sided ideal J always is a full idempotent ideal: $J = AeA$, hence we turn to the following situation:

Let A be any ring, e an idempotent, $\mathfrak{C} = eA$ the corresponding projective A-module with endomorphism ring $C = eAe$ and $\mathfrak{B} = A/AeA$ the quotient of A by the two-sided ideal generated by $\mathfrak{C}$, with B being the endomorphism ring of $\mathfrak{B}$. Then $\mathfrak{C}$ obviously is a partial tilting complex; also, $\mathrm{Hom}_A^{\bullet}(\mathfrak{C}, \mathfrak{B}) = 0$ by construction of $\mathfrak{B}$ and $\mathfrak{C}$; again $\mathfrak{B}^{\perp} \cap \mathfrak{C}^{\perp} = \{0\}$, and the muitiplication map $\mathfrak{B} \otimes_A \mathfrak{B} \to \mathfrak{B}$ is an isomorphism. Hence the existence of recollement in this situation depends only on $\mathfrak{B}$ being or not a partial tilting complex.

Corollary 14. *Let A be any ring, e an idempotent, $\mathfrak{C} = eA$ the corresponding projective A-module with endomorphism ring $C = eAe$ and $\mathfrak{B} = A/AeA$ the quotient of A by the two-sided ideal generated by $\mathfrak{C}$, with B being the A-endomorphism ring of $\mathfrak{B}$. Assume $\mathfrak{B}$ has finite projective dimension over A. Then the following conditions are equivalent*:

(1) *$\mathfrak{B} = B$ and $\mathfrak{C} = Ae$ satisfy the assumptions of Theorem* 1;

(2) $\mathrm{Ext}_A^n(\mathfrak{B}, \mathfrak{B}) = 0$ *for all* $n > 0$;

(3) $\mathrm{Tor}_A^n(\mathfrak{B}_A, {}_A\mathfrak{B}) = 0$ *for all* $n > 0$;

(4) *$\varphi : A \to B$ is a homological epimorphism.* □

In particular, the condition (2) is satisfied if AeA is projective, i.e. if we are in the situation of Corollary 12.

Note that condition (3) plays a key role in the proof of Parshall's theorem [12, 2.1] on recollements in this situation. In particular, it follows that one does not need any additional assumption in order to complete Parshall's recollement situation (involving only four functors) to the usual recollement situation (as defined above, involving six functors).

For some rings an even more easy situation can be useful:

Corollary 15. *Let A be any ring, e an idempotent and $\mathfrak{B}$ and $\mathfrak{C}$ the projective A-modules $\mathfrak{B} = eA$ and $\mathfrak{C} = (1-e)A$. Assume that $\mathrm{Hom}_A(\mathfrak{C}, \mathfrak{B}) = 0$. Then $\mathfrak{B}$ and $\mathfrak{C}$ satisfy the assumptions of Theorem* 1. □

We give two examples of this situation.

First we consider again the algebra $A_1 = A$ of Example 9:

Choose $\mathfrak{B}$ the simple projective A_1-module and $\mathfrak{C}$ the other projective A_1-module. Since A_1 has finite global dimension, Corollary 15 shows that the derived module category $D^b(\mathrm{Mod}\text{-}A_1)$ admits recollement where both sides are isomorphic to the derived category of k-vectorspaces, hence the same recollement as a finite-dimensional hereditary k-algebra with two projectives. However, A_1 is not noetherian, even not finitely generated over its center, hence it is (cf. [14, proposition 9.4]) not derived equivalent to any finite-dimensional algebra. (Note that this recollement situation—in contrast to the one of Example 9—admits a symmetric recollement situation in the sense of Theorem 10.)

Now we turn to the second example to Corollary 15: Let k be any field and V a k-vectorspace of infinite dimension. Fix a basis of V and pick an element of this basis, say v_1. A_2 is a subring of the vectorspace-endomorphisms of V consisting of these endomorphisms which send v_1 to a k-multiple of v_1 and each other basis vector v to a sum of k-multiples of v and v_1. Symbolically, one can write A_2 as the 'matrix':

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & k & 0 & 0 & 0 & k \\ \cdot & 0 & k & 0 & 0 & k \\ \cdot & 0 & 0 & k & 0 & k \\ \cdot & 0 & 0 & 0 & k & k \\ \cdot & 0 & 0 & 0 & 0 & k \end{bmatrix}$$

A_2 is indecomposable as a ring. Choose a subring of A_2 which is as an abstract ring isomorphic to A_2. This can be done in such a way that this subring is the endomorphism ring of a finitely generated projective A_2-module $\mathfrak{C}$. Choose a complementary projective A_2-module $\mathfrak{B}$. Then by Corollary 14 we get a recollement of the derived module category of A_2 where on one side again the derived module category of A_2 appears.

Finally we note that Geigle and Lenzing [5] have found another type of recollement situation in their investigation of homological epimorphisms (without mentioning recollement explicitly):

Let $R \to S$ be a homological epimorphism which is also injective such that S_R has projective dimension pdim $S_R \leq 1$. Then Geigle and Lenzing proved that S_R and $(S/R)_R$ both are partial tilting complexes and $\mathrm{pdim}(S/R)_R \leq 1$, too.

Using long exact homology sequences it is now easy to show that $\mathfrak{C} = S_R$ and $\mathfrak{B} = (S/R)_R$ satisfy the assumptions in our theorem, hence there is recollement:

$$D^-(\text{Mod-End}_R(S/R)_R) \mathrel{\substack{\leftarrow\\ \rightarrow\\ \leftarrow}} D^-(\text{Mod-}A) \mathrel{\substack{\leftarrow\\ \rightarrow\\ \leftarrow}} D^-(\text{Mod-}S)\,.$$

Acknowledgment

The formulation and the proof of the theorem depends heavily on the use of so-called perpendicular categories to certain objects. This notion first appeared (in a slightly different form, within the context of module categories) in the paper [5] of Geigle and Lenzing (whereas the notion of perpendicular categories to subcategories is much older, cf. [17]). I would like to thank Geigle and Lenzing for supplying me with a preprint of [5].

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