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Determining periodic representative volumes of concrete mixtures based on the fractal analysis

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ABSTRACT

In materials evaluation, determining an appropriate Representative Volume Element (RVE) of the material is of paramount importance. This paper is an attempt to provide a quantitative determination of the RVE of concrete mixtures, using fractal analysis. The key point of this study is to consider concrete mixture as a fractal and periodic structure and the basic periodic unit cell in this material as the RVE. Based on a new analytical approach, obtained results suggest that the ratio between the RVE size and the maximum particle size of concrete is likely to be 2.4–3.7, for increasing *DF* values from 2.5 to 3; *DF* being the fractal dimension of the concrete size distribution. Additional results were proposed for laboratory concrete testing, which suggest that for ordinary concretes, standard sample sizes, whatever the shape, should be at least around 3.5 times the nominal maximum size of aggregates. Although the proposed approach is based on simple mathematical formulas, obtained results appear broadly consistent with those of other studies based on extensive laboratory testing and modeling. The scope of application of the proposed approach can be extended to numerous solid materials that consist of grains.

1. Introduction

Concrete is a composite material that macroscopically demonstrates significant heterogeneity and complexity due to the different sizes, orientations, and shapes of aggregate particles embedded in a paste matrix. To establish the properties and performance of concrete mixtures, typically determined with laboratory tests, it is advisable to measure samples with sizes sufficiently large to get reliable and repeatable readings. Otherwise, test results may be misleading with regard to representation of the properties and performance of concretes (Kim et al., 2009). The smallest volume material large enough such that measurement yields a value representative of the entire material, is referred to as the Representative Volume Element (RVE). However, according to the literature in this field, this volume is not necessarily unique since it is not only sensitive to the material, but also to the property under investigation and to an important factor, the scale of material testing.

Since the early work by Hill (1963), the concept of RVE has been gaining much attention in the last few years. For many materials, several attempts have been made to study the RVE, existence and size determination (see e.g. Stroeven et al., 2004; Gitman, 2006; Gitman et al., 2007 and references therein for an overview).

For composite materials like concrete, consisting of matrix material with embedded inclusions (as grains, fibers, crystals, etc.), a review of the literature suggests that the RVE is generally set to a square whose side length is related to the maximum inclusion size. In order to determine appropriate RVE dimensions in concrete mixtures, many attempts have been conducted. Conventionally, traditional RVE size of concrete must be at least 3–5 times the maximum aggregate size (Van Mier and Van Vliet, 2003), however, for a better understanding of the RVE in concrete mixtures, numerous studies have been conducted. For instance, to investigate the size effect on strength and fracture energy of concrete, Van Vliet and Van Mier (2000) have conducted a series of uniaxial tension experiments. Their results suggest that the RVE size should be taken larger than 3.75 perhaps even as large as 6–7 times the maximum aggregate size. Bažant and Novak proposed that the concrete RVE must be equal to ℓ^{nd}, ℓ been the characteristic length of 2.7–3.0 times the maximum aggregate size; nd = 1, 2 or 3 is the number of spatial dimensions in which the structure is scaled (Gitman et al., 2007). In order to representatively measure electromagnetic properties of building materials as concrete, Robert (1998) showed that material sample dimensions need to be at least three times greater than the maximum aggregate size. The French building research center CSTB has shown that computations converge when the RVE size is 10 times higher than the largest size aggregate, to ensure the applicability of continuum mechanics (Mounajed, 2002). Huet (1999) has shown that the specimen to reach the RVE does not depend upon the maximum inclusion size only,

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but also on other factors such as specimen shape, inclusions content and the contrast between properties of the specimen constituents, so that a higher contrast implies a larger representative volume. Stroeven and Stroeven (2001) demonstrated also that, if for structure insensitive properties (such as stiffness), the rule of thumb is to have minimum sample dimensions exceeding the maximum aggregate size by a factor of 4–5; this size must be larger for structure sensitive properties (such as fracture). In the same sense, Kim et al. (2009) stated that relatively to undamaged state, in damaged state a larger RVE of an asphalt concrete is required, because additional heterogeneities must be considered as cracks, localized yielding and possibly aggregate movements.

The noteworthy observation from the aforementioned studies is the various sizes of the concrete REV, related to the investigated parameter for which the RVE is estimated, the type of investigation (experimental and/or modeling), the material characteristics, and so on.

In this paper we propose another way to determine the RVE size of concrete material, based on the so-called representative unit cell approach, according to which, in a structure assumed to be periodic the RVE coincides with the smallest cell of periodicity (see e.g. Geindreau and Auriault, 1999; Teixeira-Dias et al., 2004; Zeman and Šejnoha, 2007; Sýkora et al., 2009). Although, there is a certain debate regarding this approach, insofar as a RVE may contain one or some number of periodic cells (see Pindera et al., 2009), a unit cell based approach will be followed in this study, by assuming concrete as a heterogeneous, periodic and infinite structure that can be generated by periodic repetition of a basic unit cell, which will be regarded as the RVE of the concrete and that will be called *periodic RVE* throughout this paper. Meaning that, concrete will be treated as a periodic array of repeated unit cells, with periodic boundary conditions to ensure continuity at all boundaries (Mueller, 1997). We consider also, as stated by Drago and Pindera (2007), that under macroscopically uniform loading, the response of an arbitrary unit cell, which is the required RVE here, will be identical of the entire material.

To determine the periodic RVE of concrete mixture, namely the smallest periodic cell that can reconstitute the whole of material by translation in space, a new mathematical approach is undertaken in this study. According to this approach, firstly, we determine analytically the periodic RVE of a polydisperse granular material characterized by a power-law Particle Size Distribution (PSD), based on which we subsequently determine numerically the periodic RVE of a concrete, by considering this material as a packing of multi-sized particles (representing all its solid ingredients), that the PSD will be assumed fractal (as we will see later).

The remainder of this paper is organized as follows. The paper begins with a brief overview of fractals and fractal features of combined PSDs of concrete mixtures. Next, analytical equations will be introduced in order to determine the periodic RVE in an assembly of particles having a fractal size distribution. Thereafter, on the basis of findings obtained in first step, and under certain simplifying assumptions regarding concrete characteristics, the periodic RVE of concrete mixtures will be numerically determined. The proposed approach will correlate the required RVE, in terms of size and particle numbers, with concrete mix design parameters, such as size distribution, particle size and solid volume fraction. We conclude the paper by discussing the obtained results.

2. Fractals and fractal nature of concrete mixes

Fractals can be defined as disordered systems that are self similar independent of scale of observation. Their fundamental property is a non-integer dimension called fractal dimension, which can measure the degree of irregularity of the system (Diez-Orrite et al., 2005; Verbovšek, 2009). The scale invariance of fractals implies that they are characterized by a power-law relationship of form (Zolfaghari and Hajabbasi, 2008):

$$N(\ell > x) \propto x^{-DF} \tag{1}$$

where $N(x)$ is the number of objects with size ℓ greater than a predetermined size x and the exponent DF is commonly referred to as the fractal dimension. The fractal dimension can be calculated as the slope of linear regression best-fit line $\log N(x)$ vs. $\log x$ data (Vallejo and Lobo-Guerrero, 2009). In grain gradations, if a cumulative distribution follows a geometric law as in Eq. (1), the derived fractal dimension can measure the complexity of particle distribution in nature, and can provide a description of how much space a particle set fills (Yang and Juo, 2001).

This paper builds on the findings of previous researches (Lecomte and Thomas, 1992; Chouicha, 2006), according to which ideal grading curves of concrete can be transformed into straight-lines power-law of the form given in Eq. (1). Such a transformation can be achieved by going through the following steps, where a cumulative grading curve in terms of weight, is transformed into a size distribution in terms of number of particles larger than a specified size, as shown below:

- From the mass PSD, the number of aggregate particles of a particular size ϕ_i denoted by E_{ϕ_i}, can be computed from the weight residue on sieve ϕ_i mesh denoted by R_{ϕ_i}, as:

$$E_{\phi_i} = R_{\phi_i}/(\rho v_{\phi_i}) \tag{2}$$

  where v_{ϕ_i} is the volume per grain of size ϕ_i and ρ the mass density of grains.
- Therefore, assuming invariant density and spherical shape of concrete particles, $E_{C\phi_i}$, the number of grains in size class ϕ_i and greater, can be determined as:

$$E_{C\phi_i}(\phi \geqslant \phi_i) = \sum_{j=1}^{i} E_{\phi_j} = \frac{6}{\rho\pi}\sum_{j=1}^{i} R_{\phi_j}\phi_j^{-3} \tag{3}$$

- The plot of the fit straight line $\log E_{C\phi_i}$ against $\log \phi_i$ through the data points generated by Eq. (3). The general regression equation of the obtained lines can be written as follows:

$$\log E_{C\phi_i} = c - DF\log\phi_i \text{ or } E_{C\phi_i} = C\phi_i^{-DF} \tag{4}$$

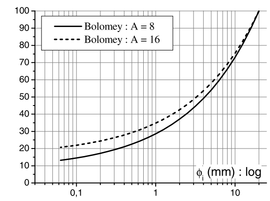
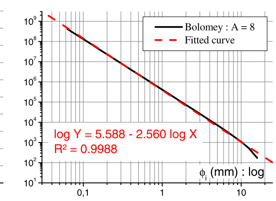
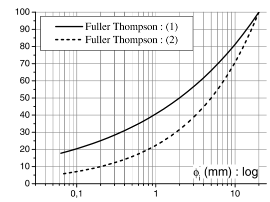
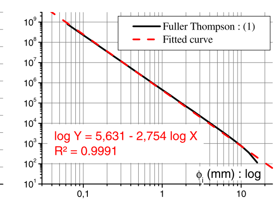
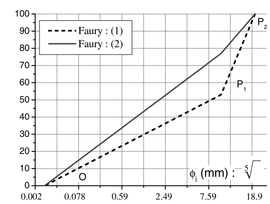
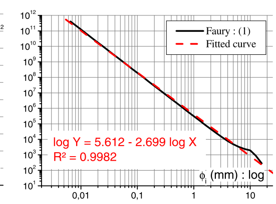
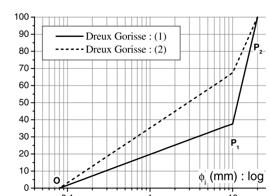
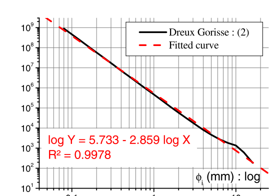
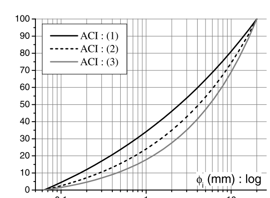
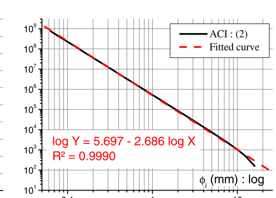
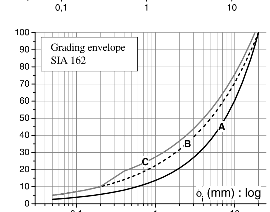
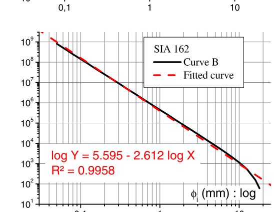
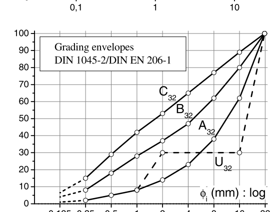
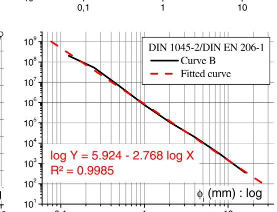
  where $C = 10^c$ is a proportionality coefficient.

To get a better insight into the fractal feature of concrete size distributions, we report in Table 1 conversion results of some ideal grading curves extensively used, into power-law curves obtained as described above, where $P_{C\phi_i}$ is the cumulative weight fraction passing through sieve opening ϕ_i; d and D are resp. the minimum and maximum particle size. Noting that, although mix design methods given in Table 1 are focused on optimizing the particle packing density of concrete (to improve its overall performance), there exist some differences between these methods, as including or not fine cementitious material or by taking into account or not some factors as aggregate characteristics, wall effect exerted by the form or certain expected concrete properties.

In Table 1, the achieved good fitting results of $\log E_{C\phi_i}$ vs. $\log \phi_i$ (correlation coefficients $R^2 \approx 1$) confirm the fractal feature of many ideal PSDs used for the mix proportioning of concrete. Similar results can be found for other mix design methods.

Furthermore, as concrete ingredients must be proportioned to get a combined grading as close as possible to an optimum grading, it is reasonably inferred that the PSD of a solid concrete mix should be fractal in nature. The clear benefit of this, is that the PSD of the solid concrete skeleton can be adequately generated based on only the values of DF and the particle sizes d and D.

Table 1
Fractal dimensions corresponding to some concrete mix designs, obtained by transforming ideal grading curves, expressed in terms of percent passing (by weight) $P_{C\phi_i} = f(\phi_i)$ in log-normal coordinates, on best-fitted straight lines $E_{C\phi_i}(\phi \geqslant \phi_i) = f(\phi_i)$ in log-log coordinates. (See below-mentioned references for further information).

Particle size distribution function	Limit parameters	DF and R^2 values		Ideal grading $P_{C\phi_i}$ (%)	Grain number distribution $E_{C\phi_i}$: log
Bolomey (French method) (Dreux and Festa, 1998) $P_{C\phi_i} = A + (100-A)\sqrt{\frac{\phi_i}{D}}$	A = 8,..., 16	2.56 (R^2 = 0.999)			
Fuller Thompson (Sobolev and Amirjanov, 2007) $P_{C\phi_i} = 100\left(\frac{\phi_i}{D}\right)^q$	q = 0.3 q = 0.5	2.75 (R^2 = 0.999) 2.56 (R^2 = 0.999)	(1) (2)		
Faury (French method) (Dreux and Festa, 1998) $P_{C\phi_i}: \begin{bmatrix} O\left(\frac{d}{0\%}\right);\ d = 6\,\mu m \\ P_1\left(\frac{D/2}{A+17\sqrt[5]{D}+\frac{B}{R/D-0.75}}\right) \\ P_2\left(\frac{D}{100\%}\right) \end{bmatrix}$	A = 22, B = 1 negligible wall effect (R/D tends to ∞) A = 38, B = 2 important wall effect (R/D tends to 1)	2.70 (R^2 = 0.998) 2.79 (R^2 = 1.000)	(1) (2)		
Dreux Gorisse (French method) (Dreux and Festa, 1998) $P_{C\phi_i}: \begin{bmatrix} O\left(\frac{d}{0\%}\right);\ d = 80\,\mu m \\ P_1\left(\frac{D/2 \text{ if } D \leq 20\text{ mm}}{50-\sqrt{D}+K_S+K_P}\right) \\ P_2\left(\frac{D}{100\%}\right) \end{bmatrix}$	$K_S + K_P \approx -8$ $K_S + K_P \approx +22$	2.63 (R^2 = 0.988) 2.86 (R^2 = 0.988)	(1) (2)		
American standard ACI 207 1 R-70 (De Larrard, 1987) $P_{C\phi_i} = 100\left(\frac{\phi_i^q - d^q}{D^q - d^q}\right)$	q = 0.20 q = 0.37 q = 0.50	2.85 (R^2 = 0.999) 2.69 (R^2 = 0.999) 2.56 (R^2 = 0.999)	(1) (2) (3)		
Swiss standard SIA 162/1 (1989) Curve A: $P_{C\phi_i} = 50\left(\phi_i/D + \sqrt{\phi_i/D}\right)$ Curve B: $P_{C\phi_i} = 100\sqrt{\phi_i/D}$ Curve C: $P_{C\phi_i}$ = 5 % above curve B, for $\phi_i \geq 0.4$ mm	Curve A Curve B Curve C	2.44 (R^2 = 0.996) 2.61 (R^2 = 0.996) 2.76 (R^2 = 0.987)			
German standard DIN 1045-2 (2001) $P_{C\phi_i} = 100\frac{\left(\frac{\phi_i}{D}\right)^n - \left(\frac{0.125}{D}\right)^n}{1-\left(\frac{0.125}{D}\right)^n}$	Curve A: n = 2/3 Curve B: n = 1/4 Curve C: n approaches 0	2.34 (R^2 = 0.999) 2.77 (R^2 = 0.998) 3.00 (R^2 = 0.999)			

3. Research approach

3.1. Proposed approach description

The primary objective of this research is to develop a simple mathematical approach to determine the RVE sizes of concrete mixtures assumed as fractal and periodic structures. Hence, the required RVE is defined geometrically in terms of a periodic unit cell which can provide sufficient information about the material and can reconstruct the entire concrete mixture by translation in space. The determination of the RVE size involves two steps: (1) analytical determination of the RVE of a granular material that is fractal in size distribution. In this step, equations will be established to select the suitable RVE that must contain an assembly of a minimum, yet sufficient, number of grains to guarantee the representativity of the studied structure, and (2) on the basis of findings obtained in (1), the numerical determination of the RVE of concrete mixtures, by approximating concrete system as a packing of particles representing the entire ingredients of concrete (aggregates, cementitious material,...), that the gradation mix is implicitly assumed fractal, as advanced in the previous section. The next sections will describe in more detail the proposed approach.

3.2. RVE of a granular matter

We consider in this section, polydisperse granular mixtures with power-law size distributions noted $M^{DF}_{d/D}$, defined through two factors controlling the PSD, namely *DF* value and the extreme particle sizes *d* and *D*. Each of these mixtures will be divided into *n* grain fractions, of size decreasing in geometric progression from ϕ_1 for the largest particles, to ϕ_n for the smallest ones; in such a way that the *i*th grain fraction will contain a mono-sized assembly of grains of size ϕ_i and of number E_{ϕ_i}, as it is shown next:

$$\forall i \in [1, \mathrm{n}] : \left[\begin{array}{l} \phi_i > \phi_{i+1} \text{ such as } \phi_1 = D \textit{ and } \phi_n = d \\ \phi_i/\phi_{i+1} = \lambda \Rightarrow \phi_i = D\lambda^{1-i} \end{array}\right. \tag{5}$$

3.2.1. Minimum numbers of grains

As highlighted above, $M^{DF}_{d/D}$ will be composed of *n* classes of mono-sized grains, to which corresponds the set of grain numbers $\{E_{\phi_1}, \ldots, E_{\phi_n}\}$ that we can derive from Eq. (4) as follows:

$$E_{\phi_i} = \begin{cases} C\phi_i^{-DF}, & i = 1 \\ C(\phi_i^{-DF} - \phi_{i-1}^{-DF}), & \forall i \in [2, \mathrm{n}] \end{cases} \tag{6}$$

Or, equivalently, when replacing ϕ_i by $D\lambda^{1-i}$:

$$E_{\phi_i} = \begin{cases} CD^{-DF}, & i = 1 \\ CD^{-DF}\lambda^{DF(i-2)}(\lambda^{DF} - 1), & \forall i \in [2, \mathrm{n}] \end{cases} \tag{7}$$

To determine the minimum numbers of grains contained in a RVE, we follow Meier et al. (2008) in assuming that a periodic RVE based on a PSD must contain minimum, yet sufficient, number of grains, selected to include at least one grain in a given size fraction. To satisfy this requirement, the target RVE corresponding to a $M^{DF}_{d/D}$, must contain the set of minimum numbers of grains noted $\{N_{\phi_1}, \ldots, N_{\phi_n}\}$, in such a way that:

$$\forall M^{DF}_{d/D} : \left[\begin{array}{lll} (i) \underset{1 \leqslant j \leqslant n}{Min}\, N_{\phi_j} = 1 & \Rightarrow & \left| \begin{array}{l}\text{to obtain an} \\ \text{irreducible unit cell}\end{array}\right. \\ (ii) \forall i \in [1, \mathrm{n}] : N_{\phi_i} \propto E_{\phi_i} & \Rightarrow & \left| \begin{array}{l}\text{to reproduce the} \\ \text{same fractal PSD}\end{array}\right. \end{array}\right. \tag{8}$$

By combining both conditions given in Eq. (8), we can formulate the following unique condition:

$$\forall i \in [1, \mathrm{n}] : N_{\phi_i} = E_{\phi_i} / \underset{1 \leqslant j \leqslant n}{Min}\, E_{\phi_j} \tag{9}$$

In this relation, one way to determine the denominator is to study the ratio between successive numbers $E_{\phi_{j+1}}$ and E_{ϕ_j}. Therefore, by using Eq. (7), we obtain:

$$E_{\phi_{j+1}}/E_{\phi_j} = \begin{cases} \lambda^{DF} - 1, & \text{if } j = 1 \quad \text{(a)} \\ \lambda^{DF}, & \text{if } j \in [2, \mathrm{n}] \quad \text{(b)} \end{cases} \tag{10}$$

From which we notice that:

(a) For *j* = 1, if $\lambda^{DF} - 1 > 1$ (or $\lambda^{DF} > 2$) we can write:

$$\begin{array}{l} E_{\phi_2}/E_{\phi_1} > 1 \Rightarrow Min(E_{\phi_1}, E_{\phi_2}) = E_{\phi_1} \\ \text{Overwise, if } \lambda^{DF} \leqslant 2 \Rightarrow Min(E_{\phi_1}, E_{\phi_2}) = E_{\phi_2} \end{array} \tag{10a}$$

(b) λ^{DF} is always > 1, then $\forall j \in [2, \mathrm{n}]$ we can write:

$$E_{\phi_{j+1}}/E_{\phi_j} > 1 \Rightarrow E_{\phi_2} < \cdots < E_{\phi_n} \Rightarrow \underset{2 \leqslant j \leqslant n}{Min}\, E_{\phi_j} = E_{\phi_2} \tag{10b}$$

It accordingly implies that $Min(E_{\phi_1}, \ldots, E_{\phi_n})$ is equal to either E_{ϕ_1} or E_{ϕ_2}, and one can state that:

$$\forall M^{DF}_{d/D} : \underset{1 \leqslant j \leqslant n}{Min}\, E_{\phi_j} = \begin{cases} E_{\phi_2} & \text{when } \lambda^{DF} \leqslant 2 \\ E_{\phi_1} & \text{when } \lambda^{DF} > 2 \end{cases} \tag{11}$$

Substituting this result and expressions of E_{ϕ_i} from Eq. (7) into Eq. (9), yields:

$$N_{\phi_i} = \begin{cases} \left.\begin{array}{ll} (\lambda^{DF} - 1)^{-1}, & i = 1 \\ \lambda^{DF(i-2)}, & \forall i \in [2, \mathrm{n}] \end{array}\right\} & \text{if } \lambda^{DF} \leqslant 2 \\ \left.\begin{array}{ll} 1, & i = 1 \\ \lambda^{DF(i-2)}(\lambda^{DF} - 1), & \forall i \in [2, \mathrm{n}] \end{array}\right\} & \text{if } \lambda^{DF} > 2 \end{cases} \tag{12}$$

If replacing λ^{i-1} by D/ϕ_i in Eq. (12), we again obtain:

$$\forall i \in [2, \mathrm{n}] : N_{\phi_i} = \begin{cases} (D/\phi_{i-1})^{DF}, & \text{if } \lambda^{DF} \leqslant 2 \\ (\lambda^{DF} - 1)(D/\phi_{i-1})^{DF}, & \text{if } \lambda^{DF} > 2 \end{cases} \tag{13}$$

As additional information, we can determine $N_{C\phi_i}$, the cumulative amount of grains of size greater than ϕ_i, and N_T, the total amount of grains in a RVE, as follows:

$$N_{C\phi_i}(\phi \geqslant \phi_i) = \sum_{j=1}^{i} N_{\phi_j} \text{ such as } N_T = \sum_{j=1}^{n} N_{\phi_j} \tag{14}$$

If we substitute Eq. (12) into Eq. (14), we obtain according to the term λ^{DF}:

$$N_{C\phi_i} = \begin{cases} \dfrac{1}{\lambda^{DF} - 1} + \displaystyle\sum_{j=2}^{i} \lambda^{DF(j-2)}, & \text{if } \lambda^{DF} \leqslant 2 \\ (\lambda^{DF} - 1)\left[\dfrac{1}{\lambda^{DF} - 1} + \displaystyle\sum_{j=2}^{i} \lambda^{DF(j-2)}\right], & \text{if } \lambda^{DF} > 2 \end{cases} \tag{15}$$

In these equations, the sum is in the form of a geometric series with a common ratio of λ^{DF}. We can use a simple formula for summing all terms in this series, by calculating the difference: $sum - sum \times \lambda^{DF}$. We obtain then:

$$\sum_{j=2}^{i} \lambda^{DF(j-2)} = \frac{1 - \lambda^{DF(i-1)}}{1 - \lambda^{DF}} \; (\text{where } \lambda^{DF} \neq 1) \tag{16}$$

Substituting both this result and $\lambda^{i-1} = D/\phi_i$ into Eq. (15), yields equations which determine $N_{C\phi_i}$ and N_T:

$$N_{C\phi_i} = \begin{cases} \dfrac{\lambda^{DF(i-1)}}{\lambda^{DF} - 1} \text{ or } \dfrac{(D/\phi_{i-1})^{DF}}{\lambda^{DF} - 1}, & \text{if } \lambda^{DF} \leqslant 2 \\ \lambda^{DF(i-1)} \text{ or } (D/\phi_{i-1})^{DF}, & \text{if } \lambda^{DF} > 2 \end{cases} \tag{17}$$

3.2.2. RVE determination

First, we determine the absolute volume noted Rve, which is the volume of the solid matter in the particles of an RVE, excluding the volume of voids between particles. Rve can be expressed as:

$$Rve = \sum_{i=1}^{n} N_{\phi_i} v_{\phi_i} \tag{18}$$

Assuming an invariant particle shape, the volume of a single grain v_{ϕ_i} will be expressed through a shape factor ξ as follows:

$$\begin{aligned} v_{\phi_i} &= \xi \phi_i^3 \\ &= v_{\phi_1} \lambda^{3(1-i)} \text{ or } v_D \lambda^{3(1-i)}, \text{ by replacing } \phi_i = D\lambda^{1-i} \end{aligned} \tag{19}$$

Substituting Eq. (19) and expressions for N_{ϕ_i} from Eq. (12) into Eq. (18), leads to the following set of equations:

$$Rve = \begin{cases} v_D \left[\dfrac{1}{\lambda^{DF} - 1} + \lambda^{3-2DF} \displaystyle\sum_{i=2}^{n} \lambda^{i(DF-3)} \right], & \text{if } \lambda^{DF} \leqslant 2 \\ v_D \left[1 + \dfrac{\lambda^{DF} - 1}{\lambda^{2DF-3}} \displaystyle\sum_{i=2}^{n} \lambda^{i(DF-3)} \right], & \text{if } \lambda^{DF} > 2 \end{cases} \tag{20}$$

In these expressions, the sum can be found as set out in the previous section. We obtain then:

$$\sum_{i=2}^{n} \lambda^{i(DF-3)} = \begin{cases} \dfrac{\lambda^{2(DF-3)} - \lambda^{(n+1)(DF-3)}}{1 - \lambda^{DF-3}}, & \text{if } DF \neq 3 \\ n-1, & \text{if } DF = 3 \end{cases} \tag{21}$$

Substituting the so obtained sum and $\lambda^{i-1} = D/\phi_i$ into Eq. (20) yields the following sets of equations:

(i) Case: $DF \neq 3$

$$Rve = \begin{cases} v_D \left[\dfrac{1}{\lambda^{DF} - 1} + \dfrac{1 - (D/d)^{DF-3}}{\lambda^3 - \lambda^{DF}} \right], & \text{if } \lambda^{DF} \leqslant 2 \\ v_D \left[1 + \dfrac{1 - (D/d)^{DF-3}}{(\lambda^3 - \lambda^{DF})(\lambda^{DF} - 1)^{-1}} \right], & \text{if } \lambda^{DF} > 2 \end{cases} \tag{22}$$

(ii) Case: $DF = 3$

$$Rve = \begin{cases} v_D \left[\dfrac{1}{\lambda^{DF} - 1} + \dfrac{\lambda^{3-2DF}}{\log \lambda} \log(D/d) \right], & \text{if } \lambda^{DF} \leqslant 2 \\ v_D \left[1 + \dfrac{\lambda^{DF} - 1}{\lambda^{2DF-3} \log \lambda} \log(D/d) \right], & \text{if } \lambda^{DF} > 2 \end{cases} \tag{23}$$

Through considering the packing density of grains, one can estimate the corresponding bulk volume using Eqs. (22) and (23), which is the required RVE. It is also worth noting that particle packing densities can be predicted through a wide variety of existing tools and models, in which results are in most cases governed by the specific properties of the granular material (size range; grading; grain shape, surface roughness, strength…); the packing procedure and applied compaction level; the boundary conditions (wall effect due to the container) and so on (see Kwan and Mora, 2001; Lecomte, 2006; Stroeven et al., 2011).

3.2.3. Influence of the boundary conditions

Grain numbers obtained through Eqs. (12) may not be whole numbers. This is due to the fact that periodic boundary conditions were considered (He, 2010), i.e. no boundary effect; parts of grains missing to have whole grain numbers are assumed to belong to the neighboring cells. Otherwise, if applying rigid boundary conditions, the Rve will contain full grains. In this case, the grain number N_{ϕ_i} can be rounded to $\overline{N}_{\phi_i}$ ($\overline{x}$ means the smallest integer greater than or equal to x), which can take any value in the interval: $\overline{N}_{\phi_i} \in [N_{\phi_i}, N_{\phi_i} + 1[$. Accordingly, to $\overline{N}_{\phi_i}$ will correspond the absolute representative volume $\overline{Rve}$, in such a way that:

$$\overline{Rve} \in \left[Rve, Rve + \sum_{i=1}^{n} v_{\phi_i} \right[\tag{24}$$

By replacing v_{ϕ_i} from Eq. (19) and then summing the obtained series, Eq. (24) can be simplified as:

$$\overline{Rve} \in \left[Rve, Rve + v_D(\lambda^3 - d/D)/(\lambda^3 - 1) \right[\tag{25}$$

Noting that, a number of calculations have been carried out in order to assess the influence of considering fractional or whole grain numbers on the RVE quantification. Results show relatively small differences between these two cases.

3.3. RVE of concrete mixtures: numerical results

In this section we will present some numerical results in which concrete RVEs are determined, based on the foregoing findings. However, in view of the complexity of the concrete structure, we had to make certain assumptions regarding this material, for simplification and for keeping a minimum number of parameters in our approach.

According to the adopted assumptions, concrete mixture will be viewed as a packing of solid spherical particles, which encompasses the whole size range of the concrete skeleton. The particle packing density of fresh concrete put in work, before any chemical reaction can occur, will be predicted through a formula, based on experimental data of measuring void content of a dry granular mix following a fractal PSD, as described by Chouicha (2006) (we omit the role of water presence on the arrangement of particles in the wet concrete mix). According to this procedure (see Fig. 1 for experiment details), the packing density δ of a granular mixture $M_{d/D}^{DF}$ can be expressed as follows:

$$\delta = 1 - V_C = 1 - \left[V_{C(d)} - A_I \exp\left(\frac{-(\Gamma - \Gamma_{opt})^2}{2w^2} \right) \right] \tag{26}$$

where V_C denotes the void content of the packing mixture; $V_{C(d)}$ denotes that of particles in class of one-size d; A_I a compaction index quantifying the effectiveness of compaction effort applied; while Γ, Γ_{opt} and w are parameters depending on DF and d/D extend (see Chouicha (2006) for details on how to calculate Eq. (26) terms).

On the other hand, we will assume also, that densities of both freshly mixed and hardened concrete are nearly equal. Accordingly, the RVE could be evaluated as below, where δ denotes in this case the volumetric fraction of solid ingredients in the concrete:

$$RVE = Rve/\delta \tag{27}$$

However, it is well known the effect of container walls on the dry packing of particles that tends to decrease the packing density at the interface between grains and walls. Such an effect also takes place in granular suspensions such as fresh concrete (Ferraris and Brower, 2001). Therefore, relation in (26) will be corrected to take into account container wall effect, by means of the following expression (De Larrard, 1999):

$$\tilde{\delta} = (k_w \delta) V_P + \delta(1 - V_P) \tag{28}$$

where $\tilde{\delta}$ is the mean packing density of the whole mixture in the container, k_w a disturbance coefficient ($k_w < 1$) and V_P the perturbed interfacial volume in the container with a lower density than in the bulk of the material. V_P can be calculated under the assumption that due the wall effect, the packing density is affected within a distance of $D/2$ from the wall. Hence, in a unit total volume of mixture, for a cylinder or cubic container, V_P can be obtained as (Lecomte, 2006):

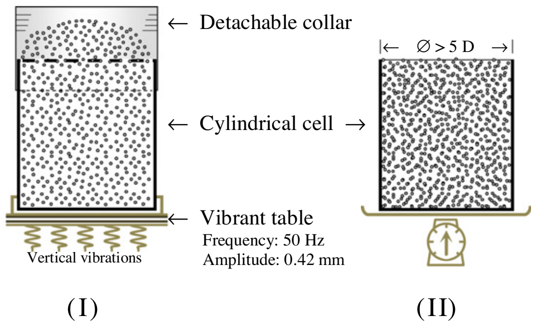


Fig. 1. Principle of packing density measurement (I) before vibration: crushed aggregate introduced into a container with a detachable collar (II) after vibration: container without collar weighed (Chouicha, 2006).

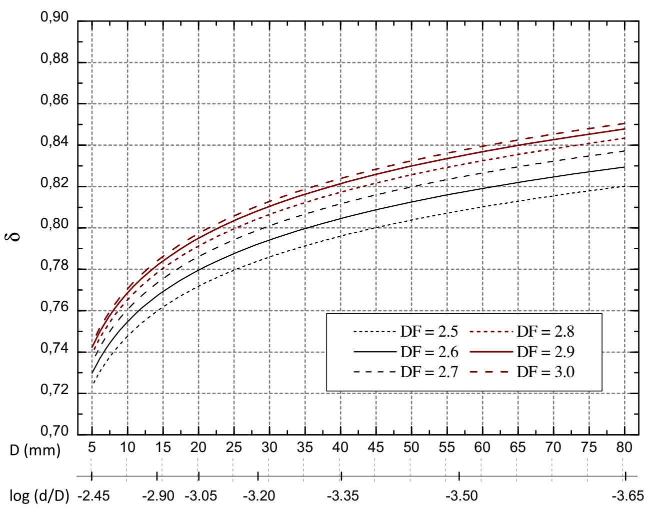


Fig. 2. Volumetric fractions of solid ingredients in fresh concrete mixtures versus *d/D* and *DF* values, calculated using relation in Eq. (26) (by considering fine particles in a flocculated state).

$$V_P = \frac{\mathbf{S}(\frac{D}{2})}{\mathbf{V}} = \begin{cases} 1-(1-\frac{D}{2A})(1-\frac{D}{A})^2 & (cube: A^3) \\ 1-(1-\frac{D}{2H})(1-\frac{D}{\Phi})^2 & (cylinder: \Phi \times \mathrm{H}) \end{cases} \tag{29}$$

where **V** is the container volume and **S** the container surface in contact with the material.

Noting that, in Eq. (27), in addition to the boundary effect, other factors can affect the solid volume fraction δ, as the consistency of fresh concrete, condition of placing, presence of an admixture in the concrete and so on (Dreux and Festa, 1998).

The RVE will have two different shapes, cubic of side A_{RVE} and cylindrical of diameter Φ_{RVE} (slenderness ratio 2). Further, the smaller cross sectional sizes A_{RVE} and Φ_{RVE} will be related to D through multiplier factors, as shown below:

$$\begin{bmatrix} A_{RVE} \\ \Phi_{RVE} \end{bmatrix} = \sqrt[3]{RVE} \times \begin{bmatrix} 1 \\ \sqrt[3]{2/\pi} \end{bmatrix} = D \times \begin{bmatrix} f_A \\ f_\Phi \end{bmatrix} \tag{30}$$

In the present approach, ideally, concrete particles are described by their true sizes (different from sieving sizes); however determining such size is highly complex. To overcome this, we propose a particle size ratio λ close enough to unity to ensure size continuity, as 1.1 ratio. Since this choice is somewhat arbitrary, we assess the influence of other values of λ (up to 2) on the obtained results.

The finest particle size in the dry concrete mixture will be equal to an average particle size of about 6 μm. Moreover, in wet state, we will consider the phenomenon of agglomeration of fines below a certain threshold size. In this case, spherical agglomerates will be considered as finer particles in the wet mix, whose smallest size will be equal to an average size of about 18 μm (Baron and Ollivier, 1996).

Based on the aforementioned assumptions, and for the sake of clarity, the obtained numerical results will be summarized graphically. In Fig. 2 we show the particle volume fractions in the concrete (without boundary effect). Figs 3a and b present resp. the plots of the grain numbers N_{ϕ_i} and N_T required to fulfill the relevant RVE requirements. Figs 4a and b show schematically the relationship between the absolute volume of the solid ingredients in a concrete RVE and some geometrical parameters of the mixture, as the grain sizes d and D, the particle size ratio λ and *DF*. Figs 5a and b show resp. the RVE sizes and the corresponding multiplier factors

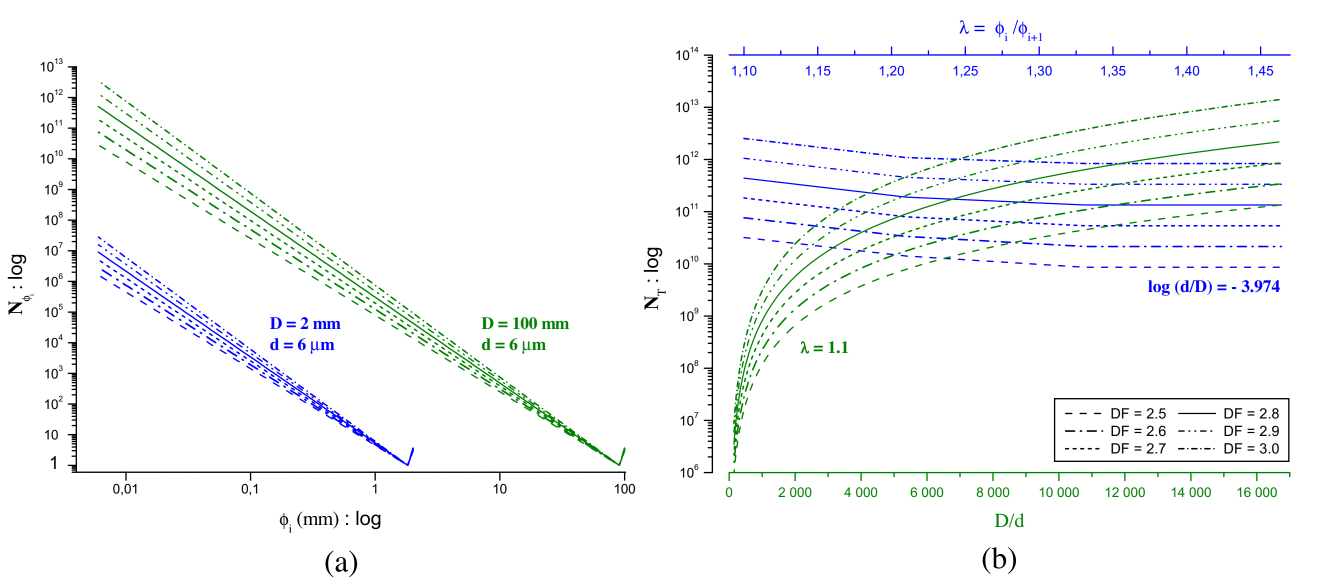


Fig. 3. Graphic representations of: (a) the amount of grains in each individual size ϕ_i of the dry solid skeleton of a RVE, related to *DF* and *D* (chosen equal to 2 and 100 mm) for d = 6 μm (data calculated from Eq. (12)); (b) the total number of grains in a RVE, related to *DF*, the size ratio λ and the grain sizes *d* and *D* (inferred from data using Eq. (17)).

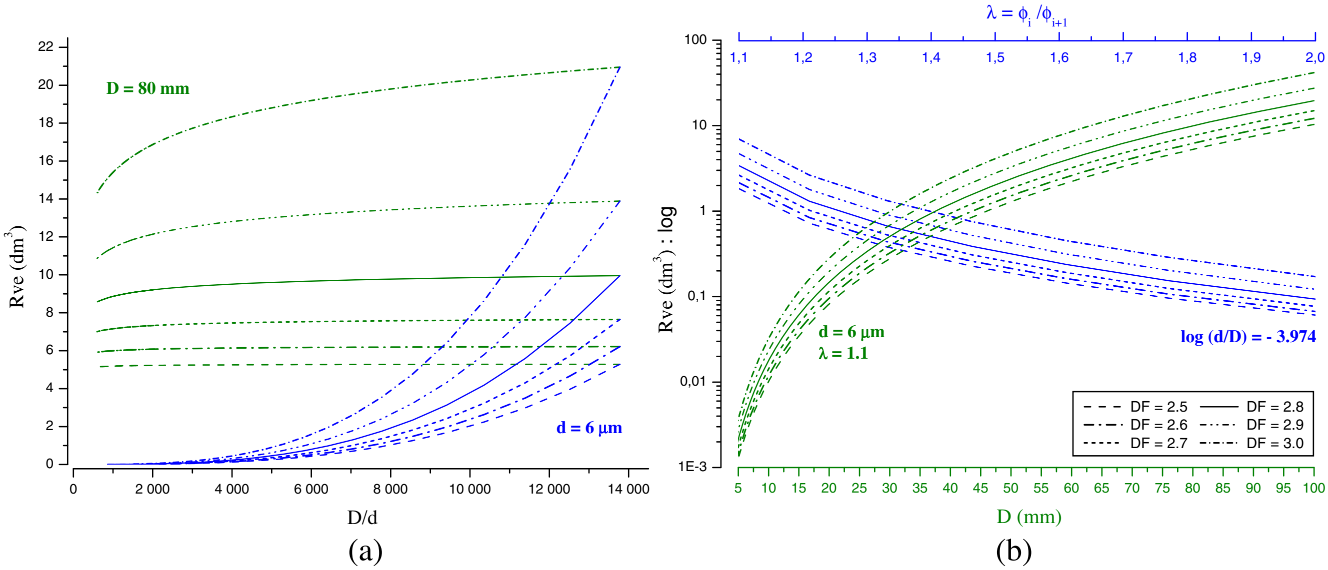


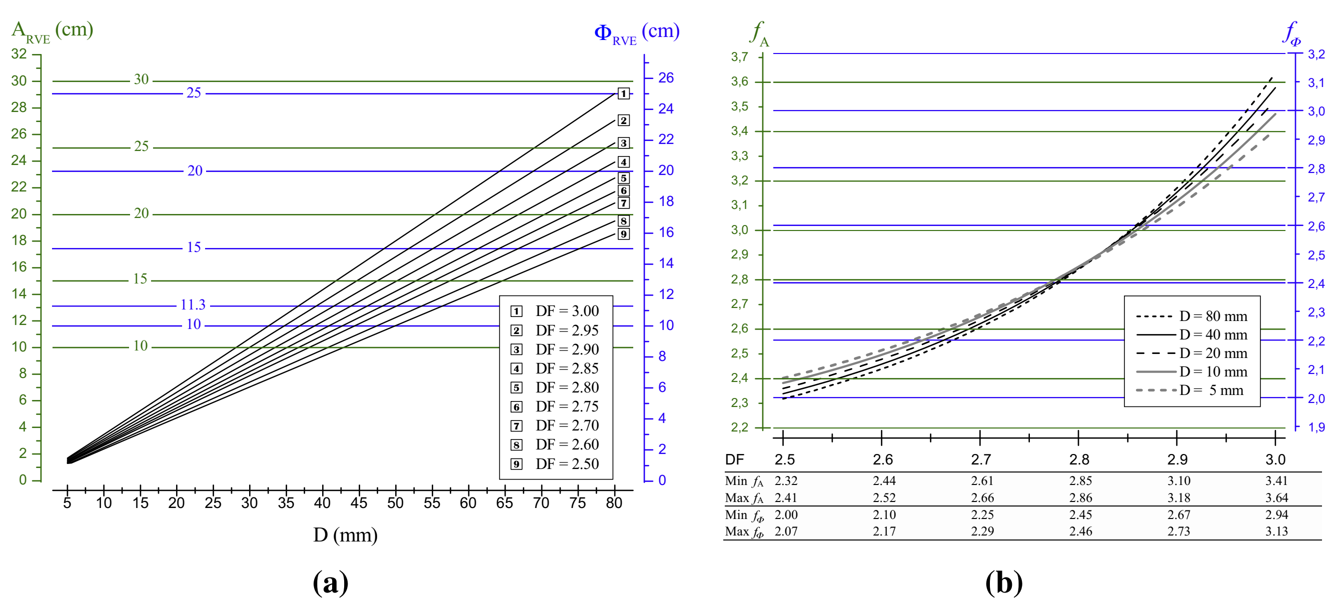
Fig. 4. Graphic representations of the absolute volume of solid concrete ingredients Rve, expressed in terms of *DF* and: in (a) the extend size *d*/*D* and the extreme particle sizes *d* and *D*; in (b) the largest particle size *D* and the size ratio λ (from data using Eqs. (22) and (23)).

(without wall effect), while Fig. 6 displays the RVE sizes of concrete, if taking into account the wall effect exerted by the container. Container will be sized and shaped as standard molds commonly used in concrete laboratory testing (in accordance to NF EN 12390-1 standard (AFNOR, 2001)).

Furthermore, in concrete technology, specific aggregate sizes, as determined by sieve size standards, must be used. Accordingly, on the basis of the data in Fig. 6, we will try to determine the allowable aggregate *size Dn* that corresponds to a given standardized mold opening size denoted X_i; *Dn* being the nominal maximum size of aggregate defined in accordance to NF EN 12620 standard (AFNOR, 2008). Hence, in Fig. 6, if we project on the *D*-axis a value of a RVE size equal to a given X_i, we can determine a corresponding value of *D*. We consider then, that the allowable aggregate size *Dn* that can admit the mold of opening size X_i must be at or below *D*. It accordingly implies that we can deduce X_i/Dn ratios, which permit to re-determine RVE sizes in terms of *Dn* and X_i for making standardized concrete samples for laboratory testing. The obtained results are summarized in Table 2.

4. Discussion

Simple analytical formulas have been proposed to estimate the periodic RVE of granular materials and concrete mixtures, both in terms of size and grain numbers. Concerning concrete RVE determination, based on data sets generated from our analytical equations and presented in graphical form, the following general remarks can be made:



DF	2.5	2.6	2.7	2.8	2.9	3.0
Min f_A	2.32	2.44	2.61	2.85	3.10	3.41
Max f_A	2.41	2.52	2.66	2.86	3.18	3.64
Min f_Φ	2.00	2.10	2.25	2.45	2.67	2.94
Max f_Φ	2.07	2.17	2.29	2.46	2.73	3.13

Fig. 5. (a) Plots of RVE sizes expressed in terms of *D* and *DF* (without boundary effect). (b) The corresponding multiplier factors f_A and f_Φ that represent the ratio between the RVE size and *D*. The numerical values at the bottom of the graph refer to the upper and lower limits of f_A and f_Φ (for each *DF* value and *D* spanning the size range from 5 to 80 mm). In (a) and (b) data are calculated using Eqs. (22), (23), (26), (27), and (30).

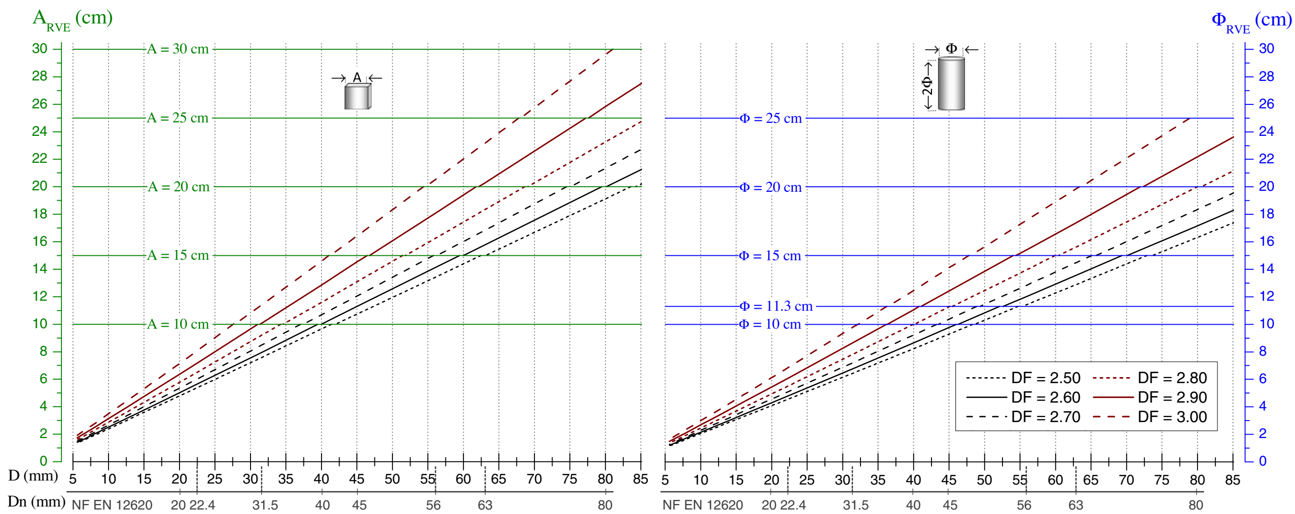


Fig. 6. RVE sizes of concrete expressed in terms of *Dn* and *DF* values, related to the cross-sectional shape and size of the mold and by considering the wall effect exerted by the container. Data calculated using Eqs. (22), (23) and (26)–(30), such as the disturbance coefficient k_w in Eq. (28) is taken equal to 0.73 for crushed aggregates (De Larrard, 1999).

Table 2
Allowed nominal maximum size of aggregates *Dn* that can admit a mold of opening size X_i, of cubic (cub.) or cylindrical (cyl.) shape, and the corresponding X_i/Dn ratio. Values deduced on the basis of Fig. 6.

Mold opening size X_i (cm)				DF											
				2.50		2.60		2.70		2.80		2.90		3.00	
i	**Cub.**	Cyl.		**Cub.**	Cyl.	**Cub.**	Cyl.	**Cub.**	Cyl.	**Cub.**	Cyl.	**Cub.**	Cyl.	Cub.	Cyl.
1	**10**	10	*Dn* (mm) ≤	**40**	45	**31.5**	45	**31.5**	40	**31.5**	31.5	**22.4**	31.5	**22.4**	31.5
2	**15**	11.3		**56**	45	**56**	45	**45**	45	**45**	45	**45**	40	**40**	31.5
3	**20**	15		**80**	63	**63**	63	**63**	63	**63**	56	**56**	45	**45**	45
4	**25**	20		—	80	**80**	80	**80**	80	**80**	63	**63**	63	**63**	63
5	**30**	25		—	—	—	—	—	—	—	80	**80**	80	**80**	63
$\underset{1\le i\le 5}{Min}(X_i/Dn)$				**2.5**	2.3	**2.7**	2.3	**3.2**	2.4	**3.2**	2.6	**3.4**	2.9	**3.8**	3.2
$\underset{1\le i\le 5}{Max}(X_i/Dn)$				**2.7**	2.6	**3.2**	2.6	**3.4**	2.6	**3.4**	3.2	**4.5**	3.4	**4.5**	4.0

— indicates that *Dn* > 80 mm

– In Figs. 3–5, it is shown that the main factors controlling the RVE (in terms of size and grain numbers) are the volumetric fraction and size distribution of the solid phase in the concrete mixture; the largest size, the size extend and size ratio of concrete particles.
– Clearly, in Fig. 4a, for the same extent size *d*/*D*, significant differences were observed between the influence of the size *D* rather than that of *d* on the RVE quantification. It can explain why RVE size must be preferentially related to *D*.
– Fig. 4b shows the marked influence of the ratio $\lambda = \phi_i/\phi_{i+1}$ on the RVE estimation; in the case when λ is close to 1 and when concrete particles are designed in terms of sieving sizes ($\sqrt[20]{10}$, $\sqrt[10]{10}$, 2 etc. according to the considered standard).
– The RVE sizes depicted in Figs. 5a and 6 (with and without wall effect) are found to grow uniformly and quite linearly with *D* and *DF* values. Note that, the selected range of *DF* values (2.5–3) covers a wide range of concrete characteristics that can lead to different sizes of the RVE, roughly in the ratio of 1:3 (see, e.g., Fig. 7 where RVE size and some physical properties of concrete are plotted against *DF* values). This suggests that, *DF* value must be considered as a parameter when determining the RVE.

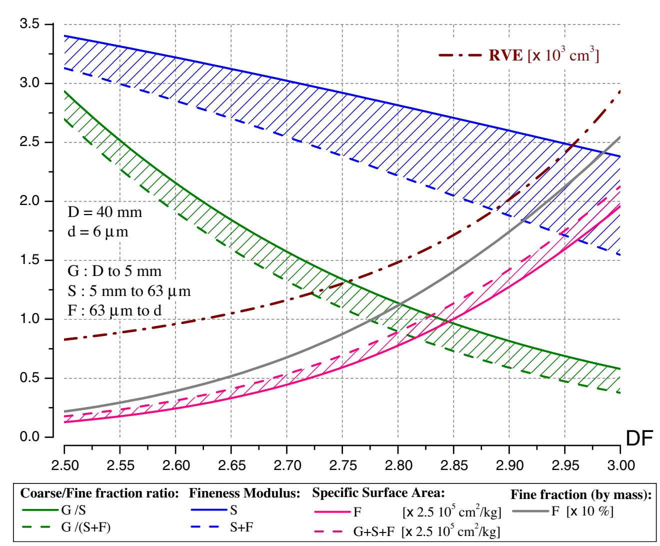


Fig. 7. Plot showing the relationship between *DF* value of the solid concrete mixture distribution and some concrete physical properties, derived from data simply calculated from fractal PSDs (by assuming no agglomeration and invariant density and shape of all particles).

– Data taken from Fig. 5b shows that, from a material point of view, the ratio between the three-dimensional RVE size and the largest aggregate size should be within the range of 2.4–3.7, for *DF* values ranging from 2.5 to 3 (without boundary effect). Furthermore, if we consider, as stated by Lecomte and Thomas, 1992, that ordinary concrete mixtures correspond generally to $DF \leqslant 2.8$, we can suggest from f_A data in Fig. 5b, that the minimum length of a cubic sample of ordinary concrete must be nearly 2.9 times more elevated than the maximum aggregate size. When comparing these findings with outcomes from previous studies quoted earlier in this paper, we can consider that the obtained RVE sizes are slightly lower but close enough to those reported in these studies.
– As specific sizes of sample and aggregate must be used for concrete testing, we have also try to extend our results to locate the maximum allowable aggregate size *Dn* (in terms of nominal size) that corresponds to a given standard mold size X_i and, accordingly, establish the relationship between these two dimensions. Data derived from Fig. 6, and displayed in Table 2 indicates that, whatever the mold shape, correlation values between *Dn* and X_i must be at least 4.0–4.5 in the general case if $DF \leqslant 3$ and, at least 3.5 in the case of ordinary concretes, i.e. $DF \leqslant 2.8$. These values seem to be rather close to those traditionally used for classical mechanical testing protocols. For instance French NF EN 12390-1 standard recommends the value of 3.5, whatever mold shape (AFNOR, 2001).

5. Conclusion

This paper aims to provide a novel analytical approach to estimate the RVE of concrete mixtures using the fractal analysis. The starting point of this study is to consider concrete mixture as a periodic and fractal structure that the RVE coincides with the smallest cell of periodicity in this material. The main results of the proposed approach state as follows:

- Through mathematical formulas, the proposed approach correlates the RVE, in terms of size and particle numbers, with some concrete mix design parameters, such as size distribution, particle size and solid volume fraction.
- The PSD of the concrete mixture, i.e. *DF* value, must be considered as a parameter when selecting the suitable RVE.
- The RVE size must be at least 2.4–3.7 times the maximum particle size in the concrete, for increasing *DF* values from 2.5 to 3 (without considering boundary effect).
- For laboratory concrete testing, standard sample size, whatever mold shape, must be at least 3.5 times the nominal maximum size of aggregates, in the case of ordinary concrete mixtures, i.e. $DF \leqslant 2.8$.

Despite the fact that the RVE used in this study is only related to the material that composes it, and despite the number of simplifying assumptions adopted; obtained results seem to be close enough to those of previous studies based primarily on laborious and time consuming laboratory testing and modeling.

This paper has illustrated the usefulness of the fractal analysis to predict the RVE of granular materials and concrete mixtures, via simple mathematical formulas. The scope of application of the proposed approach can be extended to numerous solid materials that consist of grains and that the PSD may be fractal.

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