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Approximate symmetries and conservation laws for Itô and Stratonovich dynamical systems

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Abstract

A new definition for the approximate symmetries of Itô dynamical system is given. Determining systems of approximate symmetries for Itô and Stratonovich dynamical systems have been obtained. It has been shown that approximate conservation laws can be found from the approximate symmetries of stochastic dynamical systems which do not arise from a Hamiltonian. The results have been applied to an example.

Keywords: Itô and Stratonovich dynamical systems; Approximate symmetry; Fokker–Planck equation; Conservation laws

1. Introduction

There has been a growing interest in the literature to extend Lie's theorems to stochastic differential equations (see, for instance, [1–6]). The symmetry definition given in [5] maps Itô differential to another Itô differential. Therefore, determining systems for the symmetries of the Itô and the Stratonovich stochastic dynamical systems become deterministic (i.e., no Wiener terms appear). Here we adopt the approach to symmetries of stochastic systems given in [5] and apply it for determining approximate symmetries.

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Approximate symmetry analysis of the deterministic differential equations has been developed by Baikov et al. [7] (see also [8]). It has been extended by incorporating resonances occurring in deterministic dynamical systems in [9]. Approximate symmetries of the stochastic dynamical systems have not been studied in the literature. Therefore we will investigate this problem in the sequel.

Here we consider the Itô dynamical system of the form

$$dx_i = \left(f_i^0(\mathbf{x},t) + \varepsilon f_i^1(\mathbf{x},t)\right)dt + \sqrt{\varepsilon}\, c_{i\alpha}\, dB_\alpha \quad (i = 1,\dots,n;\ \alpha = 1,\dots,r), \tag{1}$$

where $f_i^0(\mathbf{x},t) + \varepsilon f_i^1(\mathbf{x},t)$ is an approximate drift vector, $c_{i\alpha}$ is a constant diffusion matrix, dB_α is a vector valued Wiener process, and $\varepsilon \ll 1$ is a small positive perturbation parameter.

We begin Section 2 by giving the definitions of approximate symmetries. This allows us to obtain deterministic hierarchy of determining system involving random variables.

It has been shown that the approximate symmetries of the Fokker–Planck equation and the approximate symmetries of the Itô systems are related. Furthermore it has been shown [5] that approximate conservation laws can be obtained from approximate symmetries without resorting to Noether's theorem. We illustrate this result by an example of a concrete problem.

In what follows, the summation convention applies to repeated indices.

2. Approximate symmetries of stochastic dynamical systems

Consider a one-parameter family of curves

$$z = \phi_\varepsilon(t)$$

parametrized by ε.

If ε is a small continuous (deterministic) parameter, then the above family of curves defines what is called in [7] an *approximate transformation*. It is usually written, by replacing t by a, in the form

$$\bar{x} = \phi(x,a,\varepsilon),$$

where x denotes the transformed point.

If one replaces the deterministic parameter ε by a random variable ω, one obtains what is called a *stochastic process* (sometimes termed also a random process). It is written in the form

$$X_\omega(t).$$

Different stochastic processes are distinguished by choosing ω from different probability spaces.

2.1. Approximate symmetries of deterministic equations

We outline here the main notions of the theory of approximate symmetries in the first order of precision. Recall the approximate equation $f \approx g$ means that

$$f(x,\varepsilon) = g(x,\varepsilon) + o(\varepsilon)$$

or equivalently

$$f(x,\varepsilon) = g(x,\varepsilon) + O(\varepsilon^2).$$

Given a function $f(x,a,\varepsilon)$, we will choose the standard representative of the class of all functions $g(x,a,\varepsilon) \approx f(x,a,\varepsilon)$ in the form

$$f_0(x,a) + \varepsilon f_1(x,a). \tag{2}$$

An *approximate transformation*

$$\bar{x}^i \approx f_0^i(x,a) + \varepsilon f_1^i(x,a), \quad i = 1,\dots,n, \tag{3}$$

is the set of all invertible transformations

$$\bar{x}^i = f^i(x,a,\varepsilon) \tag{4}$$

such that

$$f^i(x,a,\varepsilon) \approx f_0^i(x,a) + \varepsilon f_1^i(x,a). \tag{5}$$

It is assumed that the functions $f_0^i(x,a)$ and $f_1^i(x,a)$ are defined and regular in a neighborhood of $a = 0$ and that, in this neighborhood,

$$f_0^i(x,a) = x^i, \qquad f_1^i(x,a) = 0$$

if and only if $a = 0$.

We say that Eq. (3) defines a one-parameter *approximate transformation group* if any representation (4) of (3) satisfies the group property

$$f\big(f(x,a,\varepsilon),b,\varepsilon\big) \approx f(x,c,\varepsilon), \quad c = \phi(a,b). \tag{6}$$

Upon introducing the canonical parameter a, the group property (6) can be written in the form

$$f\big(f(x,a,\varepsilon),b,\varepsilon\big) \approx f(x,a+b,\varepsilon). \tag{7}$$

In this definition, unlike the usual group property, f does not necessarily denote the same function at each occurrence. Specifically, the approximate group property can be equivalently written in the form

$$f\big(g(x,a,\varepsilon),b,\varepsilon\big) \approx h(x,a+b,\varepsilon) \tag{8}$$

with any functions $f \approx g \approx h$.

Example 1. Let us consider the one-dimensional case ($n = 1$) and consider the family of functions approximately equal to

$$f(x,a,\varepsilon) = x + a\left(1 + \varepsilon x + \frac{1}{2}\varepsilon a\right).$$

The function f has the standard form (2), $f = f_0(x,a) + \varepsilon f_1(x,a)$ with

$$f_0(x,a) = x + a, \qquad f_1(x,a) = ax + \frac{a^2}{2}.$$

The family in question forms an approximate transformation group since

$$f\big(f(x,a,\varepsilon),b,\varepsilon\big) = f(x,a+b,\varepsilon) + \frac{ab(2x+a)}{2}\,\varepsilon^2.$$

The *generator* of an approximate transformation group (3) is the set of all first-order linear differential operators

$$X = \xi^i(x,\varepsilon)\frac{\partial}{\partial x^i}$$

such that $\xi^i(x,\varepsilon) \approx \xi_0^i(x) + \varepsilon\xi_1^i(x)$, where

$$\xi_0^i(x) = \frac{\partial f_0^i(x,a)}{\partial a}\bigg|_{a=0}, \quad \xi_1^i(x) = \frac{\partial f_1^i(x,a)}{\partial a}\bigg|_{a=0}, \quad i = 1,\ldots,n.$$

It is convenient to identify X with its *canonical representative*:

$$X = \big(\xi_0^i(x) + \varepsilon\xi_1^i(x)\big)\frac{\partial}{\partial x^i}. \tag{9}$$

Let

$$X = X_0 + \varepsilon X_1 \tag{10}$$

be a given approximate generator, where

$$X_0 = \xi_0^i(x)\frac{\partial}{\partial x^i}, \qquad X_1 = \xi_1^i(x)\frac{\partial}{\partial x^i}.$$

The corresponding approximate transformation group

$$\bar{x}^i = \bar{x}_0^i + \varepsilon\bar{x}_1^i, \quad i = 1,\ldots,n,$$

is determined by the following system of equations [7]:

$$\frac{d\bar{x}_0^i}{da} = \xi_0^i(\bar{x}_0), \quad \bar{x}_0^i\big|_{a=0} = x^i, \tag{11}$$

$$\frac{d\bar{x}_1^i}{da} = \sum_{k=1}^n \bar{x}_1^k \cdot \left[\frac{\partial\xi_0^i(x)}{\partial x^k}\right]_{x=\bar{x}_0} + \xi_1^i(\bar{x}_0), \quad \bar{x}_1^i\big|_{a=0} = 0. \tag{12}$$

They are called the *approximate Lie equations*.

Example 2. Let $n = 2$ and let

$$X = (1+\varepsilon x^2)\frac{\partial}{\partial x} + \varepsilon xy\frac{\partial}{\partial y}.$$

Here $\xi_0(x,y) = (1,0)$ and $\xi_1(x,y) = (x^2, xy)$, and the approximate Lie equations (11)–(12) are written as

$$\frac{d\bar{x}_0}{da} = 1, \quad \frac{d\bar{y}_0}{da} = 0, \quad \bar{x}_0|_{a=0} = x,\ \bar{y}_0|_{a=0} = y,$$
$$\frac{d\bar{x}_1}{da} = (\bar{x}_0)^2, \quad \frac{d\bar{y}_1}{da} = \bar{x}_0\bar{y}_0, \quad \bar{x}_1|_{a=0} = 0,\ \bar{y}_1|_{a=0} = 0.$$

Integration yields

$$\bar{x} \approx x + a + \varepsilon\left(ax^2 + a^2x + \frac{a^3}{3}\right), \qquad \bar{y} \approx y + \varepsilon\left(axy + \frac{a^2}{2}y\right).$$

Let us denote by $z \equiv (z^1, \ldots, z^N) = (x, u, u_{(1)}, \ldots, u_{(k)})$ the set of independent variables $x = (x^1, \ldots, x^n)$ and dependent variables $x = (u^1, \ldots, u^m)$ together with the partial derivatives $u_{(1)}, \ldots, u_{(k)}$ of u with respect to x of the respective orders $1, \ldots, k$. Consider an approximate differential equation of order k,

$$F(z, \varepsilon) \equiv F_0(z) + \varepsilon F_1(z) \approx 0. \tag{13}$$

We include here also the systems of equations assuming that F_0 and F_1 can be vector valued functions. Let G be a one-parameter approximate transformation group and let its prolongation to the derivatives involved in Eq. (13) have the form

$$\bar{z}^i \approx f(z, a, \varepsilon) \equiv f_0^i(z, a) + \varepsilon f_1^i(z, a), \quad i = 1, \ldots, N. \tag{14}$$

We say that Eq. (13) is *approximately invariant* if the equation $F(f(z, a, \varepsilon), \varepsilon) = o(\varepsilon)$ holds whenever $z = (z^1, \ldots, z^N)$ satisfies (13). We write this condition as follows:

$$F\big(f(z, a, \varepsilon), \varepsilon\big)\big|_{(13)} = o(\varepsilon).$$

Consider an approximate transformation group of the independent and dependent variables with the generator

$$X = X^0 + \varepsilon X^1, \tag{15}$$

where

$$X^0 = \xi_0^i(x, u)\frac{\partial}{\partial x^i} + \eta_0^\alpha(x, u)\frac{\partial}{\partial u^\alpha}, \qquad X^1 = \xi_1^i(x, u)\frac{\partial}{\partial x^i} + \eta_1^\alpha(x, u)\frac{\partial}{\partial u^\alpha}. \tag{16}$$

We will write the prolongation of X to the derivatives involved in Eq. (13) in the following form:

$$\tilde{X} = \tilde{X}^0 + \varepsilon \tilde{X}^1 \equiv \zeta_0^k(z)\frac{\partial}{\partial z^k} + \varepsilon \zeta_1^k(z)\frac{\partial}{\partial z^k}. \tag{17}$$

Equation (13) is approximately invariant under the approximate transformation group with the generator (15) if and only if

$$\big[\tilde{X}^0 F_0(z) + \varepsilon\big(\tilde{X}^1 F_0(z) + \tilde{X}^0 F_1(z)\big)\big]\big|_{(13)} = o(\varepsilon). \tag{18}$$

Equation (18) is the *determining equation* for infinitesimal approximate symmetries and can be written in the form (see, e.g., [10, Section 9.5.2])

$$\tilde{X}^0 F_0(z) = \lambda(z) F_0(z), \tag{19}$$

$$\tilde{X}^1 F_0(z) + \tilde{X}^0 F_1(z) = \lambda(z) F_1(z). \tag{20}$$

The factor $\lambda(z)$ is determined by (19) and then substituted into (20). The latter equation must hold for all solutions of $F_0(z) = 0$.

It follows from the determining equations that if $X = X^0 + \varepsilon X^1$ is an approximate symmetry with $X^0 \neq 0$, then the operator

$$X^0 = \xi_0^i(x,u)\frac{\partial}{\partial x^i} + \eta_0^\alpha(x,u)\frac{\partial}{\partial u^\alpha} \tag{21}$$

is an exact symmetry for the unperturbed equation

$$F_0(z) = 0.$$

The corresponding approximate symmetry generator $X = X^0 + \varepsilon X^1$ for the perturbed equation (13) is called a *deformation of the infinitesimal symmetry* X^0 of Eq. (21) caused by the perturbation $\varepsilon F_1(z)$.

Example 3. Consider the following perturbation of the wave equation:

$$u_{tt} - u_{xx} - u_{yy} + \varepsilon u_t = 0. \tag{22}$$

The operator

$$X = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x} - \frac{\varepsilon}{2}xu\frac{\partial}{\partial u}$$

is one of *exact* symmetries for Eq. (22). An example of an approximate symmetries of Eq. (22) is

$$X = (t^2 + x^2 + y^2)\frac{\partial}{\partial t} + 2tx\frac{\partial}{\partial x} + 2ty\frac{\partial}{\partial y} - tu\frac{\partial}{\partial u} - \frac{\varepsilon}{2}(t^2 + x^2 + y^2)u\frac{\partial}{\partial u}.$$

The symmetry X^0 of the unperturbed equation (21) is called a *stable symmetry* if there exists X^1 such that $X = X^0 + \varepsilon X^1$ is an approximate symmetry for the perturbed equation (13). In particular, if the most general symmetry Lie algebra of Eq. (21) is stable, we say that the perturbed equation (13) *inherits the symmetries of the unperturbed equation.*

2.2. Approximate symmetries of the Itô dynamical system

Definition 1. If infinitesimal transformations

$$\bar{x}_i \approx x_i + a\big(\xi_i^0(\mathbf{x},t) + \varepsilon\xi_i^1(\mathbf{x},t)\big), \qquad \bar{t} \approx t + a\big(\tau^0(\mathbf{x},t) + \varepsilon\tau^1(\mathbf{x},t)\big) \tag{23}$$

leave Eq. (1) and the identities (see [11])

$$\text{(i)}\quad dB_\alpha\, dB_\beta = \delta_{\alpha\beta}\, dt, \qquad \text{(ii)}\quad dt\, dB_\alpha = 0, \qquad \text{(iii)}\quad dt\, dt = 0, \tag{24}$$

approximately form invariant, i.e.,

$$d\bar{x}_i = \big[\bar{f}_i^0(\bar{\mathbf{x}},t) + \varepsilon \bar{f}_i^1(\bar{\mathbf{x}},t)\big]d\bar{t} + \sqrt{\varepsilon}\, c_{i\alpha}(\bar{\mathbf{x}},t)\, d\bar{B}_\alpha$$
$$(i = 1,\dots,n;\ \alpha = 1,\dots,r), \tag{25}$$

then they are called approximate symmetry transformations for the Itô dynamical system. Here a is a group parameter.

Why do we require (24) to remain invariant under the transformations (23)? To be able to provide an answer to this question, we now consider the evolution of the sufficiently smooth scalar function $I(\mathbf{x},t)$ under the flow of the Itô equations (1), i.e., Itô 's formula (see, e.g., [11] and [12])[1]

$$dI(\mathbf{x},t) = I_{,t}\,dt + I_{,j}\,dx_j + \frac{1}{2}I_{,jk}\,dx_j\,dx_k. \tag{26}$$

Substitution of dx_i defined by (1) into Eq. (26) yields

$$\begin{aligned} dI &= \big(I_{,t} + \big(f_j^0 + \varepsilon f_j^1\big)I_{,j}\big)\,dt + \sqrt{\varepsilon}\,I_{,j}c_{j\alpha}\,dB_\alpha + \frac{\varepsilon}{2}I_{,jk}c_{j\alpha}c_{k\beta}\,dB_\alpha\,dB_\beta \\ &\quad + \frac{1}{2}I_{,jk}\big[\big(f_j^0 + \varepsilon f_j^1\big)\big(f_k^0 + \varepsilon f_k^1\big)\,dt\,dt + \sqrt{\varepsilon}\,\big(f_j^0 + \varepsilon f_j^1\big)c_{k\beta}\,dt\,dB_\beta \\ &\quad + \sqrt{\varepsilon}\,\big(f_k^0 + \varepsilon f_k^1\big)c_{j\alpha}\,dt\,dB_\alpha\big]. \end{aligned} \tag{27}$$

Using the identities given in (24) in (27) leads to Itô differential of the scalar function $I(\mathbf{x},t)$,

$$dI = \left[I_{,t} + f_j^0 I_{,j} + \varepsilon\left(f_j^1 I_{,j} + \frac{1}{2}c_{j\alpha}c_{k\alpha}I^m_{,jk}\right)\right]dt + \sqrt{\varepsilon}\,c_{j\alpha}I_{,j}\,dB_\alpha. \tag{28}$$

Form invariance of (1) under transformations (23) requires that Itô differential of $\bar{I}(\bar{\mathbf{x}},\bar{t})$ should read as

$$d\bar{I} = \left[\bar{I}_{,\bar{t}} + \bar{f}_j^0 \bar{I}_{,j} + \varepsilon\left(\bar{f}_j^1 \bar{I}_{,j} + \frac{1}{2}c_{j\alpha}c_{k\alpha}\bar{I}_{,jk}\right)\right]d\bar{t} + \bar{I}_{,j}\bar{g}_{j\alpha}\,d\bar{B}_\alpha. \tag{29}$$

For this to happen the identities (24) have to remain form invariant under (23):

$$\text{(i)}\quad d\bar{B}_\alpha\,d\bar{B}_\beta = \delta_{\alpha\beta}\,d\bar{t}, \qquad \text{(ii)}\quad d\bar{t}\,d\bar{B}_\alpha = 0, \qquad \text{(iii)}\quad d\bar{t}\,d\bar{t} = 0. \tag{30}$$

This justifies the idea behind the symmetry definition.

We now proceed to seek the determining system for the symmetries of (1). To achieve this we have to calculate each term in (29) in terms of the original variables $\mathbf{x}$ and t,

$$\begin{aligned} &\bar{f}_i^m\big(\mathbf{x} + a\big(\xi_i^0 + \varepsilon\xi_i^1\big), t + a(\tau^0 + \varepsilon\tau^1)\big) \\ &\quad = f_i^m(\mathbf{x},t) + a\big[\xi_j^0 f^m_{i,j} + \tau^0 f^m_{i,t} + \varepsilon\big(\xi_j^1 f^m_{i,j} + \tau^1 f^m_{i,t}\big)\big] + O(a^2). \end{aligned} \tag{31}$$

The Itô differential of (23) reads

$$\begin{aligned} &\text{(i)}\quad d\bar{x}_i = dx_i + a\big(d\xi_i^0 + \varepsilon d\xi_i^1\big) + O(a^2), \\ &\text{(ii)}\quad d\bar{t} = dt + a(d\tau^0 + \varepsilon d\tau^1) + O(a^2), \end{aligned} \tag{32}$$

where

$$\begin{aligned} d\xi_i^m &= \left[\xi^m_{i,t} + f_j^0\xi^m_{i,j} + \varepsilon\left(f_j^1\xi^m_{i,j} + \frac{1}{2}c_{j\alpha}c_{k\alpha}\xi^m_{i,jk}\right)\right]dt + \sqrt{\varepsilon}\,c_{j\alpha}\xi^m_{i,j}\,dB_\alpha, \\ d\tau^m &= \left[\tau^m_{,t} + f_j^0\tau^m_{,j} + \varepsilon\left(f_j^1\tau^m_{,j} + \frac{1}{2}c_{j\alpha}c_{k\alpha}\tau^m_{,jk}\right)\right]dt + \sqrt{\varepsilon}\,c_{j\alpha}\tau^m_{,j}\,dB_\alpha. \end{aligned}$$

[1] We will use the comma for the partial derivative with respect to the coordinate appearing in the subscript, e.g., $I_{,t} = \partial I/\partial t$.

Rendering (32)(ii) back to the right-hand side of (30)(i) yields

$$d\bar{B}_\alpha\, d\bar{B}_\beta = dB_\alpha\, dB_\beta + a\bigg[\tau^0_{,t} + f^0_j\tau^0_{,j} + \varepsilon\bigg(\tau^1_{,t} + f^0_j\tau^1_{,j} + \varepsilon f^1_j\tau^0_{,j} + \frac{1}{2}c_{j\alpha}c_{k\alpha}\tau^0_{,jk}\bigg) + \varepsilon^{1/2}c_{j\gamma}(\tau^0 + \varepsilon\tau^1)_{,j}\frac{dB_\gamma}{dt}\bigg]dB_\alpha\, dB_\beta.$$

Let us first set $\beta = \alpha$ in above equation and then Taylor expansion leads to

$$d\bar{B}_\alpha = dB_\alpha + \frac{1}{2}a\bigg[\tau^0_{,t} + f^0_j\tau^0_{,j} + \varepsilon\bigg(\tau^1_{,t} + f^0_j\tau^1_{,j} + f^1_j\tau^0_{,j} + \frac{1}{2}c_{j\alpha}c_{k\alpha}\tau^0_{,jk}\bigg) + \varepsilon^{1/2}c_{j\gamma}(\tau^0 + \varepsilon\tau^1)_{,j}\frac{dB_\gamma}{dt}\bigg]dB_\alpha. \tag{33}$$

Note, that the infinitesimal transformation law given in (33) has been partially captured in [3] by considering so-called projectable symmetries, i.e., by imposing the restriction $\tau = \tau(t)$, and hence obtaining the similar restriction of the infinitesimal transformations of probability density function. In [4], the authors obtain exactly the same formula but they rely on a theorem given in [12].

We continue our calculations and consider the remaining conditions. Let us consider (30)(ii). Substituting (32)(ii) and (33) into the left-hand side of (30)(ii) we obtain

$$d\bar{t}\, d\bar{B}_\alpha = g_{j\alpha}\tau_{,j}\, dt = 0.$$

This leads to

$$c_{j\alpha}\tau^0_{,j} = 0, \qquad c_{j\alpha}\tau^1_{,j} = 0. \tag{34}$$

Rendering (34) back into (33) yields

$$d\bar{B}_\alpha = dB_\alpha + \frac{1}{2}a\bigg[\tau^0_{,t} + f^0_j\tau^0_{,j} + \varepsilon\bigg(\tau^1_{,t} + f^0_j\tau^1_{,j} + \varepsilon f^1_j\tau^0_{,j} + \frac{1}{2}c_{j\alpha}c_{k\alpha}\tau^0_{,jk}\bigg)\bigg]dB_\alpha. \tag{35}$$

Equation given in (30)(iii) do not introduce new constraints on $\tau(\mathbf{x}, \mathbf{t})$ and $dB_\alpha(t)$.

Remark. Note, that the transformation law (35) for the infinitesimal increments of Wiener processes is different from the one given in [4] (namely, cf. (35) with Eq. (20) in [4]) and it generalizes the result obtained in [3]. Notice also that the first condition (34) was imposed in [4] as a sufficient condition to get rid of stochastic terms thus simplifying the determining equations. On the contrary, we show that this condition is not only sufficient but also necessary and that Eq. (34) follows from Eq. (24) of the symmetry definition.

We now substitute (31), (32) and (35) into equation given in (25) to obtain

$$dx_i = \big(f^0_i + \varepsilon f^1_i\big)\, dt + \sqrt{\varepsilon}\, c_{i\alpha}\, dB_\alpha + a\bigg[T_i - Z_i + \frac{\sqrt{\varepsilon}}{2}c_{i\alpha} Q\, dB_\alpha\bigg],$$

where

$$T_i = \bigg[\xi_j^0 f_{i,j}^0 + \tau^0 f_{i,t}^0 + \tau_{,t}^0 f_i^0 + f_i^0 f_j^0 \tau_{,j}^0 + \varepsilon\bigg(\xi_j^1 f_{i,j}^0 + \tau^1 f_{i,t}^0 + f_i^0 \tau_{,t}^1 + f_j^0 \tau_{,j}^1 + f_j^1 \tau_{,j}^0 + \frac{1}{2} c_{j\alpha} c_{k\alpha} \tau_{,jk}^0\bigg) + \xi_j^0 f_{i,j}^1 + \tau^0 f_{i,t}^1 + \tau_{,t}^0 f_i^1 + f_i^1 f_j^0 \tau_{,j}^0\bigg] dt,$$

$$Z_i = \bigg[\xi_{i,t}^0 + f_j^0 \xi_{i,j}^0 + \varepsilon\bigg(\xi_{i,t}^1 + f_j^0 \xi_{i,j}^1 + f_j^1 \xi_{i,j}^0 + \frac{1}{2} c_{j\alpha} c_{k\alpha} \xi_{i,jk}^0\bigg)\bigg] dt + \sqrt{\varepsilon}\big(c_{j\alpha}\xi_{i,j}^0 + \varepsilon c_{j\alpha}\xi_{i,j}^1\big)\, dB_\alpha,$$

$$Q = \tau_{,t}^0 + f_j^0 \tau_{,j}^0 + \varepsilon\bigg(\tau_{,t}^1 + f_j^0 \tau_{,j}^1 + f_j^1 \tau_{,j}^0 + \frac{1}{2} c_{j\alpha} c_{k\alpha} \tau_{,jk}^0\bigg).$$

We deduce from these equations the following theorem.

Theorem 1. *The infinitesimal transformations* (23) *provide approximate symmetries for the Itô equations* (1) *if and only if the infinitesimals* $\xi_i(\mathbf{x}, t)$ *and* $\tau(\mathbf{x}, t)$ *satisfy the following determining equations*:

$$\begin{aligned} &\xi_{i,t}^0 + f_j^0 \xi_{i,j}^0 - \xi_j^0 f_{i,j}^0 - \tau^0 f_{i,t}^0 - f_i^0 \tau_{,t}^0 - f_i^0 f_j^0 \tau_{,j}^0 = 0, \\ &c_{j\alpha}\xi_{i,j}^0 - \frac{c_{i\alpha}}{2}\big(\tau_{,t}^0 + f_j^0 \tau_{,j}^0\big) = 0, \qquad c_{j\alpha}\tau_{,j}^0 = 0, \end{aligned} \tag{36}$$

and

$$\begin{aligned} &\xi_{i,t}^1 + f_j^0 \xi_{i,j}^1 - \xi_j^1 f_{i,j}^0 - \tau^1 f_{i,t}^0 - f_i^0 \tau_{,t}^1 - f_i^0 f_j^0 \tau_{,j}^1 \\ &\quad = -f_j^1 \xi_{i,j}^0 + \xi_j^0 f_{i,j}^1 + \tau^0 f_{i,t}^1 + f_i^1 \tau_{,t}^0 + f_i^1 f_j^0 \tau_{,j}^0 + f_i^0 f_j^1 \tau_{,j}^0 \\ &\qquad + \frac{1}{2} f_i^0 c_{j\alpha} c_{k\alpha} \tau_{,jk}^0 - \frac{1}{2} c_{j\alpha} c_{k\alpha} \xi_{i,jk}^0, \\ &c_{j\alpha}\xi_{i,j}^1 - \frac{c_{i\alpha}}{2}\big(\tau_{,t}^1 + f_j^0 \tau_{,j}^1\big) = \frac{c_{i\alpha}}{2}\bigg(f_j^1 \tau_{,j}^0 + \frac{1}{2} c_{j\beta} c_{k\beta} \tau_{,jk}^0\bigg), \qquad c_{j\alpha}\tau_{,j}^1 = 0. \end{aligned} \tag{37}$$

Notice that this system does not involve Wiener terms (i.e., dB_α) and hence, it is deterministic.

2.3. Approximate symmetries of Stratonovich dynamical systems

The Stratonovich dynamical system with a small parameter ε reads as

$$dx_i = \big(f_i^0(\mathbf{x}, t) + \varepsilon f_i^1(\mathbf{x}, t)\big)\, dt + \sqrt{\varepsilon}\, c_{i\alpha}(\mathbf{x}, t) \circ dB_\alpha, \tag{38}$$

where $i = 1, \ldots, n$; $\alpha = 1, \ldots, r$, and $\circ$ denotes the Stratonovich derivative.

Theorem 2. *The infinitesimal transformations* (23) *provide approximate symmetries for the Stratonovich dynamical system* (38) *if and only if the infinitesimals* $\xi_i(\mathbf{x}, t)$ *and* $\tau(\mathbf{x}, t)$ *satisfy the following determining equations*:

$$\begin{aligned} &\xi_{i,t}^0 + f_j^0 \xi_{i,j}^0 - \xi_j^0 f_{i,j}^0 - \tau^0 f_{i,t}^0 - f_i^0 \tau_{,t}^0 - f_i^0 f_j^0 \tau_{,j}^0 = 0, \\ &c_{j\alpha}\xi_{i,j}^0 - \frac{c_{i\alpha}}{2}\big(\tau_{,t}^0 + f_j^0 \tau_{,j}^0\big) = 0, \qquad c_{j\alpha}\tau_{,j}^0 = 0, \end{aligned} \tag{39}$$

and

$$\begin{aligned}
&\xi^1_{i,t} + f^0_j \xi^1_{i,j} - \xi^1_j f^0_{i,j} - \tau^1 f^0_{i,t} - f^0_i \tau^1_{,t} - f^0_i f^0_j \tau^1_{,j} \\
&\quad = -f^1_j \xi^0_{i,j} + \xi^0_j f^1_{i,j} + \tau^0 f^1_{i,t} + f^1_i \tau^0_{,t} + f^1_i f^0_j \tau^0_{,j} + f^0_i f^1_j \tau^0_{,j}, \\
&c_{j\alpha}\xi^1_{i,j} - \frac{c_{i\alpha}}{2}\big(\tau^1_{,t} + f^0_j \tau^1_{,j}\big) = \frac{c_{i\alpha}}{2}\big(f^1_j \tau^0_{,j}\big), \qquad c_{j\alpha}\tau^1_{,j} = 0.
\end{aligned} \tag{40}$$

The proof of this theorem is similar to that of Theorem 1. For the sake of brevity we omit the proof. However, one should notice that the Stratonovich differential obeys the standard chain rule [11]. This, in turn, leads to

$$\begin{aligned}
d\bar{B}_\alpha &= dB_\alpha + \frac{1}{2} a\big[\tau^0_{,t} + f^0_j \tau^0_{,j} + \varepsilon\big(\tau^1_{,t} + f^0_j \tau^1_{,j} + f^1_j \tau^0_{,j}\big)\big] \circ dB_\alpha, \\
d\xi^m_i &= \big[\xi^m_{i,t} + f^0_j \xi^m_{i,j} + \varepsilon f^1_j \xi^m_{i,j}\big]\,dt + \sqrt{\varepsilon}\, c_{j\alpha}\xi^m_{i,j}\, dB_\alpha, \\
d\tau^m &= \big[\tau^m_{,t} + f^0_j \tau^m_{,j} + \varepsilon f^1_j \tau^m_{,j}\big]\,dt + \sqrt{\varepsilon}\, c_{j\alpha}\tau^m_{,j}\, dB_\alpha,
\end{aligned}$$

which is used in the proof of Theorem 2.

Let us define the vector fields

$$\mathbf{D}^m = \frac{\partial}{\partial t} + f^m_j \frac{\partial}{\partial x_j}, \quad \mathbf{X}^m = \tau^m \frac{\partial}{\partial t} + \xi^m_k \frac{\partial}{\partial x_k}, \quad \mathbf{C}_\alpha = c_{l\alpha}\frac{\partial}{\partial x_l}, \quad m = 0, 1. \tag{41}$$

Equation (40) of the determining system can be rewritten now in the following form:

$$\begin{aligned}
&[\mathbf{D}^0, \mathbf{X}^0] = \big(\tau^0_{,t} + f^0_j \tau^0_{,j}\big)\mathbf{D}^0, \qquad [\mathbf{C}_\alpha, \mathbf{X}^0] = \frac{1}{2}\big(\tau^0_{,t} + f^0_j \tau^0_{,j}\big)\mathbf{C}_\alpha, \\
&[\mathbf{D}^0, \mathbf{X}^1] = -[\mathbf{D}^1, \mathbf{X}^0] + \big(\tau^1_{,t} + f^0_j \tau^1_{,j} + f^1_j \tau^0_{,j}\big)\mathbf{D}^0 + \big(\tau^0_{,t} + f^0_j \tau^0_{,j}\big)\mathbf{D}^1, \\
&[\mathbf{C}_\alpha, \mathbf{X}^1] = \frac{1}{2}\big(\tau^1_{,t} + f^0_j \tau^1_{,j} + f^1_j \tau^0_{,j}\big)\mathbf{C}_\alpha,
\end{aligned} \tag{42}$$

where the expressions in the left-hand sides are the Lie brackets.

3. The relation between approximate symmetries of the Fokker–Planck equation and the Itô system

To every Itô stochastic ordinary differential equation there corresponds a deterministic partial differential equation called the *Fokker–Planck equation*. Its normalizable solution is the probability distribution of the solution to the stochastic differential equation (see [13] and [14]).

For the Itô system (1), the associate Fokker–Planck equation has the form

$$F \equiv p_{,t} + f^0_k p_{,k} + f^0_{k,k} p + \varepsilon\left(f^1_k p_{,k} + f^1_{k,k} p - \frac{1}{2} c_{k\alpha} c_{l\alpha} p_{,kl}\right) = 0. \tag{43}$$

Here, we will discuss a correspondence between the approximate symmetries of Eqs. (1) and (43). Since the Fokker–Planck equation is a deterministic partial differential equation its approximate symmetries are found by using the method given in Section 2.1. Namely, we look for the approximate group generator (15)–(16) written in the form

$$X=\tau^0(\mathbf{x},t,p)\frac{\partial}{\partial t}+\xi_i^0(\mathbf{x},t,p)\frac{\partial}{\partial x_i}+\Pi^0(\mathbf{x},t,p)\frac{\partial}{\partial p}+\zeta_t^0\frac{\partial}{\partial p_{,t}}$$
$$+\varepsilon\left[\tau^1(\mathbf{x},t,p)\frac{\partial}{\partial t}+\xi_i^1(\mathbf{x},t,p)\frac{\partial}{\partial x_i}+\Pi^1(\mathbf{x},t,p)\frac{\partial}{\partial p}\right]$$

and consider its second prolongation:

$$X_{(2)}=\tau^0(\mathbf{x},t,p)\frac{\partial}{\partial t}+\xi_i^0(\mathbf{x},t,p)\frac{\partial}{\partial x_i}+\Pi^0(\mathbf{x},t,p)\frac{\partial}{\partial p}+\zeta_t^0\frac{\partial}{\partial p_{,t}}+\zeta_i^0\frac{\partial}{\partial p_{,i}}$$
$$+\zeta_{ij}^0\frac{\partial}{\partial p_{,ij}}+\varepsilon\left[\tau^1(\mathbf{x},t,p)\frac{\partial}{\partial t}+\xi_i^1(\mathbf{x},t,p)\frac{\partial}{\partial x_i}+\Pi^1(\mathbf{x},t,p)\frac{\partial}{\partial p}+\zeta_t^1\frac{\partial}{\partial p_{,t}}\right.$$
$$\left.+\zeta_i^1\frac{\partial}{\partial p_{,i}}+\zeta_{ij}^1\frac{\partial}{\partial p_{,ij}}\right],$$

where

$$\zeta_t^m=\Pi_{,t}^m+p_{,t}\Pi_{,p}^m-p_{,k}\left(\xi_{k,t}^m+p_{,t}\xi_{k,p}^m\right)-p_{,t}\left(\tau_{,t}^m+p_{,t}\tau_{,p}^m\right),$$
$$\zeta_i^m=\Pi_{,i}^m+p_{,i}\Pi_{,p}^m-p_{,k}\left(\xi_{k,i}^m+p_{,i}\xi_{k,p}^m\right)-p_{,t}\left(\tau_{,i}^m+p_{,i}\tau_{,p}^m\right),$$
$$\zeta_{ij}^m=D_j\left(\zeta_i^m\right)-p_{,it}\tau_{,j}^m-p_{,ik}\xi_{k,j}^m-p_{,ik}p_{,j}\xi_{k,p}^m-p_{,it}p_{,j}\tau_{,p}^m.$$

Here, $m=0,1$ and

$$D_j=\frac{\partial}{\partial x_j}+p_{,j}\frac{\partial}{\partial p}.$$

The approximate invariance criterion reads (cf. Eq. (18) and [10, Section 9.5.2])

$$X_{(2)}(F)|_{F=o(\varepsilon)}=o(\varepsilon).$$

It leads to the following determining system:

$$c_{i\alpha}c_{l\alpha}\xi_{k,i}^0+c_{k\alpha}c_{j\alpha}\xi_{l,j}^0-c_{i\alpha}c_{l\alpha}\left(\tau_{,t}^0+f_i^0\tau_{,i}^0\right)$$
$$+\varepsilon\left[c_{i\alpha}c_{l\alpha}\xi_{k,i}^1+c_{k\alpha}c_{j\alpha}\xi_{l,j}^1-c_{i\alpha}c_{l\alpha}\left(\tau_{,t}^1+f_i^1\tau_{,i}^0+f_i^0\tau_{,i}^1\right)\right]=0, \tag{44}$$

$$\xi_{k,t}^0-\left(\tau^0f_k^0\right)_{,t}+f_i^0\xi_{k,i}^0-\xi_i^0f_{k,i}^0-f_i^0f_k^0\tau_{,i}^0$$
$$+\varepsilon\left[\left(\tau^0+\tau^1f_k^0\right)_{,t}+f_i^1\xi_{k,i}^0+f_i^0\xi_{k,i}^1-\xi_i^0f_{k,i}^1-\xi_i^1f_{k,i}^0\right.$$
$$\left.-\frac{1}{2}c_{i\alpha}c_{j\alpha}\xi_{k,ij}^0-f_i^0f_k^1\tau_{,i}^0-f_i^1f_k^0\tau_{,i}^0-f_i^0f_k^0\tau_{,i}^1+c_{i\alpha}c_{j\alpha}\Pi_{,i}^1\right]=0, \tag{45}$$

$$\Pi_{,t}^{10}+f_i^0\Pi_{,i}^{10}+\left(\tau^0f_{i,i}^0\right)_{,t}+f_{k,ki}^0\xi_i^0+f_{i,i}^0f_m^0\tau_{,m}^0$$
$$+\varepsilon\left[\Pi_{,t}^{11}+f_i^0\Pi_{,i}^{11}+f_i^0\Pi_{,i}^{10}-\frac{1}{2}c_{i\alpha}c_{j\alpha}\Pi_{,ij}^{10}+\left(\tau^0f_{i,i}^1\right)_{,t}+\left(\tau^1f_{i,i}^0\right)_{,t}\right.$$
$$\left.+f_{k,ki}^0\xi_i^1+f_{k,ki}^1\xi_i^0+f_{i,i}^1f_m^0\tau_{,m}^0+f_{i,i}^0f_m^1\tau_{,m}^0+f_{i,i}^0f_m^0\tau_{,m}^1\right]=0, \tag{46}$$

and

$$\Pi^2_{,t} + f^0_k \Pi^2_{,k} + f^0_{k,k}\Pi^2 + \varepsilon\left(f^1_k \Pi^2_{,k} + f^1_{k,k}\Pi^2 - \frac{1}{2}c_{k\alpha}c_{l\alpha}\Pi^2_{,kl}\right) = 0, \tag{47}$$

where

$$\Pi^0(\mathbf{x},t,p) + \varepsilon\Pi^1(\mathbf{x},t,p) = \left(\Pi^{10}(\mathbf{x},t) + \varepsilon\Pi^{11}(\mathbf{x},t)\right)p + \Pi^2(\mathbf{x},t,\varepsilon),$$
$$\tau^m = \tau^m(\mathbf{x},t), \quad \xi^m_j = \xi^m_j(\mathbf{x},t), \quad m = 0,1.$$

Let us suppose that $\xi^m_i(\mathbf{x},t)$ and $\tau^m(\mathbf{x},t)$ satisfy (36)–(37). We now want to determine conditions under which they also satisfy (44)–(46). Equation (44) can be rewritten as

$$c_{l\alpha}T^0_k + c_{k\alpha}T^0_l + \varepsilon\left(c_{l\alpha}T^1_k + c_{k\alpha}T^1_l\right) = 0,$$

where

$$T^0_i = c_{j\alpha}\xi^0_{i,j} - \frac{c_{i\alpha}}{2}\left(\tau^0_{,t} + f^0_j\tau^0_{,j}\right), \qquad T^1_i = c_{j\alpha}\xi^1_{i,j} - \frac{c_{i\alpha}}{2}\left(\tau^1_{,t} + f^1_j\tau^0_{,j} + f^0_j\tau^1_{,j}\right).$$

Since $\xi_i(\mathbf{x},t)^m$ and $\tau(\mathbf{x},t)^m$ satisfy (37), we have $T^m_k = 0$, $m = 0,1$.

Differentiating (44) with respect to x_l and summing with (45) one obtains

$$c_{l\alpha}Q^0_{,l} = 0, \qquad c_{l\alpha}Q^1_{,l} = 0, \tag{48}$$

where

$$Q^0 = \Pi^{10} + \xi^0_{i,i} - f^0_i\tau^0_{,i}, \qquad Q^1 = \Pi^{11} + \xi^1_{i,i} - f^1_i\tau^0_{,i} - f^0_i\tau^1_{,i}.$$

Differentiating (44) with respect to x_k and x_l, then differentiating (45) with respect to x_k and finally summing the resulting equations with Eq. (46) one obtains

$$Q^0_{,t} + f^0_k Q^0_{,k} = 0, \qquad Q^1_{,t} + f^0_k Q^1_{,k} + f^1_k Q^0_{,k} = 0. \tag{49}$$

Solutions to Eqs. (48) and (49) provide a relation between the approximate symmetries of the Fokker–Planck equation and the approximate symmetries of Itô system. In case of constant solutions C_1 and C_2 we find

$$\Pi^{10} = -\xi^0_{i,i} + f^0_i\tau^0_{,i} + C_1, \qquad \Pi^{11} = -\xi^1_{i,i} + f^1_i\tau^0_{,i} + f^0_i\tau^1_{,i} + C_2.$$

Hence, we have just proven the following theorem.

Theorem 3. *Let*

$$X = \tau^0\frac{\partial}{\partial t} + \xi^0_j\frac{\partial}{\partial x_j} + \varepsilon\left(\tau^1\frac{\partial}{\partial t} + \xi^1_j\frac{\partial}{\partial x_j}\right)$$

be the generator of the approximate symmetry of the Itô system (1). *Then*

$$Y = X + \left[C_1 - \xi^0_{i,i} + f^0_i\tau^0_{,i} + \varepsilon\left(C_2 - \xi^1_{i,i} + f^1_i\tau^0_{,i} + f^0_i\tau^1_{,i}\right)\right]p\frac{\partial}{\partial p}. \tag{50}$$

Likewise, one can easily show that if Y is an approximate symmetry of the Fokker–Planck equation (43) then X is an approximate symmetry of the Itô system (1).

4. Approximate conservation laws

Conserved quantities of Stratonovich dynamical systems were considered in [1] and [2] without recourse to Hamiltonian formulation. Here we give an alternative theorem for approximate conservation laws. For more details, see [15].

Definition 2. We call $I^0(\mathbf{x},t)+\varepsilon I^1(\mathbf{x},t)$ an approximate conserved quantity if

$$I^0(\mathbf{x},t)+\varepsilon I^1(\mathbf{x},t)\approx C,\quad C=\text{const},$$

on the sample paths of the stochastic dynamical system.

The Itô differential of the conserved quantity $I(\mathbf{x},t)$ can be easily obtained from (28), namely

$$\begin{aligned} &dI^0+\varepsilon\, dI^1\\ &\quad=\left[I^0_{,t}+f^0_jI^0_{,j}+\varepsilon\big(I^1_{,t}+f^0_jI^1_{,j}+f^1_jI^0_{,j}\big)+\frac{\sqrt{\varepsilon}}{2}\big(c_{j\alpha}c_{k\alpha}I^0_{,jk}+\varepsilon c_{j\alpha}c_{k\alpha}I^1_{,jk}\big)\right]dt\\ &\qquad+\sqrt{\varepsilon}\,\big(c_{j\alpha}I^0_{,j}+\varepsilon c_{j\alpha}I^1_{,j}\big)\,dB_\alpha=O(\varepsilon^2). \end{aligned}$$

This leads to the following partial differential equations:

$$I^0_{,t}+f^0_jI^0_{,j}=0,\quad c_{j\alpha}I^0_{,j}=0,\quad I^1_{,t}+f^0_jI^1_{,j}=-f^1_jI^0_{,j},\quad c_{j\alpha}I^1_{,j}=0 \tag{51}$$

for an approximate conserved quantity to satisfy. One can easily show that an approximate conserved quantity of the Stratonovich dynamical system (38) should also satisfy (51).

Using the vector fields given in (41), Eq. (51) can be rewritten as

$$\mathbf{D}^0(I^0)=0,\qquad \mathbf{C}^0_\alpha(I^0)=0, \tag{52}$$

$$\mathbf{D}^0(I^1)=-\mathbf{D}^1(I^0),\qquad \mathbf{C}^0_\alpha(I^1)=0. \tag{53}$$

Theorem 4. *Suppose that Itô dynamical system* (1) *admits a stable approximate symmetry vector field of the form*

$$X=X_0+\varepsilon X_1=\tau^0\frac{\partial}{\partial t}+\xi^0_i\frac{\partial}{\partial x_i}+\varepsilon\left(\tau^1\frac{\partial}{\partial t}+\xi^1_i\frac{\partial}{\partial x_i}\right).$$

Then an approximate conservation law $I(\mathbf{x},t)=I^0(\mathbf{x},t)+\varepsilon I^1(\mathbf{x},t)$ can be found as

$$I^0(\mathbf{x},t)=\xi^0_{i,i}-f^0_i\tau^0_{,i}+P^0,\qquad I^1(\mathbf{x},t)=\xi^1_{i,i}-f^1_i\tau^0_{,i}-f^0_i\tau^1_{,i}+P^1, \tag{54}$$

where P^0 and P^1 satisfy Eqs. (45) *and* (46).

Proof. It follows immediately from the comparison between Eq. (51) and Eqs. (48) and (49). □

Theorem 5. *Let*

$$\operatorname{div}\mathbf{D}^m=0\quad(m=0,1)$$

and let

$$\mathbf{X}_1 = \mathbf{X}_1^0 + \varepsilon \mathbf{X}_1^1, \quad \ldots, \quad \mathbf{X}_n = \mathbf{X}_n^0 + \varepsilon \mathbf{X}_n^1$$

be the linearly independent symmetry vector fields satisfying the properties

$$\tau^0_{,t} + f^0_j \tau^0_{,j} = 0, \qquad \tau^1_{,t} + f^0_j \tau^1_{,j} + f^1_j \tau^0_{,j} = 0. \tag{55}$$

Then the approximate conserved quantity $I^0(\mathbf{x},t) + \varepsilon I^1(\mathbf{x},t)$ can be obtained from the approximate symmetries as follows:

$$\begin{aligned} I^0(\mathbf{x},t) &= \mathbf{X}_1^0 \rfloor \mathbf{X}_2^0 \rfloor \cdots \rfloor \mathbf{X}_n^0 \rfloor \Omega, \\ I^1(\mathbf{x},t) &= \mathbf{X}_1^1 \rfloor \mathbf{X}_2^0 \rfloor \cdots \rfloor \mathbf{X}_n^0 \rfloor \Omega + \mathbf{X}_1^0 \rfloor \mathbf{X}_2^1 \cdots \rfloor \mathbf{X}_n^0 \rfloor \Omega + \cdots \\ &\quad + \mathbf{X}_1^0 \rfloor \mathbf{X}_2^0 \rfloor \cdots \rfloor \mathbf{X}_n^1 \rfloor \Omega, \end{aligned} \tag{56}$$

where $\rfloor$ is the interior product and $\Omega = dx_1 \wedge \cdots \wedge dx_n$ is the volume form.

Proof. Let us calculate the Lie derivative of I^0 with respect to $\mathbf{D}^0$ to obtain

$$\begin{aligned} \mathbf{D}^0(I^0) \equiv \mathcal{L}_{\mathbf{D}^0} I^0 &= \left(\mathcal{L}_{\mathbf{D}^0}\mathbf{X}_1^0\right) \rfloor \mathbf{X}_2^0 \rfloor \cdots \rfloor \mathbf{X}_n^0 \rfloor \Omega + \cdots \\ &\quad + \mathbf{X}_1^0 \rfloor \left(\mathcal{L}_{\mathbf{D}^0}\mathbf{X}_2^0\right) \rfloor \cdots \rfloor \mathbf{X}_n^0 \rfloor \Omega + \mathbf{X}_1^0 \rfloor \mathbf{X}_2^0 \rfloor \cdots \rfloor \mathbf{X}_n^0 \rfloor (\mathcal{L}_{\mathbf{D}^0}\Omega). \end{aligned} \tag{57}$$

Using (55) in (42) we find

$$\left[\mathbf{D}^0, \mathbf{X}_j^0\right] \equiv \mathcal{L}_{\mathbf{D}^0}\mathbf{X}_j^0 = 0 \quad (j = 1, \ldots, n).$$

Since

$$\mathcal{L}_{\mathbf{D}^0}\Omega = \operatorname{div}\mathbf{D}^0\Omega$$

and $\operatorname{div}\mathbf{D}^0 = 0$, one has $\mathcal{L}_{\mathbf{D}^0}\Omega = 0$. Therefore all the terms on the right-hand side of (57) vanish. We now calculate the Lie derivative of I^0 with respect to $\mathbf{C}_\alpha$ to obtain

$$\begin{aligned} \mathbf{C}_\alpha(I^0) \equiv \mathcal{L}_{\mathbf{C}_\alpha} I^0 &= \left(\mathcal{L}_{\mathbf{C}_\alpha}\mathbf{X}_1^0\right) \rfloor \mathbf{X}_2^0 \rfloor \cdots \rfloor \mathbf{X}_n^0 \rfloor \Omega + \cdots \\ &\quad + \mathbf{X}_1^0 \rfloor \left(\mathcal{L}_{\mathbf{C}_\alpha}\mathbf{X}_2^0\right) \rfloor \cdots \rfloor \mathbf{X}_n^0 \rfloor \Omega + \mathbf{X}_1^0 \rfloor \mathbf{X}_2^0 \rfloor \cdots \rfloor \mathbf{X}_n^0 \rfloor (\mathcal{L}_{\mathbf{C}_\alpha}\Omega). \end{aligned} \tag{58}$$

Using (55) in (42) we find

$$\left[\mathbf{C}_\alpha, \mathbf{X}_j^0\right] \equiv \mathcal{L}_{\mathbf{C}_\alpha}\mathbf{X}_j^0 = 0 \quad (j = 1, \ldots, n).$$

Furthermore, $\mathcal{L}_{\mathbf{C}_\alpha}\Omega = 0$. Therefore all the terms on the right-hand side of (58) also vanish. Hence, Eq. (52) holds for

$$\mathbf{X}_1^0 \rfloor \mathbf{X}_2^0 \rfloor \cdots \rfloor \mathbf{X}_n^0 \rfloor \Omega.$$

Similar calculations for I^1 given in (56) lead to the equation

$$\begin{aligned} \mathcal{L}_{\mathbf{D}^0} I^1 &= -\left(\mathcal{L}_{\mathbf{D}^1}\mathbf{X}_1^0\right) \rfloor \mathbf{X}_2^0 \rfloor \cdots \rfloor \mathbf{X}_n^0 \rfloor \Omega - \mathbf{X}_1^0 \rfloor \left(\mathcal{L}_{\mathbf{D}^1}\mathbf{X}_2^0\right) \rfloor \cdots \rfloor \mathbf{X}_n^0 \rfloor \Omega + \cdots \\ &\quad - \mathbf{X}_1^0 \rfloor \mathbf{X}_2^0 \rfloor \cdots \rfloor \left(\mathcal{L}_{\mathbf{D}^1}\mathbf{X}_n^0\right) \rfloor \Omega. \end{aligned}$$

It can be rewritten as

$$\mathcal{L}_{\mathbf{D}^0} I^1 = -\mathcal{L}_{\mathbf{D}^1}\left[\mathbf{X}_1^0 \rfloor \mathbf{X}_2^0 \rfloor \cdots \rfloor \mathbf{X}_n^0 \rfloor \Omega\right].$$

Since

$$\mathbf{X}_1^0 \rfloor \mathbf{X}_2^0 \rfloor \cdots \rfloor \mathbf{X}_n^0 \rfloor \Omega = I^0,$$

we arrive at the equation

$$\mathcal{L}_{\mathbf{D}^0} I^1 = -\mathcal{L}_{\mathbf{D}^1} I^0.$$

It can be easily shown that

$$\mathcal{L}_{\mathbf{C}_\alpha} I^1 = 0.$$

Hence I^1 given in (56) satisfies (53). This completes the proof. □

5. An application

Let us consider the following stochastic dynamical system:

$$dx_1 = x_2\,dt, \qquad dx_2 = \big(-x_1 + \varepsilon f(x_1, x_2)\big)\,dt + \sqrt{\varepsilon}\,dB. \tag{59}$$

For this problem, we have

$$f_1^0 = x_2, \quad f_2^0 = -x_1, \quad f_1^1 = 0, \quad f_2^1 = f(x_1, x_2),$$
$$c_{11} = c_{12} = c_{22} = 0, \quad c_{21} = 1.$$

The determining equations (39) for Eq. (59) are written as

$$\xi^0_{1,t} + x_2\xi^0_{1,1} - x_1\xi^0_{1,2} - \xi^0_2 - x_2\tau^0_{,t} - x_2^2\tau^0_{,1} = 0,$$
$$\xi^0_{2,t} + x_2\xi^0_{2,1} - x_1\xi^0_{2,2} + \xi^0_1 + x_1\tau^0_{,t} + x_1x_2\tau^0_{,1} = 0,$$
$$\tau^0_{,2} = 0, \quad \xi^0_{1,2} = 0, \quad \xi^0_{2,2} - \frac{1}{2}\big(\tau^0_{,t} + x_2\tau^0_{,1},\big) = 0,$$

and yield

$$\tau^0 = C_1, \quad \xi_1^0 = C_2\cos t + C_3\sin t, \quad \xi_2^0 = C_3\cos t - C_2\sin t.$$

Now the determining equations (40) become

$$\xi^1_{1,t} + x_2\xi^1_{1,1} - x_1\xi^1_{1,2} - \xi^1_2 - x_2\tau^1_{,t} - x_2^2\tau^1_{,1} = 0,$$
$$\xi^1_{2,t} + x_2\xi^1_{2,1} - x_1\xi^1_{2,2} + \xi^1_1 + x_1\tau^1_{,t} + x_1x_2\tau^1_{,1} = \xi^0_1 f_{,1} + \xi^0_2 f_{,2},$$
$$\tau^1_{,2} = 0, \quad \xi^1_{1,2} = 0, \quad \xi^1_{2,2} - \frac{1}{2}\big(\tau^1_{,t} + x_2\tau^1_{,1}\big) = 0.$$

It follows that

$$\xi_1^1 = \frac{3}{2}\dot{\tau}^1 x_1 + Z(t), \qquad \xi_2^1 = \frac{1}{2}\dot{\tau}^1 x_2 + \frac{3}{2}\ddot{\tau}^1 x_1 + \dot{Z}(t),$$

where τ^1 satisfies the equation

$$\begin{aligned} &x_1\frac{d}{dt}\left(\frac{3}{2}\ddot{\tau}^1 + 2\tau^1\right) + 2\ddot{\tau}^1 x_2 + \ddot{Z} + Z \\ &\quad = (C_2\cos t + C_3\sin t)f_{,1} + (C_3\cos t - C_2\sin t)f_{,2}, \end{aligned} \tag{60}$$

where the dot denotes the differentiation with respect to t. It is manifest from Eq. (60) that we have to distinguish the following two cases:

(I) $f(x_1, x_2) = Ax_1 + Bx_2 + Cx_1^2 + Dx_1x_2 + Ex_2^2$,
(II) $f(x_1, x_2)$ is an arbitrary function.

The first case involves the following subcases:

I(i) $f(x_1, x_2) = Ax_1 + Bx_2 + C(x_1^2 - x_2^2)$. Then the approximate symmetry vector fields are:

$$\begin{aligned}\mathbf{X}_1 = {} & \cos t \frac{\partial}{\partial x_1} - \sin t \frac{\partial}{\partial x_2} + \varepsilon \biggl[\left(At \sin t - Bt \cos t - \frac{3}{2} C x_1 \cos t \right) \frac{\partial}{\partial x_1} \\ & + \left(At \sin t - Bt \cos t - \frac{1}{2} C x_2 \cos t + \frac{3}{2} C x_1 \sin t \right) \frac{\partial}{\partial x_2} \biggr],\end{aligned} \tag{61}$$

$$\begin{aligned}\mathbf{X}_2 = {} & \sin t \frac{\partial}{\partial x_1} + \cos t \frac{\partial}{\partial x_2} + \varepsilon \biggl[\left(At \cos t + Bt \sin t - \frac{3}{2} C x_1 \sin t \right) \frac{\partial}{\partial x_1} \\ & + \left(At \cos t + Bt \sin t - \frac{1}{2} C x_2 \sin t - \frac{3}{2} C x_1 \cos t \right) \frac{\partial}{\partial x_2} \biggr],\end{aligned} \tag{62}$$

$$\mathbf{X}_3 = \varepsilon \left(\cos t \frac{\partial}{\partial x_1} - \sin t \frac{\partial}{\partial x_2} \right), \qquad \mathbf{X}_4 = \varepsilon \left(\sin t \frac{\partial}{\partial x_1} + \cos t \frac{\partial}{\partial x_2} \right). \tag{63}$$

I(ii) $f(x_1, x_2) = Ax_1 + Bx_2 + Dx_1x_2 + C(x_1^2 - x_2^2)$. The approximate symmetry vector fields are $\mathbf{X}_2$, $\mathbf{X}_3$ and $\mathbf{X}_4$.

I(iii) $f(x_1, x_2) = Ax_1 + Bx_2 + Cx_1^2 + Ex_2^2$. The approximate symmetry vector fields are $\mathbf{X}_1$, $\mathbf{X}_3$ and $\mathbf{X}_4$.

I(iv) $f(x_1, x_2) = Ax_1 + Bx_2 + Cx_1^2 + Dx_1x_2 + Ex_2^2$. The approximate symmetry vector fields are $\mathbf{X}_3$ and $\mathbf{X}_4$.

In Case II, i.e., when $f(x_1, x_2)$ is an arbitrary function, it follows from Eq. (60) that the approximate symmetries are $\mathbf{X}_3$ and $\mathbf{X}_4$.

Theorem 5 furnishes the following approximate conserved quantity for the system (59) in the case I(i):

$$\begin{aligned}& I^0 = \mathbf{X}_1^0 \rfloor \mathbf{X}_2^0 \rfloor \Omega = 1, \quad \Omega = dx_1 \wedge dx_2, \\ & I^1 = \mathbf{X}_1^1 \rfloor \mathbf{X}_2^0 \rfloor \Omega + \mathbf{X}_1^0 \rfloor \mathbf{X}_2^1 \rfloor \Omega = 3Cx_1 + (B - A)t \cos(2t) - (A + B) \sin(2t).\end{aligned}$$

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