On the Hyers–Ulam–Rassias stability of functional equations in \( n \)-variables

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Abstract

In this paper we investigate a generalization of the Hyers–Ulam–Rassias stability for a functional equation of the form \( f(\phi(X)) = \phi(X)f(X) + \psi(X) \) and the stability in the sense of Ger for the functional equation of the form \( f(\phi(X)) = \phi(X)f(X) \), where \( X \) lie in \( n \)-variables. As a consequence, we obtain a stability result in the sense of Hyers–Ulam–Rassias, Gavruta, and Ger for some well-known equations such as the gamma, beta, and \( G \)-function type’s equations.

Keywords: Functional equation; Gamma, beta, and \( G \)-function; Hyers–Ulam stability; Hyers–Ulam–Rassias stability

1. Introduction

In 1940, the stability problem raised by S.M. Ulam [27] was solved by D.H. Hyers in [6]. The result of Hyers has been significantly generalized to the unbounded case by Th.M. Rassias [18], and this has later been extended by P. Gavruta [4], R. Ger [5], Th.M. Rassias and others (cf. [3, 7–11, 19–26]).
The functional equation in which we are interested in this article is derived from the gamma functional equation
\[ f(x + 1) = xf(x), \]
which was considered by S.-M. Jung [11–13]. This equation generalizes the gamma type functional equation
\[ f(x + p) = \phi(x)f(x) \]
and the beta type functional equation
\[ f(x + p, y + p) = \psi(x, y)f(x, y) \]
in which the former has been recently extended to
\[ f(\phi(x)) = \phi(x)f(x) + \psi(x) \]
by T. Trif [26].

In this paper, we will investigate a generalization of the Hyers–Ulam–Rassias stability in the sense of Gávruta and the stability in the sense of Ger for the functional equations
\[ f(\varphi(X)) = \phi(X)f(X) + \psi(X), \]  
\[ f(\varphi(X)) = \phi(X)f(X), \]  
(respectively, where \( \varphi, \phi, \psi \) are given functions, while \( f \) is the unknown function and \( X \) depends upon \( n \)-variables. Namely, the aim of this paper is the extension to the domain of \( n \)-variables and applications of well-known results concerning gamma, G, Schröder, and beta types functional equations.

In Section 2, we study the stability in the sense of Gávruta for the functional equations (1.1), (1.2).

In Section 2’, we consider the special case of Section 2 with \( \varphi(X) = X + P \).

In Section 3, our results shown in Sections 2, 2’ are applied to the gamma, G, Schröder, beta types functional equations and some examples suitably restricted to a domain in one or two variables.

In Section 4, we consider the stability in the sense of Ger for the functional equation (1.2), and we provide applications to the gamma, beta, G-functional equations, and some examples.

Throughout this paper, let \( B \) be a Banach space over the field \( K \), where \( K \) will be either the field \( R \) of real numbers or the field \( C \) of complex numbers. Each positive real number \( \delta \) is fixed. \( R_+ \) denotes the set of all nonnegative real numbers. Given the nonempty set \( S \) and the function \( \varphi : S^n \to S^n \), we put \( \varphi_0(X) := X \) and \( \varphi_n(X) := \varphi(\varphi_{n-1}(X)) \) for all positive integers \( n \) and all points \( X \in S^n \). The functions \( \phi : S^n \to K \setminus \{0\} \), \( \psi : S^n \to B \), and \( \varepsilon : S^n \to R_+ \) are defined.

2. Generalization of Hyers–Ulam stability of Eqs. (1.1) and (1.2)

Let \( \varphi, \phi, \varepsilon \) be given functions such that
\[ \omega(X) := \sum_{k=0}^{\infty} \frac{\varepsilon(\varphi_k(X))}{\prod_{j=0}^{k} |\phi(\varphi_j(X))|} < \infty, \quad \forall X \in S^n. \]  
(2.1)

**Theorem 1.** Let the functions \( \varphi, \phi, \varepsilon \) satisfy the condition (2.1). If a function \( f : S^n \to B \) satisfies the inequality
\[ \|f(\varphi(X)) - \phi(X)f(X) - \psi(X)\| \leq \varepsilon(X), \quad \forall X \in S^n, \]  
(2.2)
then there exists an unique solution \( g : S^n \to B \) of Eq. (1.1) such that
\[ \|g(X) - f(X)\| \leq \omega(X). \]  
(2.3)
Proof. For any $X \in S^n$ and for every positive integer $n$, let $\omega_n : S^n \to R_+$ and $g_n : S^n \to B$ be the functions defined by

$$\omega_n(X) := \sum_{k=0}^{n-1} \epsilon(\phi_k(X)) \prod_{j=0}^{k} |\phi(\phi_j(X))|$$

and

$$g_n(X) := \frac{f(\phi_n(X))}{\prod_{j=0}^{n-1} \phi(\phi_j(X))} - \sum_{k=0}^{n-1} \frac{\psi(\phi_k(X))}{\prod_{j=0}^{k} |\phi(\phi_j(X))|}$$

for all $X \in S^n$, respectively.

By (2.2), it follows that

$$\|f(\phi(X)) - f(X) - \psi(X)\| = \frac{\epsilon(\phi(X))}{|\phi(X)|}$$

for all $X \in S^n$.

Substituting $X$ by $\phi_n(X)$ in this inequality, and then dividing both sides of the obtained inequality by $\prod_{j=0}^{n-1} |\phi(\phi_j(X))|$, we get

$$\|g_{n+1}(X) - g_n(X)\| = \frac{\epsilon(\phi_n(X))}{\prod_{j=0}^{n} |\phi(\phi_j(X))|}.$$  \hspace{1cm} (2.4)

By induction on $n$ we prove that

$$\|g_n(X) - f(X)\| \leq \omega_n(X)$$  \hspace{1cm} (2.5)

for all $X \in S^n$, and for all positive integers $n$. For the case $n = 1$, the inequality (2.5) is an immediate consequence of (2.2).

Assume that the inequality (2.5) holds true for some $n$. Then we obtain the inequality for $n+1$. This is an immediate consequence of

$$\|g_{n+1}(X) - f(X)\| \leq \|g_{n+1}(X) - g_n(X)\| + \|g_n(X) - f(X)\|$$

$$\leq \frac{\epsilon(\phi_n(X))}{\prod_{j=0}^{n} |\phi(\phi_j(X))|} + \omega_n(X) = \omega_{n+1}(X).$$

We claim that $\{g_n(X)\}$ is a Cauchy sequence. Indeed, by (2.4) and (2.1), we have for $n > m$ that

$$\|g_n(X) - g_m(X)\| \leq \sum_{k=m}^{n-1} \|g_{k+1}(X) - g_k(X)\| \leq \sum_{k=m}^{n-1} \frac{\epsilon(\phi_k(X))}{\prod_{j=0}^{k} |\phi(\phi_j(X))|} \to 0$$

as $m \to \infty$.

Hence, we can define a function $g : S^n \to B$ by

$$g(X) := \lim_{n \to \infty} g_n(X).$$  \hspace{1cm} (2.6)

From the definition of $g_n$, we have $g_n(\phi(X)) = \phi(X)g_{n+1}(X) + \psi(X)$ and therefore the function $g$ satisfies (1.1).

We show from (2.5) that $g$ satisfies the inequality (2.3) as follows:

$$\|g(X) - f(X)\| = \lim_{n \to \infty} \|g_n(X) - f(X)\| \leq \lim_{n \to \infty} \omega_n(X) = \omega(X), \quad \forall X \in S^n.$$
If \( h : S^n \to B \) is another such function, which satisfies (1.1) and (2.3), then we have

\[
\| g(X) - h(X) \| = \| g(\varphi_n(X)) - h(\varphi_n(X)) \| \cdot \prod_{j=0}^{n-1} \frac{1}{|\varphi_j(X)|}
\]

\[
\leq 2\omega_n(\varphi_n(X)) \cdot \prod_{j=0}^{n-1} \frac{1}{|\varphi_j(X)|}
\]

\[
= 2 \left( \sum_{k=0}^{\infty} \frac{\epsilon(\varphi_{n+k}(X))}{\prod_{j=0}^{n+k} |\varphi_j(X)|} \right) \cdot \prod_{j=0}^{n-1} \frac{1}{|\varphi_j(X)|}
\]

\[
= 2 \sum_{k=n}^{\infty} \frac{\epsilon(\varphi_k(X))}{\prod_{j=0}^{k} |\varphi_j(X)|}
\]

for all \( X \in S^n \) and all positive integers \( n \), which tends to zero as \( n \to \infty \), since \( \omega(X) \) is bounded. This implies the uniqueness of \( g \).

Setting \( \epsilon(X) = \delta \) in Theorem 1, we have the Hyers–Ulam stability of Eq. (1.1).

Let the functions \( \varphi, \phi \) satisfy

\[
\mu(X) := \sum_{k=0}^{\infty} \prod_{j=0}^{k} \frac{1}{|\varphi_j(X)|} < \infty, \quad \forall X \in S^n.
\]

Corollary 1. Let \( \varphi, \phi \) satisfy condition (2.7). If a function \( f : S^n \to B \) satisfies the inequality

\[
\| f(\varphi(X)) - \phi(X)f(X) - \psi(X) \| \leq \delta
\]

for all \( X \in S^n \), then there exists an unique solution \( g : S^n \to B \) of Eq. (1.1) such that

\[
\| g(X) - f(X) \| \leq \delta \mu(X).
\]

Theorem 2 and Corollary 2 follow immediately from Theorem 1 and Corollary 1 with \( \psi(X) = 0 \).

Theorem 2. Let \( \varphi, \phi, \epsilon \) satisfy condition (2.1). If a function \( f : S^n \to B \) satisfies the inequality

\[
\| f(\varphi(X)) - \phi(X)f(X) \| \leq \epsilon(X), \quad \forall X \in S^n,
\]

then there exists an unique solution \( g : S^n \to B \) of Eq. (1.2) satisfying (2.3) for all \( X \in S^n \).

Corollary 2. Let \( \varphi, \phi \) satisfy condition (2.7). If a function \( f : S^n \to B \) satisfies the inequality

\[
\| f(\varphi(X)) - \phi(X)f(X) \| \leq \delta, \quad \forall X \in S^n,
\]

then there exists an unique solution \( g : S^n \to B \) of Eq. (1.2) satisfying (2.9) for all \( X \in S^n \).
2’. Stability in the case \( \varphi(X) = X + P \) of Eq. (1.1)

We consider a special case of Section 2 as follows, that is
\[
\varphi(X) = X + P,
\]
where \( X = (x_1, x_2, \ldots, x_n) \), \( P = \left( p_1, p_2, \ldots, p_n \right) \in (0, \infty)^n \), and each positive real number \( x_i \) is a variable, each positive real number \( p_i \) is fixed, and \( n \) is a natural number. The statement \( X > 0 \) means that each component \( x_i \) of \( X \) lies in the interval \((0, \infty)\), and the statement \( X > N_0 \) means that \( x_i > n_0 \) for each component \( x_i \) of \( X \) and for a fixed natural number \( n_0 \).

Then, all results shown in Section 2 are replaced with the analogous results in the form \( \varphi(X) = X + P \).

Let the functions \( \varphi \) and \( \varepsilon \) satisfy the inequalities
\[
\omega'(X) := \sum_{k=0}^{\infty} \frac{\varepsilon(X + kP)}{\prod_{j=0}^{k} |\varphi(X + jP)|} < \infty, \quad \text{and}
\]
\[
\mu'(X) := \sum_{k=0}^{\infty} \frac{1}{\prod_{j=0}^{k} |\varphi(X + jP)|} < \infty, \quad \forall X > 0.
\]

**Theorem 2’.** Let \( \varphi, \varepsilon \) satisfy condition (2.1’). If a function \( f : (0, \infty)^n \to R \) satisfies the inequality
\[
|f(X + P) - \varphi(X) f(X)| \leq \varepsilon(X), \quad \forall X > N_0,
\]
then there exists an unique solution \( g : (0, \infty)^n \to R \) of Eq. (1.2) such that
\[
|g(X) - f(X)| \leq \omega'(X), \quad \forall X > N_0.
\]

**Proof.** Setting \( S = (0, \infty) \), \( B = R \), \( \varphi(X) = X + P \) in Theorem 2, then the claimed result of this theorem is satisfied except for the condition that replaces \( X \in S^n \) by \( X > N_0 \). For this, we define the new function \( g_0 : (n_0, \infty)^n \to R \) by
\[
g_0(X) := \lim_{n \to \infty} g_n(X)
\]
in substituting \( g \) defined in (2.6) for \( g_0 \).

Now, we extend the function \( g_0 \) to the domain \((0, \infty)^n\). We define for each \( 0 < X \leq N_0 \),
\[
g(X) := \frac{g_0(X + kP)}{\prod_{n=1}^{k} \varphi(X + nP)},
\]
where \( k \) is the smallest natural number satisfying the inequalities \( x_i + kp_i > n_0 \) for each \( i \).

Then, \( g(X + P) = \varphi(X) g(X) \) for all \( X > 0 \) and \( g(X) = g_0(X) \) for all \( X > N_0 \). Also the inequality
\[
|g(X) - f(X)| < \omega'(X)
\]
holds for all \( X > 0 \). \( \square \)
Corollary 2'. Let \( \phi \) satisfies condition (2.7'). If a function \( f : (0, \infty)^n \to R \) satisfies the inequality
\[
|f(X + P) - \phi(X)f(X)| \leq \delta, \quad \forall X > N_0,
\]
then there exists an unique solution \( g : (0, \infty)^n \to R \) of Eq. (1.2) with
\[
|g(X) - f(X)| \leq \delta \mu'(X), \quad \forall X > N_0.
\]

3. Applications to the gamma type, the \( G \)-function, Schröder, and the beta type functions

The results shown in the Sections 2, 2' can be applied to the well-known stability results for the gamma, \( G \), beta, Schröder functional equations, and also to certain generalized forms. It suffices to show how to bring the Eq. (1.1) into the concrete forms of those functional equations.

3.1. The beta type functional equations

We restrict the functional equation (1.1) in the case of a double variable. Then, we can obtain the same results for the beta type functional equation, as follows:
\[
f(x + p, y + q) = \phi(x, y)f(x, y) + \psi(x, y),
\]
\[
f(x + p, y + q) = \phi(x, y)f(x, y),
\]
\[
f(x + 1, y + 1)^{-1} = \frac{(x + y)(x + y + 1)}{xy}f(x, y)^{-1},
\]
\[
f(x + 1, y + 1) = \frac{xy}{(x + y)(x + y + 1)}f(x),
\]
which provide some special cases of Eq. (1.1).

The beta function \( B(x, y) = \int_0^1 t^{x-1}(1 - t)^{y-1} dt \) is a solution of the beta functional equation (3.3'), which is closely related to the gamma function \( \Gamma(x) \). The relationship between them is given by \( B(x, y) = (\Gamma(x)\Gamma(y))/\Gamma(x + y) = B(y, x) \).

In the case of 2-variables, the following Corollaries 3, 4 follow from Theorems 1, 2' with \( S = (0, \infty), B = R \).

The condition (2.1) with \( \psi(x, y) = (x + p, y + q) \) applies
\[
\omega_B(x, y) := \sum_{k=0}^{\infty} \frac{\varepsilon(x + kp, y + kq)}{\prod_{j=0}^{k} |\phi(x + jp, y + jq)|} < \infty, \quad \forall x, y > 0.
\]

Corollary 3. Let \( \phi, \varepsilon \) satisfy condition (3.4). If a function \( f : (0, \infty) \times (0, \infty) \to R \) satisfies the inequality
\[
|f(x + p, y + q) - \phi(x, y)f(x, y) + \psi(x, y)| \leq \varepsilon(x, y), \quad \forall x, y > n_0;
\]
then there exists an unique solution \( g : (0, \infty) \times (0, \infty) \to R \) of Eq. (3.1) with
\[
|g(x, y) - f(x, y)| \leq \omega_B(x, y), \quad \forall x, y > n_0.
\]
Corollary 4 [10]. Let \( \phi, \varepsilon \) satisfy condition (3.4). If a function \( f : (0, \infty) \times (0, \infty) \to R \) satisfies the inequality
\[
|f(x + p, y + q) - \phi(x, y)f(x, y)| \leq \varepsilon(x, y), \quad \forall x, y > n_0;
\]
then there exists an unique solution \( g : (0, \infty) \times (0, \infty) \to R \) of Eq. (3.2) such that
\[
|g(x, y) - f(x, y)| \leq \omega_{\varepsilon}(x, y), \quad \forall x, y > n_0.
\]

The following Corollary 5 follows from Corollary 4 with \( \phi(x, y) = ((x + y)(x + y + 1))/xy \) and \( p = q = 1 \). The condition (3.4) is replaced by
\[
\omega_{\varepsilon_1}(x, y) := \sum_{k=0}^{\infty} \varepsilon(x + k, y + k)
\times \prod_{j=0}^{k} \frac{(x + j)(y + j)}{(x + j) + (y + j)((x + j) + (y + j) + 1)} < \infty \tag{3.5}
\]
for all \( x, y > 0 \).

Corollary 5 [15]. Let a function \( \varepsilon \) satisfies condition (3.5). If the function \( f : (0, \infty) \times (0, \infty) \to R \) satisfies the inequality
\[
|f(x + 1, y + 1) - \frac{(x + y)(x + y + 1)}{xy}f(x, y)| \leq \varepsilon(x, y), \quad \forall x, y > n_0;
\]
then there exists an unique reciprocal of the beta functional equation \( g : (0, \infty) \times (0, \infty) \to R \) of Eq. (3.3) with
\[
|g(x, y) - f(x, y)| \leq \omega_{\varepsilon_1}(x, y), \quad \forall x, y > n_0.
\]

Remark 1. Corollaries 3–5 with \( \varepsilon(x) = \delta \) imply the Hyers–Ulam stability of Eqs. (3.1)–(3.3).

3.2. The gamma type functional equations

We restrict the functional equation (1.1) for the case of a single variable. Then, we can obtain similar results for the gamma, \( G \), Schröder functional equations, as follows:
\[
f(\varphi(x)) = \phi(x)f(x) + \psi(x), \tag{3.6}
\]
\[
f(\varphi(x)) = xf(x), \tag{3.6'}
\]
\[
f(\varphi(x)) = cf(x) \quad c: \text{constant}, \tag{3.6''}
\]
\[
f(x + 1) = x(f(x) + 1), \tag{3.7}
\]
\[
f(x + 1) = (x + 1)f(x), \tag{3.7'}
\]
\[
f(x + p) = \phi(x)f(x), \tag{3.8}
\]
\[
f(x + 1) = \phi(x)f(x), \tag{3.8'}
\]
\[
f(x + 1) = xf(x), \tag{3.8''}
\]
which are special cases of Eq. (1.1).
The gamma function given by \( \Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \) is a solution of the gamma functional equation (3.8″).

**Remark 2.** The Hyers–Ulam stability and the generalized Hyers–Ulam–Rassias stability for all of the above functional equations follow immediately from Theorems 1, 2′ with \( S = (0, \infty), B = \mathbb{R} \) or \( K \), \( p = 1 \), \( \psi(x) = 0 \), \( \phi(x) = x \), \( \varphi_j(x) = x + j \), \( \varepsilon(x) = \delta \) by restricting to a single variable. Since the Hyers–Ulam stability and the generalized Hyers–Ulam–Rassias stability of Eqs. (3.8), (3.8′), (3.8″) are studied in the papers [1,10–14,17,26].

Equation (3.6′) can be considered as the generalized form of Schröder functional equation (3.6″). In the case \( c > 1 \), Trif proved the Hyers–Ulam stability of Eq. (3.6″).

Restricting the condition (2.1) to a single variable, we get for all \( x \in S \)

\[
\omega'(x) := \sum_{k=0}^{\infty} \frac{\varepsilon(\varphi_k(x))}{\prod_{j=0}^{k} |\phi(\varphi_j(x))|} < \infty, \tag{2.1′}
\]

\[
\omega_\phi(x) := \sum_{k=0}^{\infty} \frac{\varepsilon(\varphi_k(x))}{\prod_{j=0}^{k} |\varphi_j(x)|} < \infty, \tag{3.9}
\]

\[
\omega_c(x) := \sum_{k=0}^{\infty} \frac{\varepsilon(\varphi_k(x))}{c^{k+1}} < \infty \quad c: \text{positive constant.} \tag{3.10}
\]

**Corollary 6** [26]. Let \( \varphi, \phi, \varepsilon \) satisfy condition (2.1′). If a function \( f : S \to B \) satisfies the inequality

\[
\| f(\varphi(x)) - \phi(x) f(x) - \psi(x) \| \leq \varepsilon(x),
\]

then there exists an unique solution \( g : S \to B \) of the functional equation (3.6) with

\[
\| g(x) - f(x) \| \leq \omega'(x).
\]

**Corollary 7.** Let a function \( f : S \to B \) satisfies the inequality

\[
\| f(\varphi(x)) - \phi(x) f(x) - \psi(x) \| \leq \delta,
\]

where

\[
\mu_\delta(x) := \sum_{k=0}^{\infty} \frac{1}{\prod_{j=0}^{k} |\varphi_j(x)|} < \infty
\]

for all \( x \) in \( S \).

Then, there exists an unique solution \( g : S \to B \) of the functional equation (3.6) with

\[
\| g(x) - f(x) \| \leq \delta \mu_\delta(x).
\]

**Corollary 8.** Let \( \varphi, \varepsilon \) satisfy condition (3.9). If a function \( f : S \to B \) satisfies the inequality

\[
\| f(\varphi(x)) - x f(x) \| \leq \varepsilon(x),
\]

then there exists an unique solution \( g : S \to B \) of the functional equation (3.6) with

\[
\| g(x) - f(x) \| \leq \delta \mu_\delta(x).
\]
then there exists an unique solution $g : S \to B$ of Eq. (3.6') such that

$$\| g(x) - f(x) \| \leq \omega_\gamma(x).$$

**Corollary 9.** Let $\varphi$ satisfies condition $\| \varphi_j(x) \| > \| \varphi(x) \| > 1$ for all $j$. If a function $f : S \to B$ satisfies the inequality

$$\| f(\varphi(x)) - x f(x) \| \leq \delta,$$

then there exists an unique solution $g : S \to B$ of Eq. (3.6') such that

$$\| g(x) - f(x) \| \leq \frac{\delta \| \varphi(x) \|}{x(\| \varphi(x) \| - 1)}.$$

In particular, if $\varphi$ satisfies the inequality $\| \varphi_j(x) \| > \| x \| > 1$, then there exists an unique solution $g : S \to B$ of the generalized Schröder functional equation (3.6') satisfying

$$\| g(x) - f(x) \| \leq \frac{\delta}{\| x \| - 1}.$$

**Corollary 10.** Let $\varphi, \varepsilon$ satisfy condition (3.10). If a function $f : S \to B$ satisfies the inequality

$$\| f(\varphi_j(x)) - c f(x) \| \leq \varepsilon(x),$$

then there exists the unique Schröder function $g : S \to B$ satisfying

$$\| g(x) - f(x) \| \leq \omega_\varepsilon(x).$$

**Corollary 11 [26].** Let $c > 1$. If a function $f : S \to B$ satisfies the inequality

$$\| f(\varphi_j(x)) - c f(x) \| \leq \delta,$$

then there exists the unique Schröder function $g : S \to B$ satisfying

$$\| g(x) - f(x) \| \leq \frac{\delta}{c - 1}.$$

The condition (3.9) with $\varphi_j(x) = x + j$ can be represented by

$$\omega_{\gamma_1}(x) := \sum_{k=0}^{\infty} \frac{\varepsilon(x+k)}{\Gamma(k+1)} < \infty, \quad \forall x > 0. \tag{3.11}$$

**Corollary 12.** Let $\varepsilon$ satisfy condition (3.11). If a function $f : (0, \infty) \to K$ satisfies the inequality

$$\| f(x + 1) - x (f(x + 1)) \| \leq \varepsilon(x) \quad \text{or} \quad \| f(x + 1) - (x + 1) f(x) \| \leq \varepsilon(x),$$

then there exists an unique solution $g : (0, \infty) \to K$ of Eq. (3.7) or (3.7') respectively with

$$\| g(x) - f(x) \| \leq \omega_{\gamma_1}(x).$$

**Remark 3.** The Hyers–Ulam stability of Eqs. (3.7) and (3.7') follows immediately from Corollary 12 with $\varepsilon(x) = \delta$. 
3.3. The G-functional equation

The G-function introduced by E.W. Barnes \[2\],
\[ G(z) = (2\pi)^{\frac{1}{2}} e^{-\frac{z(z-1)}{2}} e^{-\gamma (z-1)} \prod_{k=1}^{\infty} \left[ \left( 1 + \frac{z-1}{k} \right)^k e^{1-z+\frac{z-1}{2}} \right], \]
satisfy the equation \( G(x+1) = \Gamma(x)G(x) \) and \( \Gamma(1) = G(1) = 1 \), where \( \gamma \) is the Euler-Mascheroni’s constant defined by \( \gamma = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \log n \right) \approx 0.577215664 \ldots \)

The properties and values of G-function depend on those of the gamma function. Since the double gamma function \( \Gamma_2 \) is defined by the reciprocal of the G-function (see \[2\]),
\[ \Gamma_2(x) = \frac{1}{G(x)} \]
and its functional equation is \( \Gamma_2(x+1) = \Gamma_2(x)/\Gamma(x) \). Therefore, the stability problem for the G-function is equivalent to the stability for the reciprocal of the double gamma function.

Putting \( \phi(x) = \Gamma(x) \) and \( p = 1 \) in Eq. (3.6), we obtain
\[ f(x+p) = \Gamma(x)f(x) + \psi(x), \quad (3.12) \]
\[ f(x+p) = \Gamma(x)f(x), \quad (3.13) \]
\[ f(x+1) = \Gamma(x)f(x). \quad (3.14) \]
Equation (3.14) will be called the G-functional equation from the definition of G-function.

The condition (2.1') in a single variable is represented by
\[ \omega_{G_p}(x) := \sum_{k=0}^{\infty} \frac{\epsilon(x+k)}{\prod_{j=0}^{k} |\Gamma(x+j)|} < \infty, \quad (3.15) \]
\[ \omega_G(x) := \sum_{k=0}^{\infty} \frac{\epsilon(x+k)}{\prod_{j=0}^{k} |\Gamma(x+j)|} < \infty. \quad (3.16) \]

**Corollary 13.** Let the functions \( \Gamma, \epsilon \) satisfy condition (3.15). If a function \( f : (0, \infty) \to R \) satisfies the inequality
\[ |f(x+p) - \Gamma(x)f(x) - \psi(x)| \leq \epsilon(x), \quad \forall x > n_0, \]
then there exists an unique solution \( g : (0, \infty) \to R \) of Eq. (3.12) with
\[ |g(x) - f(x)| \leq \omega_{G_p}(x), \quad \forall x > n_0. \]

**Corollary 14** \[16\]. Let the functions \( \Gamma, \epsilon \) satisfy condition (3.15). If a function \( f : (0, \infty) \to R \) satisfies the inequality
\[ |f(x+p) - \Gamma(x)f(x)| \leq \epsilon(x), \quad \forall x > n_0, \]
then there exists an unique solution \( g : (0, \infty) \to R \) of Eq. (3.13) with
\[ |g(x) - f(x)| \leq \omega_{G_p}(x), \quad \forall x > n_0. \]
**Corollary 15** [16]. Let the functions $\Gamma, \varepsilon$ satisfy condition (3.16). If a function $f : (0, \infty) \to \mathbb{R}$ satisfies the inequality
\[ |f(x + 1) - \Gamma(x)f(x)| \leq \varepsilon(x), \quad \forall x > n_0, \]
then there exists a unique $G$-function $g : (0, \infty) \to \mathbb{R}$ satisfying Eq. (3.14) with
\[ |g(x) - f(x)| \leq \omega_G(x), \quad \forall x > n_0. \]

**Remark 4.** The Hyers–Ulam stability of Eqs. (3.13), (3.14) follows immediately from Corollaries 13–15 with $\varepsilon(x) = \delta$. Stability results of $G$-functional equations are founded in [16].

**3.4. Examples**

The results of Sections 2, 3 may be applied to the following examples.

**Example 1** (Corollary). Set $\phi(X) = x_1 * x_2 * \cdots * x_n$ in Corollary 2', where $*$ is an operation on the set $S$. If a function $f : (0, \infty)^n \to \mathbb{R}$ satisfies the inequality
\[ |f(X + P) - (x_1 * x_2 * \cdots * x_n)f(X)| \leq \delta, \quad \forall X > N_0, \]
then there exists a unique solution $g : (0, \infty)^n \to \mathbb{R}$ of the equation $f(X + P) = (x_1 * x_2 * \cdots * x_n)f(X)$ with
\[
|g(X) - f(X)| \leq \begin{cases} 
\omega_n(x), & \text{if } \omega_n(x) = \sum_{k=0}^{\infty} \prod_{j=0}^{k} \frac{\delta}{(x_1 + j p_1 + \cdots + j p_n)}, \\
\frac{1}{x_1 * x_2 * \cdots * x_n} \Gamma, & \text{if } x_1 * \cdots * x_n > 1, \\
\frac{1}{x_1 * x_2 * \cdots * x_n} \delta, & \text{if } p_1 * p_2 * \cdots * p_n > 1, \\
\frac{\delta}{x_1 + x_2 + \cdots + x_n} \sum_{k=0}^{\infty} \frac{1}{k(n)^k}, & \text{if } P = (1, \ldots, 1), \ x_1 * \cdots * x_n = x_1 \cdots x_n, \\
\frac{\delta}{x_1 + \cdots + x_n} \sum_{k=0}^{\infty} \frac{1}{(km)^k}, & \text{if } P = (1, \ldots, 1), \ x_1 * \cdots * x_n = x_1 + \cdots + x_n. 
\end{cases}
\]

From Theorem 2' or Corollary 4, we have the following Example 2 (Corollary) in a single variable.

**Example 2** (Corollary [14]). If a function $f : (0, \infty) \to \mathbb{R}$ satisfies the inequality
\[ |f(x + 1) - \phi(x)f(x)| \leq \delta, \quad \forall x \geq n_0, \]
where $\phi$ is a function such that
\[ \mu_\phi := \sum_{k=0}^{\infty} \prod_{j=0}^{k} \frac{1}{|\phi(x + j)|} < \infty, \quad (3.17) \]
then there exists an unique solution \( g : (0, \infty) \to R \) of Eq. (3.8') with
\[
|g(x) - f(x)| \leq \delta \mu g(x).
\]

The definition of the function \( \phi \) in the following examples satisfies the condition (3.17). Thus the functional equation (3.8') for \( \phi(x) \) by following each cases has the Hyers–Ulam stability.

**Example 3.** \( \phi(x) = c > 1 \), where \( c \) is constant.

**Example 4.** \( \phi(x) = (1 + \frac{1}{x})^x \). Note that \( \lim_{x \to \infty} (1 + \frac{1}{x})^x = e > 1 \).

**Example 5.** \( \phi(x) = x^n \), for \( x > 1, n \in \mathbb{N} \).

**Example 6.** \( \phi(x) = \arctan(x) \), since \( \lim_{x \to \infty} \arctan(x) = \frac{\pi}{2} \).

**Example 7.** \( \phi(x) = \arcsin(x) \) for \( x > 1 \), since \( \lim_{x \to \infty} \arcsin(x) = \frac{\pi}{2} \).

Similarly, we can also consider \( \sinh(x), \cosh(x), \log(x) \) with a suitable domain for each function.

### 4. Stability in the sense of Ger for Eq. (1.2)

The following theorem provides the stability in the sense of Ger for Eq. (1.2).

**Theorem 3.** Let \( \varepsilon : S^n \to (0, 1) \) satisfies
\[
\sum_{j=0}^{\infty} \varepsilon(\phi_j(X)) < +\infty. \tag{4.1}
\]

If a function \( f : S^n \to (0, \infty) \) satisfies the inequality
\[
\left| \frac{f(\phi(X))}{\phi(X) f(X)} - 1 \right| \leq \varepsilon(X), \quad \forall X \in S^n, \tag{4.2}
\]
then there exists an unique solution \( g : S^n \to (0, \infty) \) of Eq. (1.2) with
\[
\alpha(X) \leq \frac{g(X)}{f(X)} \leq \beta(X), \tag{4.3}
\]
where \( \alpha(X) := \prod_{j=0}^{\infty} (1 - \varepsilon(\phi_j(X))) \) and \( \beta(X) := \prod_{j=0}^{\infty} (1 + \varepsilon(\phi_j(X))) \).

**Proof.** The condition (4.1) implies that \( \prod_{j=0}^{\infty} (1 \pm \varepsilon(\phi_j(X))) \) converges. Hence, we can define the functions \( \alpha, \beta \) for all \( X \in S^n \) such that \( 0 < \alpha := \prod_{j=0}^{\infty} (1 - \varepsilon(\phi_j(X))) < \prod_{j=0}^{\infty} (1 + \varepsilon(\phi_j(X))) := \beta < +\infty \), that is, these series are bounded.
For any \( X \in S^n \) and for every positive integer \( n \), we define
\[
g_n(X) = \prod_{j=0}^{n-1} \frac{f(\phi^j(X))}{\phi(\phi^j(X))}, \tag{4.4}
\]
For all positive integers \( m,n \) with \( n > m \), we have
\[
g_n(X)g_m(X) = \frac{f(\phi^{m+1}(X))}{\phi(\phi^{m+1}(X))} \cdot \frac{f(\phi^{m+2}(X))}{\phi(\phi^{m+2}(X))} \cdot \ldots \cdot \frac{f(\phi^n(X))}{\phi(\phi^n(X))} \tag{4.5}
\]
It also follows from (4.2) that
\[
0 < 1 - \varepsilon(\phi^j(X)) \leq f(\phi^{j+1}(X)) \phi(\phi^j(X))f(\phi^j(X)) \leq 1 + \varepsilon(\phi^j(X)) \tag{4.6}
\]
for all \( X \in S^n \) and \( j = 0, 1, 2, \ldots \). From (4.5) and (4.6), we get
\[
\prod_{j=m}^{n-1} (1 - \varepsilon(\phi^j(X))) \leq \frac{g_n(X)}{g_m(X)} \leq \prod_{j=m}^{n-1} (1 + \varepsilon(\phi^j(X))) \tag{4.9}
\]
for all \( X \in S^n \). This implies from (4.7), (4.9), and the definitions of \( \alpha, \beta \) that
\[
\alpha(X) \leq \frac{g_0(X)}{f(X)} \leq \beta(X), \quad \forall X \in S^n. \tag{4.10}
\]
Assume \( h : S^n \rightarrow (0, \infty) \) is another solution of Eq. (4.8) which satisfies inequality (4.10). By (4.8), we have
\[
\frac{g_0(X)}{h(X)} = \frac{g_0(\varphi_n(X))}{h(\varphi_n(X))} = \frac{g_0(\varphi_n(X))}{f(\varphi_n(X))} \cdot \frac{f(\varphi_n(X))}{h(\varphi_n(X))}
\]
for any \( X \in S^n \) and for any natural number \( n \).
Hence, we have
\[
\frac{\alpha(\varphi_n(X))}{\beta(\varphi_n(X))} \leq \frac{g_0(X)}{h(X)} \leq \frac{\beta(\varphi_n(X))}{\alpha(\varphi_n(X))}
\]
for any natural number \( n \). By the boundedness of the series \( \varepsilon \),
\[
\alpha(\varphi_n(X)) = \prod_{j=0}^{\infty} (1 - \varepsilon(\varphi_j(X))) \rightarrow 1
\]
as \( n \to \infty \). Similarly \( \beta(\varphi_n(X)) \to 1 \) as \( n \to \infty \).
Therefore, it is obvious that \( h(X) \equiv g_0(X) \).

From the proof of Theorem 3, we can see that the inequality (4.1) is a condition for the convergence of \( \alpha \) and \( \beta \). Hence the following corollary is natural.

**Corollary 16.** Let a function \( f \) satisfies inequality (4.2), such that \( \varepsilon : S^n \rightarrow (0, 1) \) is a function such that
\[
\alpha(X) := \prod_{j=0}^{\infty} (1 - \varepsilon(\varphi_j(X))) \quad \text{and} \quad \beta(X) := \prod_{j=0}^{\infty} (1 + \varepsilon(\varphi_j(X)))
\]
are bounded for all \( X \in S^n \). Then there exists an unique solution \( g : S^n \rightarrow (0, \infty) \) of Eq. (1.2) satisfying (4.3) for all \( X \in S^n \).

Restrict Theorem 3 with \( S = (0, \infty) \), \( \varphi(x, y) = (x + p, y + q) \) into two variable. Then we have the stability in the sense of Ger for the reciprocal of beta functional equation and the generalized beta functional equations as follows.

**Corollary 17** [10]. If \( f : (0, \infty) \times (0, \infty) \rightarrow R_+ \) is a function that satisfies the inequality
\[
\left| \frac{f(x+p, y+q)}{\varphi(x, y)f(x, y)} - 1 \right| \leq \varepsilon(x, y), \quad \forall x, y \geq n_0,
\]
where \( \varepsilon : (0, \infty) \times (0, \infty) \rightarrow (0, 1) \) is a function such that
\[
\sum_{j=0}^{\infty} \varepsilon(x+jp, y+jq) < +\infty,
\]
then there exists an unique solution \( g : (0, \infty) \times (0, \infty) \rightarrow R_+ \) of Eq. (3.2) with
\[
\alpha_\varepsilon(x, y) \leq \frac{g(x, y)}{f(x, y)} \leq \beta_\varepsilon(x, y),
\]
where \( \alpha_{\varepsilon}(x, y) := \prod_{j=0}^{\infty} (1 - \varepsilon(x + j p, x + j q)) \) and \( \beta_{\varepsilon}(x, y) := \prod_{j=0}^{\infty} (1 + \varepsilon(x + j p, x + j q)) \).

**Corollary 18** [10]. If \( f : (0, \infty) \times (0, \infty) \to R_+ \) is a function which satisfies the inequality

\[
\left| \frac{xy}{(x+y)(x+y+1)} \frac{f(x, y)}{f(x+1, y+1)} - 1 \right| \leq \varepsilon(x, y), \quad \forall x, y \geq n_0,
\]

where \( \varepsilon : (0, \infty) \times (0, \infty) \to (0, 1) \) is a function such that

\[
\sum_{j=0}^{\infty} \varepsilon(x+j, y+j) < +\infty,
\]

then there exists an unique solution \( g : (0, \infty) \times (0, \infty) \to R_+ \) of the functional equation (3.3) with

\[
\alpha_{\varepsilon 1}(x, y) \leq g(x, y) f(x, y) \leq \beta_{\varepsilon 1}(x, y),
\]

where \( \alpha_{\varepsilon 1}(x, y) := \prod_{j=0}^{\infty} (1 - \varepsilon(x+j, y+j)) \) and \( \beta_{\varepsilon 1}(x, y) := \prod_{j=0}^{\infty} (1 + \varepsilon(x+j, y+j)) \).

**Proof.** Apply Corollary 17 with \( p = 1, q = 1 \), substitute \( f \) to \( f^{-1} \) and \( \phi(x, y) = (x+y)(x+y+1)/(xy) \). \( \Box \)

One can restrict \( n \)-variables to one variable in Theorem 3. We can consider the stability in the sense of Ger for the gamma and generalized functional equations with \( \varepsilon(x) = \delta/x^{1+\theta} \), and \( \psi(x) = x + p, \phi(x) = x, p = 1 \).

**Corollary 19** [14]. Let \( f : (0, \infty) \to R_+ \) be a function that satisfies the inequality

\[
\left| \frac{f(x+p)}{\phi(x)f(x)} - 1 \right| \leq \varepsilon(x), \quad \forall x > n_0,
\]

where \( \varepsilon : (0, \infty) \to (0, 1) \) is a function such that

\[
\sum_{j=0}^{\infty} \varepsilon(x+j p) < +\infty,
\]

then there exists an unique solution \( g : (0, \infty) \to R_+ \) of Eq. (3.8) with

\[
\alpha_{\gamma}(x) \leq \frac{g(x)}{f(x)} \leq \beta_{\gamma}(x),
\]

where \( \alpha_{\gamma}(x) := \prod_{j=0}^{\infty} (1 - \varepsilon(x+j p)) \) and \( \beta_{\gamma}(x) := \prod_{j=0}^{\infty} (1 + \varepsilon(x+j p)) \).

**Corollary 20** [10]. Let \( \theta > 0 \) be given. If a mapping \( f : (0, \infty) \to R_+ \) satisfies the inequality

\[
\left| \frac{f(x+1)}{xf(x)} - 1 \right| \leq \frac{\delta}{x^{1+\theta}}, \quad \forall x > n_0,
\]
then there exists an unique solution $g : (0, \infty) \rightarrow (0, \infty)$ of the gamma functional equation (3.8′) such that for any $x > \max \{n_0, \delta^{1/(1+\theta)}\}$

$$\alpha(x) \leq \frac{g(x)}{f(x)} \leq \beta(x),$$

where $\alpha(x) := \prod_{j=0}^{\infty} (1 - \delta/(x + j)^{1+\theta})$ and $\beta(x) := \prod_{j=0}^{\infty} (1 + \delta/(x + j)^{1+\theta})$.

**Remark 5.** The stability in the sense of Ger for the functional equations (3.2), (3.3), (3.8), (3.8′), (3.8′′), (3.13), (3.14) has been studied in papers [10, 14, 16, 17].

### 4.1. Examples

We apply the result of Corollary 19 for $p = 1$.

**Example 8.** $\varepsilon(1 + i) = 1/(1 + i)^p$, for $p > 1$. Note that the $p-$series $\sum_{k=0}^{\infty} 1/k^p$ in the case $p > 1$ converges.

**Example 9.** $\varepsilon(1 + i) = 1/(1 + i)!$. Note that $\sum_{i=0}^{\infty} 1/(1 + i)! = e - 1$.

**References**