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Kergin approximation in Banach spaces

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Abstract

We explore the convergence of Kergin interpolation polynomials of holomorphic functions in Banach spaces, which need not be of bounded type. We also investigate a case where the Kergin series diverges.

Keywords: Kergin interpolation; Banach space

Kergin interpolation is a generalization of both the Lagrange interpolation in the one-dimensional case, and the Taylor polynomial in the case where all interpolation points coincide. In several variables, interpolation polynomials are not unique. However, Kergin [1] proved that interpolation polynomials enjoying natural properties exist and are unique:

Theorem 1 (*Kergin*). *Let $N, K \in \mathbb{N}$, $N > 0$, and $x_0, \ldots, x_K \in \mathbb{R}^N$, not necessarily distinct. There is a unique $\chi : \mathcal{C}^K(\mathbb{R}^N) \to P^K(\mathbb{R}^N)$ satisfying:*

(1) *χ is linear.*
(2) *For every $f \in \mathcal{C}^K(\mathbb{R}^N)$, every $q \in Q^k$ in $\mathbb{R}^N$, where $k \in \{0, \ldots, K\}$, and every $J \subset \{0, \ldots, K\}$ with $\#(J) = k+1$, there exists $x \in \operatorname{conv}(x_j)_{j\in J}$ such that $q(D)(\chi(f) - f)(x) = 0$.*

Here, $\mathcal{C}^K(\mathbb{R}^N)$ is the set of functions with K continuous derivatives, Q^k is the set of homogeneous polynomials of degree k, and $P^K(\mathbb{R}^N)$ is the set of polynomials of degree at most K. It fell to Micchelli [2] and Milman [3] to discover a formula for these Kergin polynomials. This formula also extends to the infinite-dimensional Banach space case, see [4,

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5]. In this case, the potential unboundedness of continuous functions, even on bounded sets bounded away from the boundary of the domain, presents new difficulties in proving convergence results. Filipsson [4] proved a convergence result for holomorphic functions bounded on a ball. Petersson also proved convergence results for entire functions on nuclear spaces, see [5], and Hilbert–Schmidt spaces, see [6].

We give the formula for the Kergin polynomial below. Let X, Y be complex Banach spaces, $U \subset X$ open and $f : U \to Y$. Define $d^0 f = f$ and

$$d^{k+1} f : U \times X^{k+1} \to Y,$$
$$d^{k+1} f(x; \xi_1, \ldots, \xi_{k+1}) = \lim_{t \to 0} \frac{1}{t}(d^k f(x + t\xi_{k+1}; \xi_1, \ldots, \xi_k) - d^k f(x; \xi_1, \ldots, \xi_k)),$$

if this limit exists. This is just the $(k+1)$th iteration of the directional derivative of f, see, e.g., [7]. Let $p_0, \ldots, p_n, \in X$. Suppose that $d^n f$ exists and is continuous. The Kergin polynomial of f of degree n is the sum

$$f(p_0) + \sum_{k=1}^{n} \int_{S_k} d^k f(s_0 p_0 + \cdots + s_k p_k; x - p_0, \ldots, x - p_{k-1}) ds_1 \cdots ds_k, \tag{1}$$

where

$$S_k = \left\{ (s_1, \ldots, s_k) \in \mathbb{R}^k : s_j \geq 0, \sum_{j=1}^{k} s_j \leq 1 \right\}$$

is the standard k-simplex, and

$$s_0 = 1 - s_1 - \cdots - s_k. \tag{2}$$

This is a Bochner, or vector-valued, integral, see, e.g., [8]. In the case where $Y = \mathbb{C}$, this is just the usual Lebesgue (or Riemann) integration.

L. Filipsson observed that Micchelli's error formula for the difference between f and its degree $(k-1)$ Kergin polynomial carries over to Banach spaces:

$$\int_{S_k} d^k f(s_0 p_0 + \cdots + s_{k-1} p_{k-1} + s_k x; x - p_0, \ldots, x - p_{k-1}) ds_1 \cdots ds_k, \tag{3}$$

where s_0 is as in (2). Given an infinite sequence $p_0, p_1, p_2, \ldots$, we define the infinite Kergin series by replacing n in (1) by ∞. Under some circumstances, this series will approximate the given function. That is the primary subject of this paper. First, we need the following proposition.

Proposition 2. *Let X be a complex Banach space and let $K \subset X$ be compact. Then*

(1) *the convex hull of K is compact, and*
(2) *the balanced hull of K is compact.*

This can be found, for example, in [9], Chapter I, page 6.

Now we can move on to approximation.

Theorem 3. *Let X and Y be complex Banach spaces, $U \subset X$ open, $V \subset U$. Suppose that the sequence $\{p_j\}$ is contained in a compact convex set $L \subset U$. Let W be the convex hull of $L \cup V$ and let W' be the balanced convex hull of $L + V$. Suppose that for some $\rho > e$, $W + \rho W' \subset U$ and $f : U \to Y$ is holomorphic. Then the Kergin series for f converges to f uniformly on compact subsets of V.*

Proof. First, we observe that if $T^k = (\mathbb{R}/\mathbb{Z})^k$ is the k-dimensional torus with the Haar probability measure dt, then

$$d^k f(a; v_0, \ldots, v_{k-1}) = \int_{T^k} f(a + v_0 e^{2\pi i t_0} + \cdots + v_{k-1} e^{2\pi i t_{k-1}}) dt, \tag{4}$$

provided each v_j is small enough so that the right-hand side of (4) is defined. Let $s = (s_1, \ldots, s_k)$, $a(s, x) = s_0 p_0 + s_1 p_1 + \cdots + s_k x$, with s_0 again as in (2). Plugging this into the error formula (3) yields

$$\left(\frac{k}{\rho}\right)^k \int_{S_k} \int_{T^k} f\left(a(s, x) + (x - p_0)\frac{\rho e^{2\pi i t_0}}{k} + \cdots + (x - p_{k-1})\frac{\rho e^{2\pi i t_{k-1}}}{k}\right) dt ds.$$

We have used homogeneity to factor out $\left(\frac{k}{\rho}\right)^k$. Define

$$c(s, t) = (x - p_0)\frac{\rho e^{2\pi i t_0}}{k} + \cdots + (x - p_{k-1})\frac{\rho e^{2\pi i t_{k-1}}}{k}.$$

Set $b(s, t) = a(s, x) + c(s, t)$. If the interpolation points p_j are in L and $x \in V$, we can see that $a(s, x) \in W$ and $c(s, t) \in \rho W'$. Thus, by hypothesis, we have that $b(s, t)$ takes values in U.

Now we restrict our attention to the case where x is in a compact subset K of V. Applying Proposition 2, plus the fact that the sum of two compact sets is compact, we see that the image of $b(s, t)$ is contained in a compact subset of U. Furthermore, this set is independent of k, and depends only on K and L. Let M be the maximum of f on this subset. This leads to the inequality

$$\left\| \left(\frac{k}{\rho}\right)^k \int_{S_k} \int_{T^k} f(b(s, t)) dt ds \right\| \le \left(\frac{k}{\rho}\right)^k M \mathrm{vol}(S_k).$$

Recall that $\mathrm{vol}(S_k) = 1/k!$, which, by Stirling's formula, is asymptotically equal to

$$\left(\frac{e}{k}\right)^k \frac{1}{\sqrt{2\pi k}}.$$

Substituting these all into the above estimate on the error of the $(k-1)$th degree Kergin polynomial yields

$$\left(\frac{e}{\rho}\right)^k \frac{M}{\sqrt{2\pi k}}.$$

Since $\rho > e$, the error goes to zero. □

This allows f to be unbounded if V contains an open set, as in the following example.

Example 1. Let $B(r) = \{x \in X : \|x\| < r\}$. Let $f : B(1) \to Y$ be holomorphic, and choose $r, r' > 0$ such that

$$\frac{1 - \max(r, r')}{r + r'} > e. \tag{5}$$

Suppose that the closure of the set of all interpolation points p_k is a compact subset of $B(r')$. Then the corresponding Kergin series for f converges to f uniformly on compact subsets of $B(r)$.

Proof. Let $U = B(1)$, $V = B(r)$, and

$$e < \rho < \frac{1 - \max(r, r')}{r + r'}.$$

Let L be the convex hull of the closure of the interpolation points. By Theorem 3, it suffices to check that $W + \rho W' \subset U$, where W and W' are as in the theorem. Since W is the convex hull of $L \cup V$, we have $W \subset B(\max(r, r'))$. Furthermore, by the triangle inequality, we have that $W' \subset B(r + r')$. Thus, if $x \in W + \rho W'$, then

$$\|x\| < \max(r, r') + \rho(r + r') \leq \max(r, r') + \frac{1 - \max(r, r')}{r + r'}(r + r') = 1.$$

In other words, $W + \rho W' \subset U$, as required. □

Note that (5) is satisfied if, for example,

$$r = r' = \frac{1}{e + 1}.$$

Observe that here, f may be unbounded even on a small ball, whereas in [4], f must be bounded. The price for such a convergence result is the stronger restriction on the interpolation points than the one found in [4]. Furthermore, Filipsson's convergence is uniform on balls, whereas Theorem 3 only shows convergence on compact sets. Now we give an example where the interpolation points are not in a compact set, and the Kergin series of an entire function diverges at the origin.

Example 2. Let $f : l^1 \to \mathbb{C}$ be defined by

$$f(x) = \sum_{n=1}^{\infty} \left(n! \prod_{k=1}^{n} x_k \right).$$

The function f is entire, i.e., it is holomorphic on all of l^1. Let $\{e_k\}$ be the standard basis for l^1 (0's everywhere except for a 1 in the kth position). Using this basis as interpolation points yields a Kergin series that diverges at the origin.

Proof. First, we show that f is entire. Let $x^0 = (x_n^0) \in l^1$. Choose n_0 so large that

$$\sum_{n > n_0} |x_n^0| < \delta < e$$

and let

$$M > \prod_{n=1}^{n_0} |x_n^0|.$$

We will show that the sum

$$\sum_{n > n_0} n! \prod_{k=1}^{n} |x_k|$$

converges uniformly near x^0, in fact on the set

$$\left\{ x : \prod_{n=1}^{n_0} |x_n| < M, \sum_{n > n_0} |x_n| < \delta \right\}.$$

The modulus of the (n_0+n)th term in the sum defining f is

$$\begin{aligned}(n_0+n)!\prod_{k=1}^{n_0+n}|x_k| &\le (n_0+n)!M\prod_{k=1}^{n}|x_{n_0+k}| \\ &\le (n_0+n)!M\left(\frac{1}{n}\sum_{k=1}^{n}|x_{n_0+k}|\right)^n \qquad (6)\\ &\le (n_0+n)!M\left(\frac{\delta}{n}\right)^n. \qquad (7)\end{aligned}$$

In (6), we used the fact that the geometric mean is no bigger than the arithmetic mean. Stirling's formula implies that (7) is asymptotically equal to

$$\begin{aligned}&\left(\frac{n_0+n}{e}\right)^{n_0+n}\sqrt{2\pi(n_0+n)}M\left(\frac{\delta}{n}\right)^n \\ &\quad=\left(\frac{n_0+n}{n}\right)^{n_0+n}\sqrt{2\pi(n_0+n)}M\frac{\delta^n n^{n_0}}{\mathrm{e}^{n+n_0}}=O\left(n^{n_0+1/2}\left(\frac{\delta}{e}\right)^n\right),\end{aligned}$$

because

$$\left(\frac{n_0+n}{n}\right)^{n_0+n}=\left(1+\frac{n_0}{n}\right)^{n_0+n}\to \mathrm{e}^{n_0}$$

as $n\to\infty$.

Hence the series of f converges uniformly near x_0 and so f is entire.

Now we show that the Kergin series diverges at the origin. Define $f_n=x_1x_2\cdots x_n$. We check by induction that

$$d^k f_n(y;e_1,e_2,\ldots,e_k)=\begin{cases}y_{k+1}\cdots y_n & \text{if } k<n,\\ 1 & \text{if } k=n,\\ 0 & \text{if } k>n.\end{cases}$$

In particular, the left-hand side is non-negative when all $y_j\ge 0$. Hence for such y

$$(-1)^k d^k f(y;-e_1,-e_2,\ldots,-e_k)=\sum_{n=1}^{\infty} n!d^k f_n(y;e_1,\ldots,e_k)\ge k!.$$

Setting $y=s_0e_1+\cdots+s_ke_{k+1}$ and integrating over S_k yields the absolute value of the $(k+1)$th term in the Kergin series at the origin, which must be at least 1. Summing up, we have a divergent series. □

Theorem 3 requires that the interpolation sequence be contained in a compact subset of U. Here, we set $U=l^1$, $V=\{0\}$. The only criterion in the theorem not satisfied is that L must be compact.

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