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Diophantine Approximation with Algebraic Points of Bounded Degree[1]

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algebraic numbers by algebraic points of bounded degree.

1. INTRODUCTION

This paper deals with the Diophantine approximation with algebraic points of bounded degree. In 1955, Roth proved his celebrated theorem that for any real algebraic number α, and every $\kappa > 2$, the inequality $|(x/y) - \alpha| > (1/\max\{|x|, |y|\}^{\kappa})$ holds for all, but finitely many $(x/y) \in \mathbb{Q}$. Roth's theorem was extended by E. Wirsing in 1968 to the approximation of the algebraic number α by algebraic numbers of bounded degree. In 1970, W. Schmidt generalized Roth's theorem to the approximation of more than one algebraic number by rational points. In this paper, we extend Schmidt's theorem to the approximation by algebraic points of bounded degree. However, we could not fully obtain Wirsing's version of Schmidt's subspace theorem. This paper thus could be regarded as a first step towards the complete understanding of Diophantine approximation with algebraic points of bounded degree. We note that this paper is motivated by the recent discovered relationship between Diophantine approximation and Nevanlinna theory. See [Ru] for the further discussion.

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2. THE MAIN RESULT

Let k be a number field of degree d. Denote by $M(k)$ the set of valuations of k and by $M_\infty(k)$ the set of archimedean valuations of k. For each valuation $v \in M(k)$, denote by k_v the completion of k with respect to v and by $n_v = [k_v : \mathbb{Q}_v]$ the local degree. Define an absolute value associated to an archimedean valuation v by

$$\|x\|_v = |x| \qquad \text{if} \quad K_v = \mathbb{R}$$

$$\|x\|_v = |x|^2 \qquad \text{if} \quad K_v = \mathbb{C}.$$

If v is non-archimedean, then v is an extension of a p-adic valuation on $\mathbb{Q}$ for some prime p; the absolute value is defined so that

$$\|x\|_v = |x|_p^{n_v}$$

if $x \in \mathbb{Q} - \{0\}$.

Let $\mathbf{x} \in k^{n+1}$. We let $\|\mathbf{x}\|_v = \max_{0 \leqslant i \leqslant n} \{\|x_i\|_v\}$. Then we define the height function

$$h(\mathbf{x}) = \frac{1}{[k : \mathbb{Q}]} \sum_{v \in M(k)} \log \|\mathbf{x}\|_v.$$

Given any polynomial f with coefficients in k, we define $\|f\|_v$ and $h(f)$ as the $\| \ \|_v$-value and absolute logarithmic height, respectively, of the point whose coordinates are the coefficients of f. As is known, $h(\mathbf{x})$ and $h(f)$ are independent of the choice of the field k. Further, $h(\lambda \mathbf{x}) = h(\mathbf{x})$ and $h(\lambda f) = h(f)$ for all $\lambda \in \overline{\mathbb{Q}}^*$.

Given a linear form $L = \sum_{i=0}^n a_j X_j$, we denote by $\bigodot_\beta L$ the β-th fold symmetric tensor product of L. This linear form $\bigodot_\beta L$ contains $\binom{n+\beta}{\beta}$ terms and the coefficient of each term is $a_0^{t_0} \cdots a_n^{t_n}$, where $t_0 + \cdots + t_n = \beta$. In this paper, we will always use the lexicographic order for the index set $(t_0, ..., t_n)$, where $t_0 + \cdots + t_n = \beta$.

Theorem (Main Theorem). *Let k be a number field and let S be a finite set of valuation including M_∞. Let $\{L_1, ..., L_q\}$ be linear forms with coefficients in k. Let r be a positive integer. Suppose that the set of linear forms $\{\bigodot_\beta L_1, ..., \bigodot_\beta L_q\}$ is in general position for each $\beta \leqslant r$. Given any $\varepsilon > 0$. Then for all but finitely many non-proportional points of $\mathbf{x} = (x_0, ..., x_n) \in \bar{k}^{n+1}$ with $[k(x_0, ..., x_n) : k] \leqslant r$, $L_i(\mathbf{x}) \neq 0$, $1 \leqslant i \leqslant q$,*

$$\frac{1}{[k(x_0, ..., x_n) : \mathbb{Q}]} \sum_{i=1}^{q} \sum_{v \in S} \sum_{\substack{w \mid v \\ w \in M(k(x_0, ..., x_n))}} \log \frac{\|\mathbf{x}\|_w \, \|L_i\|_w}{\|L_i(\mathbf{x})\|_w}$$

$$\leqslant \left(2\binom{n+r}{r} - 2 + \varepsilon\right) h(\mathbf{x})$$

Remark 1. When $n=1$, if $\{L_1, ..., L_q\}$ are distinct, then since the van der Monde determinant does not vanish, $\{\bigodot_\beta L_1, ..., \bigodot_\beta L_q\}$ is in general position for each β. Therefore, this theorem recovers Wirsing's theorem. In this case, our bound is sharp.

Remark 2. The assumption $\{\bigodot_\beta L_1, ..., \bigodot_\beta L_q\}$ is in general position for each $\beta \leqslant r$ can not be simply removed in our proof. When $n \geqslant 2$, $\{\bigodot_\beta L_1, ..., \bigodot_\beta L_q\}$ in general position does not imply that $\{\bigodot_{\beta+1} L_1, ..., \bigodot_{\beta+1} L_q\}$ is in general position. On the other hand, $\{\bigodot_\beta L_1, ..., \bigodot_\beta L_q\}$ in general position does not imply that $\{\bigodot_{\beta-1} L_1, ..., \bigodot_{\beta-1} L_q\}$ is in general position either. We include the following examples from [St].

EXAMPLE 1. Let $n=2$, $L_1 = X_0$, $L_2 = X_1$, $L_3 = X_2$, $L_4 = X_0 + 2X_1 + 4X_2$, $L_5 = X_0 - X_1 + 2X_2$, and $L_6 = X_0 + X_1 + 6X_2$. It is clear that $\{L_1, ..., L_6\}$ is in general position. Under the dictionary order, we may realize $Y_0 = X_0^2$, $Y_1 = X_0 X_1$, $Y_2 = X_0 X_2$, $Y_3 = X_1^2$, $Y_4 = X_1 X_2$, and $Y_5 = X_2^2$. Then we have

$$\bigodot_2 L_1 = Y_0$$

$$\bigodot_2 L_2 = Y_3$$

$$\bigodot_2 L_3 = Y_5$$

$$\bigodot_2 L_4 = Y_0 + 2Y_1 + 4Y_2 + 4Y_3 + 8Y_4 + 16Y_5$$

$$\bigodot_2 L_5 = Y_0 - Y_1 + 2Y_2 + Y_3 - 2Y_4 + 4Y_5$$

$$\bigodot_2 L_6 = Y_0 + Y_1 + 6Y_2 + Y_3 + 6Y_4 + 36Y_5$$

Then

$$\bigodot_2 L_1 + 4\bigodot_2 L_2 - 16\bigodot_2 L_3 - \bigodot_2 L_4 - \bigodot_2 L_5 + \bigodot_2 L_6 = 0$$

Therefore, $\bigodot_2 L_1, ..., \bigodot_2 L_6$ are not in general position.

EXAMPLE 2. Let $n=2$, $L_1=X_0$, $L_1=X_0$, $L_2=X_1$, $L_3=X_2$, $L_4=X_0+X_2$, $L_5=X_0+2X_1$, and $L_6=X_0+3X_1+X_2$. Since $L_4=L_1+L_3$, $\{L_1, ..., L_6\}$ is not in general position. Then we have

$$\bigodot_2 L_1 = Y_0$$

$$\bigodot_2 L_2 = Y_3$$

$$\bigodot_2 L_3 = Y_5$$

$$\bigodot_2 L_4 = Y_0 + Y_2 + Y_5$$

$$\bigodot_2 L_5 = Y_0 + 2Y_1 + 4Y_3$$

$$\bigodot_2 L_6 = Y_0 + 3Y_1 + Y_2 + 9Y_3 + 3Y_4 + Y_5$$

One can easily check that $\bigodot_2 L_1, ..., \bigodot_2 L_6$ are in general position.

Remark 3. The proof of the main theorem is similar to so-called Stoll's algebroid reduction method. A discussion of this method could be found in [Ru]. This proof also works for function fields. For the effective version of Ru-Wong's theorem and Wirsing's theorem of function fields, we refer to [Wa].

To prove the main theorem, we will need a special result in Ru-Wong's theorem [RW]. We recall this result:

THEOREM (Ru-Wong). *Let k be a number field. Let $\{L_1, ..., L_q\}$ be a set of hyperplanes in $\mathbb{P}^n(k)$, located in general position. Let S be a finite subset of $M(k)$ containing $M_\infty(k)$. Then for any $\varepsilon>0$,*

$$\frac{1}{[k:\mathbb{Q}]}\sum_{i=1}^{q}\sum_{v\in S}\log\frac{\|\mathbf{x}\|_v\,\|L_i\|_v}{\|L_i(\mathbf{x})\|_v}\leqslant(2n+\varepsilon)\,h(\mathbf{x}).$$

for all $\mathbf{x}\in k^{n+1}$ with $L_i(\mathbf{x})\neq 0$, $1\leqslant i\leqslant q$, except for finitely many non-proportional points.

We now prove the main theorem.

Proof. Let $L_i=\sum_{j=0}^n a_{ij}X_j$, $i=1, 2, ..., q$, be the given linear forms. Consider the points $\mathbf{x}=(x_0, ..., x_n)\in\bar{k}^{n+1}$ with $[k(x_0, ..., x_n):k]=\beta\leqslant r$. Let $E=k(x_0, ..., x_n)=k(\mathbf{x})$. Let $\sigma_1, ..., \sigma_\beta$ be the distinct embeddings of E in an algebraic closure of k. Let S_β be the symmetric group which permute $\sigma_1, ..., \sigma_\beta$. Note that $\sigma_t L_i(x_0, ..., x_n)=L_i(\sigma_t(x_0), ..., \sigma_t(x_n))$. Then

$$\prod_{t=1}^{\beta} \sigma_t L_i(x_0, ..., x_n)$$
$$= \sum_{t_0+\cdots+t_n=\beta} a_{i0}^{t_0} \cdots a_{in}^{t_n} \frac{1}{t_0! \cdots t_n!}$$
$$\times \sum_{\tau \in S_\beta} \tau\left(\prod_{l=1}^{t_0} \sigma_l(x_0) \cdots \prod_{l=t_0+\cdots+t_{n-1}+1}^{\beta} \sigma_l(x_n)\right)$$
$$= (\bigodot_\beta L_i)\left(..., \frac{1}{t_0! \cdots t_n!} \sum_{\tau \in S_\beta} \tau\left(\prod_{l=1}^{t_0} \sigma_l(x_0) \cdots \prod_{l=t_0+\cdots+t_{n-1}+1}^{\beta} \sigma_l(x_n)\right), ...\right) \tag{1}$$

Let $P = (..., (1/t_0! \cdots t_n!) \sum_{\tau\in S_\beta} \tau(\prod_{l=1}^{t_0} \sigma_l(x_0) \cdots \prod_{l=t_0+\cdots+t_{n-1}+1}^{\beta} \sigma_l(x_n)), ...)$. It is clear that

$$\frac{1}{t_0! \cdots t_n!} \sum_{\tau \in S_\beta} \tau\left(\prod_{l=1}^{t_0} \sigma_l(x_0) \cdots \prod_{l=t_0+\cdots+t_{n-1}+1}^{\beta} \sigma_l(x_n)\right) \in k.$$

Since each component of P has the above form, $P \in k^{\binom{n+\beta}{\beta}}$. From (1), we see that, for each i, $1 \leqslant i \leqslant q$,

$$\prod_{w \mid v} \|L_i(\mathbf{x})\|_w = \|(\bigodot_\beta L_i)(P)\|_v \tag{2}$$

Our next step is to compare $\|P\|_v$ and $\prod_{w \mid v} \|\mathbf{x}\|_w$. We recall that the Gauss lemma asserts that if $f_1, ..., f_m \in k[X_1, ..., X_n]$, then $\|f_1 \cdots f_m\|_v = \|f_1\|_v \cdots \|f_m\|_v$, for any non-archimedean place v. From (1), we see that

$$\prod_{t=1}^{\beta} (\sigma_t(x_0)\, a_{i0} + \cdots + \sigma_t(x_n)\, a_{in})$$
$$= \sum_{t_0+\cdots+t_n=\beta} a_{i0}^{t_0} \cdots a_{in}^{t_n} \frac{1}{t_0! \cdots t_n!}$$
$$\times \sum_{\tau \in S_\beta} \tau\left(\prod_{l=1}^{t_0} \sigma_l(x_0) \cdots \prod_{l=t_0+\cdots+t_{n-1}+1}^{\beta} \sigma_l(x_n)\right).$$

This becomes a polynomial identity after replacing the a_{ij} by variables X_{ij}. By Gauss' lemma, for every non-archimedean place v,

$$\prod_{w \mid v} \max_j \{ \|x_j\|_w \}$$
$$= \max\left\{..., \left\|\frac{1}{t_0! \cdots t_n!} \sum_{\tau \in S_\beta} \tau\left(\prod_{l=1}^{t_0} \sigma_l(x_0) \cdots \prod_{l=t_0+\cdots+t_{n-1}+1}^{\beta} \sigma_l(x_n)\right)\right\|_v, ...\right\} \tag{3}$$

The above equality is also true up to a multiple scalar for archmidean place v, since in this, the Gauss lemma reads

$$\frac{1}{4^{d^n}} \|fg\|_v \leqslant \|f\|_v \|g\|_v \leqslant 4^{d^n} \|fg\|_v,$$

for every two polynomials f, g in $k[X_1, ..., X_n]$ such that $\deg f + \deg g < d$. Thus (3) implies that

$$\sum_{w \mid v} \log \|\mathbf{x}\|_w - \log \|P\|_v$$

is zero when $w \notin M_\infty(k)$, and $O(1)$ when $w \in M_\infty(k)$. Therefore

$$[k(x_0, ..., x_n) : k]\, h(\mathbf{x}) = h(P) + O(1). \tag{4}$$

By assumption $\bigodot_\beta L_1, ..., \bigodot_\beta L_q$ are in general position. Applying Ru-Wong's theorem above, we have, for any $\varepsilon > 0$, the inequality

$$\frac{1}{[k : \mathbb{Q}]} \sum_{i=1}^{q} \sum_{v \in S} \log \frac{\|P\|_v \|\bigodot_\beta L_i\|_v}{\|(\bigodot_\beta L_i)(P)\|_v} \leqslant \left(2\binom{n+\beta}{\beta} - 2 + \varepsilon\right) h(P). \tag{5}$$

holds for all, except for finitely many non-proportional points $P \in k^{(n+\beta/\beta)}$. By (2), (3), (4), (5) and noting that, for every point P, the preimage of the map

$$(x_0, ..., x_n) \to P = \left(..., \frac{1}{t_0! \cdots t_n!} \sum_{\tau \in S_\beta} \tau\left(\prod_{l=1}^{t_0} \sigma_l(x_0) \cdots \prod_{l=t_0+\cdots+t_{n-1}+1}^{\beta} \sigma_l(x_n)\right), ...\right)$$

is finite, we have that

$$\frac{1}{[k(x_0, ..., x_n) : \mathbb{Q}]} \sum_{i=1}^{q} \sum_{v \in S} \sum_{\substack{w \mid v \\ w \in M(K(x_0, ..., x_n))}} \log \frac{\|\mathbf{x}\|_w \|L_i\|_w}{\|L_i(\mathbf{x})\|_w}$$

$$\leqslant \left(2\binom{n+\beta}{\beta} - 2 + \varepsilon\right) h(\mathbf{x})$$

holds for all, except finitely many, non-proportional points of $\mathbf{x} = (x_0, ..., x_n)$ with $x_i \in \bar{k}$, $0 \leqslant i \leqslant n$, $[k(x_0, ..., x_n) : k] = \beta \leqslant r$. ∎

As we noted, our result didn't give a full generalization of Schmidt's subspace theorem to the approximation with algebraic points of bounded degree. The following conjecture about Wirsing's type of Schmidt's subspace theorem was raised in [Ru], *Conjecture.* Let k be a number field, and let S be a finite subset of $M(k)$ containing $M_\infty(k)$. Let $\{L_1, ..., L_q\}$ be linear forms with coefficients in k, in general position. Let r be a positive integer. If $\varepsilon > 0$, and if C is a positive constant, then the set of non-proportional points of $\mathbf{x} = (x_0, x_1, ..., x_n)$ with $x_i \in \bar{k}$, $0 \leqslant i \leqslant n$, $[k(x_1/x_0, ..., x_n/x_0) : k] \leqslant r$, such that $L_i(\mathbf{x}) \neq 0$, $1 \leqslant i \leqslant q$, and

$$\frac{1}{[k(x_0, ..., x_n) : \mathbb{Q}]} \sum_{v \in S} \sum_{\substack{w \mid v \\ w \in M(K(x_0, ..., x_n))}} \sum_{j=1}^{q} \log \frac{\|\mathbf{x}\|_w \cdot \|L_j\|_w}{\|L_j(\mathbf{x})\|_w}$$
$$\geqslant \left(2n - l + 1 + \frac{l(2n - l + 1)}{2}(2r - 2) + \varepsilon\right) h(\mathbf{x}) + C$$

is contained in a finite union of linear subspaces of dimension $l - 1$.

When $r = 1$, the above conjecture is Ru-Wong's theorem [RW].

3. APPLICATIONS TO DECOMPOSABLE FORM EQUATIONS

As above, let k be an algebraic number field. Denote by $M(k)$ the set of places of k and write $M_\infty(k)$ for the set of archimedean places of k. Let S be a finite subset of $M(k)$ containing $M_\infty(k)$. An element $x \in k$ is said to be a S-integer if $\|x\|_v \leqslant 1$ for each $v \notin S$. Denote by O_S the set of S-integers. The units of O_S are called S-units. We introduce the concept of algebraic S-integers.

Definition. A point $x \in \bar{k}$ is called an algebraic S-integer if $\|x\|_w \leqslant 1$ for all places w over $k(x)$ such that $w \mid v$ for some $v \notin S$.

Let q, n, r denote positive integers with $q > 2\binom{n+r}{r} - 2$, and let $F(X) = F(X_0, ..., X_n) \in O_S[X]$ be a decomposable form of degree q. Assume that $F(X) = L_1(X) \cdots L_q(X)$ over k and that the set of linear forms $\{\bigodot_\beta L_1, ..., \bigodot_\beta L_q\}$ is in general position for each $\beta \leqslant r$, where $\bigodot_\beta$ denotes the β-th fold symmetric tensor product. For given real numbers c, m with $c > 0$, consider the solutions of the inequality

$$0 < \prod_{v \in S} \prod_{w \mid v} \|F(\mathbf{x})\|_w \leqslant c \left(\prod_{v \in S} \prod_{w \mid v} \|\mathbf{x}\|_w\right)^m, \tag{6}$$

for $\mathbf{x} = (x_0, ..., x_n) \in \bar{k}^{n+1}$ where each coordinate is an algebraic S-integer and $[k(x_0, ..., x_n) : k] \leqslant r$. If $\mathbf{x}$ is such a solution, then so is $\mathbf{x}' = \eta \mathbf{x}$ for every algebraic S-unit η. Such solutions $\mathbf{x}, \mathbf{x}'$ are called proportional.

THEOREM 2. *Suppose that* $m < q - 2\binom{n+r}{r} + 2$ *and that the set of linear forms* $\{\bigodot_\beta L_1, ..., \bigodot_\beta L_q\}$ *is in general position for each* $\beta \leqslant r$, *where* $\bigodot_\beta$ *denotes the* β*-th fold symmetric tensor product. Then the inequality* (6) *has only finitely many non-proportional solutions* $\mathbf{x} = (x_0, ..., x_n) \in \bar{k}^{n+1}$ *where each coordinate is an algebraic* S*-integer and* $[k(x_0, ..., x_n) : k] \leqslant r$.

Proof. We shall prove Theorem 2 by using our main theorem. Let $\mathbf{x} = (x_0, ..., x_n)$ be such a solution. Let $E = k(x_0, ..., x_n)$. Let $M(E)$ denote the set of places of E. For $w \in M(E)$, define and normalize $\| \ \|_w$ in a similar manner as over k above. Let T denote the set of extension to E of the places in S. Then we deduce from (6) that

$$0 < \prod_{w \in T} \|F(\mathbf{x})\|_w \leqslant c \left(\prod_{w \in T} \|\mathbf{x}\|_w \right)^m. \tag{7}$$

Let $\varepsilon > 0$ with $0 < \varepsilon < q - m - 2\binom{n+r}{r} + 2$. By our main theorem, except for finitely many non-proportional points, we have

$$\frac{1}{[E:\mathbb{Q}]} \sum_{w \in T} \sum_{j=1}^{q} \log \frac{\|\mathbf{x}\|_w \cdot \|L_j\|_w}{\|L_j(\mathbf{x})\|_w} \leqslant \left(2 \binom{n+r}{r} - 2 + \varepsilon \right) h(\mathbf{x}).$$

Since $F(\mathbf{x}) = \prod_{j=1}^{q} L_j(\mathbf{x})$, by (7), the above inequality yields

$$\frac{1}{[E:\mathbb{Q}]} \sum_{w \in T} \left((q-m) \log \|\mathbf{x}\|_w + \sum_{j=1}^{q} \log \|L_j\|_w \right) \leqslant \left(2 \binom{n+r}{r} - 2 + \varepsilon \right) h(\mathbf{x}).$$

Since the coefficients of L_j are S-integers, $\prod_{w \in T} \prod_{j=1}^{q} \|L_j\|_w \geqslant 1$. Furthermore, in view of $\mathbf{x} \in O_T^{m+1}$, we have

$$h(\mathbf{x}) \leqslant \frac{1}{[E:\mathbb{Q}]} \sum_{w \in T} \log \|\mathbf{x}\|_w.$$

Hence it follows that

$$(q-m)\, h(\mathbf{x}) \leqslant \left(2 \binom{n+r}{r} - 2 + \varepsilon \right) h(\mathbf{x}).$$

That is

$$\left(q - m - 2 \binom{n+r}{r} + 2 - \varepsilon \right) h(\mathbf{x}) \leqslant 0.$$

Since $q-m-2\binom{n+r}{r}+2-\varepsilon>0$, we have that $h(\mathbf{x})=0$. So $\mathbf{x}=\mathbf{0}$. This finishes the proof of Theorem 2.

Theorem 2 implies the following result concerning the algebraic S-integer solutions of decomposable form equations.

THEOREM 3. *Given a non-negative integer r. Let q, n, r be positive integers with* $q>2\binom{n+r}{r}-2$. *Suppose that* $m<q-2\binom{n+r}{r}+2$. *Let* $G(X)\in O_S[X]$ *be a homogeneous polynomial with total degree* $\deg G=m$. *Let* $F(X)=F(X_0,...,X_n)\in O_S[X]$ *be a decomposable forms of degree q such that* $F(X)=L_1(X)\cdots L_q(X)$ *over k. Assume that the set of linear forms* $\{\bigodot_\beta L_1,...,\bigodot_\beta L_q\}$ *is in general position for each* $\beta\leqslant r$. *Then there are only finitely many points* $\mathbf{x}\in\bar{k}^{n+1}$ *for which* x_i, $0\leqslant i\leqslant n$ *are algebraic S-integers with* $[k(x_0,...,x_n):k]\leqslant r$, *and satisfies* $F(\mathbf{x})=G(\mathbf{x})$.

Proof. By Theorem 2, the equation $F(\mathbf{x})=G(\mathbf{x})$ has only finitely many non-proportional solutions. When $\mathbf{x}_0$, $\mathbf{x}=\eta\mathbf{x}_0$ are solutions, then

$$\eta^q F(\mathbf{x}_0)=G(\eta\mathbf{x}_0).$$

This can be regarded as an equation of degree q in η with leading coefficient $F(\mathbf{x}_0)\neq 0$. However, this equation has at most q solutions in η which proves the assertion.

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