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Uniform asymptotics for the full moment conjecture of the Riemann zeta function ☆

Ghaith A. Hiary [a,*], Michael O. Rubinstein [b]

[a] University of Bristol, Department of Mathematics, Howard House, Bristol, BS8 1TW, United Kingdom
[b] Pure Mathematics, University of Waterloo, 200 University Avenue West, Waterloo, Ontario, N2L 3G1, Canada

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ABSTRACT

Conrey, Farmer, Keating, Rubinstein, and Snaith, recently conjectured formulas for the full asymptotics of the moments of *L*-functions. In the case of the Riemann zeta function, their conjecture states that the $2k$-th absolute moment of zeta on the critical line is asymptotically given by a certain $2k$-fold residue integral. This residue integral can be expressed as a polynomial of degree k^2, whose coefficients are given in exact form by elaborate and complicated formulas. In this article, uniform asymptotics for roughly the first k coefficients of the moment polynomial are derived. Numerical data to support our asymptotic formula are presented. An application to bounding the maximal size of the zeta function is considered.

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Contents

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* Corresponding author.
E-mail address: hiaryg@gmail.com (G.A. Hiary).

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1. Introduction

The absolute moments of the Riemann zeta function on the critical line are a natural statistical quantity to study in connection with value distribution questions. For example, they can be used to understand the maximal size of the zeta function. These moments are also connected to the remainder term in the general divisor problem [T].

Hardy and Littlewood proved a leading-term asymptotic for the second moment on the critical line [HL]. A few years later, in 1926, Ingham gave the full asymptotic expansion [I]. In the same article, Ingham gave a leading term asymptotic for the fourth moment. The full asymptotic expansion for the fourth moment was obtained by Heath-Brown in 1979 [HB]. In comparison, the higher moments seemed far more difficult and mysterious. Keating and Snaith, in a breakthrough, conjectured the leading-term asymptotic [KS].

Recently, however, based on number-theoretic considerations, Conrey, Farmer, Keating, Rubinstein, and Snaith, conjectured [CFKRS1,CFKRS2] the following full asymptotic expansion for the $2k$-th absolute moment of the Riemann zeta function $\zeta(s)$ on the critical line:

$$\frac{1}{T}\int_0^T \left|\zeta(1/2+it)\right|^{2k} dt \sim \frac{1}{T}\int_0^T P_k\left(\log\frac{t}{2\pi}\right) dt, \quad \text{as } T\to\infty, \tag{1}$$

where $P_k(x)$ is a polynomial of degree k^2:

$$P_k(x) =: c_0(k)x^{k^2} + c_1(k)x^{k^2-1} + \cdots + c_{k^2}(k), \tag{2}$$

given *implicitly* by the $2k$-fold residue

$$P_k(x) = \frac{(-1)^k}{k!^2}\frac{1}{(2\pi i)^{2k}}\oint\cdots\oint \frac{G(z_1,\ldots,z_{2k})\Delta^2(z_1,\ldots,z_{2k})}{\prod_{i=1}^{2k} z_i^{2k}} \times e^{\frac{x}{2}\sum_{i=1}^{k} z_i - z_{k+i}}\, dz_1\cdots dz_{2k}, \tag{3}$$

where the path of integration is around small circles enclosing $z_i = 0$, and

$$\Delta(z_1,\ldots,z_{2k}) := \prod_{1\leqslant i<j\leqslant 2k}(z_j - z_i) \tag{4}$$

is the Vandermonde determinant, and

$$G(z_1,\dots,z_{2k}) := A(z_1,\dots,z_{2k}) \prod_{i,j=1}^{k} \zeta(1+z_i-z_{k+j}) \tag{5}$$

is a product of zetas and the "arithmetic factor" (Euler product)

$$\begin{aligned} &A(z_1,\dots,z_{2k}) \\ &:= \prod_p \prod_{i,j=1}^{k} \left(1-p^{-1-z_i+z_{k+j}}\right) \int_0^1 \prod_{j=1}^{k} \left(1-\frac{e^{2\pi i\theta}}{p^{\frac{1}{2}+z_j}}\right)^{-1} \left(1-\frac{e^{-2\pi i\theta}}{p^{\frac{1}{2}-z_{k+j}}}\right)^{-1} d\theta \qquad (6) \\ &= \prod_p \sum_{j=1}^{k} \prod_{i\neq j}^{k} \frac{\prod_{m=1}^{k}(1-p^{-1+z_{i+k}-z_m})}{1-p^{z_{i+k}-z_{j+k}}}. \qquad (7) \end{aligned}$$

As pointed out by [CFKRS1], the rhs of (3) has an almost identical form to an exact expression for the moment polynomial of random unitary matrices, the difference being that $G(z_1,\dots,z_{2k})$ is replaced by the function $\prod_{i,j=1}^{k}(1-e^{z_{j+k}-z_i})^{-1}$ in the unitary case, so there is no arithmetic factor.

The CFKRS conjecture (3) agrees with the theorems of Hardy and Littlewood, Ingham, and Heath-Brown, for $k=1$ and $k=2$. It has been supported numerically; see [CFKRS1,CFKRS2,HO,RY]. The conjecture provides a method for computing the lower order coefficients of the moment polynomial $P_k(x)$. It gives, in particular, a stronger asymptotic than that of Keating and Snaith who, by carrying out an analogous computation for random unitary matrices, first predicted the leading coefficient (see [KS]):

$$c_0(k) = \frac{a_k g_k}{k^2!}, \tag{8}$$

where

$$a_k := \prod_p (1-1/p)^{k^2} F(k,k;1;1/p), \tag{9}$$

and

$$g_k := k^2! \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}. \tag{10}$$

1.1. Results

Our main theorem develops a uniform asymptotic for $c_r(k)$ in the region $0 \leqslant r \leqslant k^\beta$, for any fixed $\beta < 1$. We expect the asymptotics can be corrected so as to remain valid well beyond the first k coefficients (i.e. for $\beta \geqslant 1$), and that the methods in our paper, which are of combinatorial nature, will be helpful in deriving uniform asymptotics for the moments of other L-functions.

To state our main theorem, let us first define

$$B_k := \sum_p \frac{k\log p}{p-1} - \frac{F(k+1,k+1;2;1/p)}{F(k,k;1;1/p)} \frac{\log p}{p}, \tag{11}$$

where $F(a,b;c;t)$ is the Gauss hypergeometric function

$$F(a,b;c;t) := \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{t^n}{n!}. \tag{12}$$

In the notation of [CFKRS2], B_k is the same as $B_k(1;)$, which is given in Eqs. (2.24) and (2.43) there. The factor B_k is arithmetic in nature. It is the coefficient of the linear term in the following Taylor expansion of the arithmetic factor:

$$\log A(z_1,\dots,z_{2k}) = \log a_k + B_k \sum_{i=1}^{k} z_i - z_{k+i} + \cdots, \tag{13}$$

where it is known (see 2.7 of [CFKRS1]) that

$$a_k = A_k(0,\dots,0). \tag{14}$$

Theorem 6.2 will later furnish the following asymptotic for B_k:

$$B_k \sim 2k\log k, \quad \text{as } k\to\infty. \tag{15}$$

Main theorem. *Fix $\beta < 1$, let $0 \leqslant r \leqslant k^\beta$, and let*

$$\tau_k := 2B_k + 2\gamma k, \tag{16}$$

where $\gamma = 0.5772\dots$ is the Euler constant. Notice by (15) *we have*

$$\tau_k \sim 4k\log k, \quad \text{as } k\to\infty. \tag{17}$$

Then as $k\to\infty$, and uniformly in $0 \leqslant r \leqslant k^\beta$,

$$c_r(k) = \tau_k^r \binom{k^2}{r} \frac{a_k g_k}{k^2!}\left[1 + O\left(\frac{r^2}{k^2}\right)\right] \tag{18}$$

$$= \tau_k^r \binom{k^2}{r} c_0(k)\left[1 + O\left(k^{2(\beta-1)}\right)\right]. \tag{19}$$

Alternatively,

$$c_r(k) = \frac{\tau_k^r k^{2r}}{r!} c_0(k)\left[1 + O\left(k^{2(\beta-1)}\right)\right], \tag{20}$$

as $k\to\infty$. Asymptotic constants depend only on β.

Remarks. 1) The asymptotic formulas (18) and (19) of our theorem are actually equalities for $r = 0$, and $r = 1$. The $r = 0$ case is trivial, and the $r = 1$ case follows from either (2.71) of [CFKRS2] or (49) below. 2) For comparison, the corresponding asymptotic in the unitary case, provided in [HR], is:

$$\tilde{c}_r(k) = k^r \binom{k^2}{r} \tilde{c}_0(k)\left[1 + O\left(\frac{r^2}{k^2}\right)\right], \tag{21}$$

where $\tilde{c}_r(k)$ is the coefficient of x^{k^2-r} in the $2k$-th moment polynomial of random unitary matrices.

Although the CFKRS conjecture seems hopelessly difficult to prove, the precise nature of the asymptotic formula allows one to gain insight into the behavior of the zeta function. For example, by deriving an asymptotic for $c_r(k)$ that is applicable as r and k both tend to infinity, one can understand the true size of $\zeta(1/2+it)$. The results we present here are a step in this direction.

One difficulty in extracting uniform asymptotics for the coefficients of $P_k(x)$ from a residue like (3) is that the coefficients are given only implicitly. By comparison, both the coefficients and the roots of the moment polynomials for random unitary matrices, which correspond to the zeta function moment polynomials according to the random matrix philosophy, are known explicitly, via random matrix theory calculations. In fact, the proof of Theorem 1 of [HR], which provides complete uniform asymptotics for the coefficients in the unitary case, makes essential use of the information about the roots via a saddle-point technique. In the case of the zeta function, however, we do not have 'simple' closed form expressions for the moment polynomials.

We remark that if one directly applies the methods of this paper to the residue expression for unitary moment polynomials, given in [CFKRS1, Eq. (1.5.9)], then one encounters similar difficulties as in the zeta function (e.g. a similar difficulty in deriving asymptotics beyond the first k coefficients). The main added simplicity in the unitary case is that it does not involve an arithmetic factor.

Before delving into the careful details of the next sections, let us describe the basic idea of the proof. To this end, define

$$R(z_1,\dots,z_{2k}) := G(z_1,\dots,z_{2k}) \prod_{i,j=1}^{k} (z_i - z_{k+j}), \tag{22}$$

where, recall, $G(z_1,\dots,z_{2k}) = A(z_1,\dots,z_{2k}) \prod_{i,j=1}^{k} \zeta(1+z_i-z_{k+j})$. The extra product on the rhs in (22) is introduced in order to cancel the poles in the product of zetas in the definition of $G(z_1,\dots,z_{2k})$. This renders the function $R(z_1,\dots,z_{2k})$ analytic and non-zero in a neighborhood of the origin, where it is equal to a_k. Therefore, we may write

$$P_k(x) = \frac{(-1)^k}{k!^2} \frac{1}{(2\pi i)^{2k}} \oint \cdots \oint \frac{\Delta^2(z_1,\dots,z_{2k}) e^{\frac{x}{2}\sum_{i=1}^{k} z_i - z_{k+i}}}{\prod_{i,j=1}^{k}(z_i - z_{k+j}) \prod_{i=1}^{2k} z_i^{2k}} \times e^{\log R(z_1,\dots,z_{2k})}\, dz_1 \cdots dz_{2k}, \tag{23}$$

and consider the Taylor expansion of $\log R(z_1,\dots,z_{2k})$:

$$\log R(z_1,\dots,z_{2k}) = \log a_k + \frac{\tau_k}{2} \sum_{i=1}^{k} z_i - z_{k+i} + \cdots, \tag{24}$$

where, recall, $\tau_k = 2B_k + 2\gamma k \sim 4k\log k$, as $k\to\infty$. Also, dropping the factor $\exp(\log R(z_1,\dots,z_{2k}))$, define

$$p_k(x,0) := \frac{(-1)^k}{k!^2} \frac{1}{(2\pi i)^{2k}} \oint \cdots \oint \frac{\Delta^2(z_1,\dots,z_{2k}) e^{\frac{x}{2}\sum_{i=1}^{k} z_i - z_{k+i}}}{\prod_{i,j=1}^{k}(z_i - z_{k+j}) \prod_{i=1}^{2k} z_i^{2k}}\, dz_1 \cdots dz_{2k} \tag{25}$$

(a more general function $p_k(x,\alpha)$ will be introduced in the next section). Our basic claim is that the approximation

$$P_k(x) \approx a_k p_k(x+\tau_k, 0), \tag{26}$$

obtained from $P_k(x)$ by truncating the Taylor expansion of $\log R(z_1,\dots,z_{2k})$ at the linear term, is good enough to deduce asymptotics for the coefficients $\{c_r(k),\ 0\leqslant r\leqslant k^\beta\}$, for any fixed $\beta<1$, in the

sense the leading term asymptotic of the coefficient of x^{k^2-r}, $0 \leqslant r \leqslant k^\beta$, on either side of (26) is the same.

Notice the formula defining $p_k(x,0)$ does not involve the complicated arithmetic factor $A(z_1,\dots,z_{2k})$ present in the residue expression for $P_k(x)$. Moreover, by the results of Conrey, Farmer, Keating, Rubinstein, and Snaith, the function $p_k(x+\tau_k,0)$ can be evaluated explicitly as a polynomial in x of degree k^2. For, by property (45) later, and the formulas in Section 2.7 of [CFKRS1], we have

$$p_k(x+\tau_k,0) = \frac{g_k}{k^2!}(x+\tau_k)^{k^2}. \tag{27}$$

The idea that the linear term in the Taylor expansion of $\log R(z_1,\dots,z_{2k})$ ought to dominate over $0 \leqslant r \leqslant k^\beta$ was inspired, in part, by the analogous asymptotic (21), derived in [HR], for the moments of the characteristic polynomial of random unitary matrices.

As mentioned earlier, the main theorem of this paper shows that the coefficients of the polynomial $a_k p_k(x+\tau_k,0) = \frac{a_k g_k}{k^2!}(x+\tau_k)^{k^2}$ provide the leading asymptotics, as $k\to\infty$, for essentially the first k coefficients of $P_k(x)$. The proof of this theorem will naturally split into two main parts. In the first part, which is presented in Section 3, Section 4, and Section 5, we obtain estimates on certain functions in k, later denoted by p_k. In the second part, which is presented in Section 6, we obtain bounds on the Taylor coefficients of the logarithm of the arithmetic factor. The latter bounds (and in some cases asymptotics) are fairly involved but generally straightforward, while the former bounds are more subtle, requiring somewhat more thought. Both bounds are obtained via essentially combinatorial arguments.

1.2. Numerical verifications and an application to the maximal size of $|\zeta(1/2+it)|$

Table 1 provides numerical confirmation of our main theorem, listing values of the ratio

$$\frac{c_r(k)}{c_0(k)\binom{k^2}{r}\tau_k^r} \tag{28}$$

for $k = 10, 20, 30, 40, 50$ and $0 \leqslant r \leqslant 7$. Our theorem provides an estimate for this ratio of $1 + O((r/k)^2)$, and our table is consistent with such a remainder term, agreeing, for example, to 3–4 decimal places for $r = 2$ and $k = 50$, and 2–3 decimal places for $r = 8$ and $k = 50$.

Next, let $\beta < 1$, and, as usual, $k \in \mathbb{Z}_{\geqslant 0}$. While the asymptotic formula for $c_r(k)$ given in our main theorem holds, as $k\to\infty$, for $r < k^\beta$, it appears, numerically, that our asymptotic formula is, uniformly, an upper bound for $|c_r(k)|$ for all $0 \leqslant r \leqslant k^2$.

We therefore conjecture, for all non-negative integers k, and all $0 \leqslant r \leqslant k^2$, that:

$$\left|c_r(k)\right| \leqslant c_0(k)\binom{k^2}{r}\tau_k^r. \tag{29}$$

We have verified this conjecture numerically for all $k \leqslant 13$, $0 \leqslant r \leqslant k^2$, and all $k \leqslant 64$, $0 \leqslant r \leqslant 8$. The coefficients of the moment polynomials were computed in the former case in [RY] and in the latter case using the program developed for the computations in [CFKRS1] and [CFKRS2]. See Fig. 1 for evidence supporting this conjecture, which depicts the ratio $c_r(k)/(c_0(k)\binom{k^2}{r}\tau_k^r)$ for $k = 10$ and $0 \leqslant r \leqslant k^2$.

Assuming the bound (29), we have, by the binomial theorem and term-wise comparison, the following upper bound for $P_k(x)$, for all $k \in \mathbb{Z}_{\geqslant 0}$ and $x \in \mathbb{R}$:

$$\left|P_k(x)\right| \leqslant c_0(k)\left(|x|+\tau_k\right)^{k^2}. \tag{30}$$

Table 1
A comparison of our asymptotic formula for $c_r(k)$, for $k = 10, 20, 30, 40, 50$ and $r \leqslant 7$. The 1's are explained by the remark following the main theorem that the asymptotic formula is actually an identity for $r = 0$ and $r = 1$. We expect there to be lower terms in our asymptotic expansion, and will return to the problem of determining them in a future paper.

k	r	$c_r(k)$	$c_r(k)/(c_0(k)\binom{k^2}{r}\tau_k^r)$
10	0	3.548884925e−148	1
10	1	2.357691331e−144	1
10	2	7.702336630e−141	0.9934255388
10	3	1.649486344e−137	0.9803060865
10	4	2.604519447e−134	0.9608017974
10	5	3.233666778e−131	0.9352015310
10	6	3.287651416e−128	0.9039165203
10	7	2.814729470e−125	0.8674698258
20	0	9.404052083e−789	1
20	1	7.007560591e−784	1
20	2	2.600909647e−779	0.9986738069
20	3	6.410977573e−775	0.9960221340
20	4	1.180624032e−770	0.9920509816
20	5	1.732651855e−766	0.9867716274
20	6	2.110801042e−762	0.9802005819
20	7	2.195579847e−758	0.9723595087
30	0	2.174528185e−2019	1
30	1	6.409313254e−2014	1
30	2	9.429995281e−2009	0.9994621075
30	3	9.234275546e−2004	0.9983864033
30	4	6.770756592e−1999	0.9967738368
30	5	3.964993050e−1994	0.9946262257
30	6	1.931729883e−1989	0.9919462534
30	7	8.053463103e−1985	0.9887374636
40	0	1.878520688e−3887	1
40	1	1.450126078e−3881	1
40	2	5.592030026e−3876	0.9997132915
40	3	1.436301603e−3870	0.9991398909
40	4	2.764308226e−3865	0.9982800615
40	5	4.252265871e−3860	0.9971343131
40	6	5.445979160e−3855	0.9957034019
40	7	5.972928889e−3850	0.9939883295
50	0	3.461963190e−6425	1
50	1	5.605367518e−6419	1
50	2	4.535291006e−6413	0.9998231027
50	3	2.444917857e−6407	0.9994693125
50	4	9.879474579e−6402	0.9989387280
50	5	3.191850197e−6396	0.9982315414
50	6	8.588531004e−6391	0.9973480389
50	7	1.979690769e−6385	0.9962886003

Let $|\zeta(1/2 + it_0)| = m_T := \max_{t\in[0,T]} |\zeta(1/2 + it)|$. Lemma 3.3 of [FGH] provides:

$$m_T \leqslant 2(CT \log T)^{1/2k} \left(\frac{1}{T} \int_0^T \left|\zeta(1/2 + it)\right|^{2k} dt \right)^{1/2k} \tag{31}$$

for some absolute constant $C > 0$. Farmer, Gonek and Hughes use this inequality, combined with the Keating and Snaith leading term conjecture for the moments of zeta to estimate m_T. However, the leading term does a poor job at bounding the true size of the moments if we allow k to grow with T.

However, using our conjectured bound (30) for $P_k(x)$, we have, in whatever range of k that (1) remains valid asymptotically, that

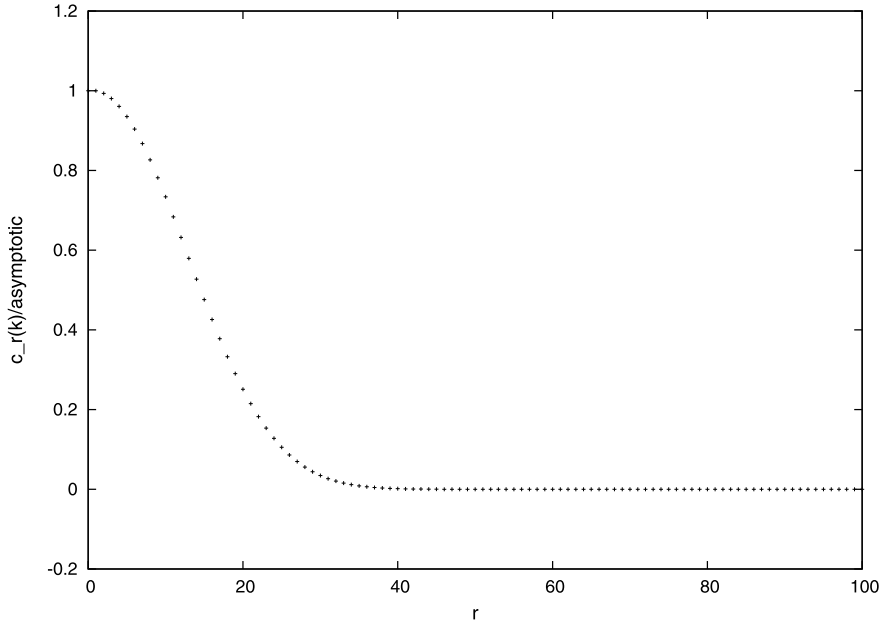

Fig. 1. We compare the ratio of $c_r(10)$, $0 < r < 100$, to our asymptotic formula. Here, $k = 10$ is relatively small, and we only get reasonable agreement for the first few r. However, the graph indicates that the asymptotic formula is, uniformly, an upper bound for $|c_r(k)|$.

$$m_T \leqslant 2\big(c_0(k)C_2 T\log T\big)^{1/2k}\left(\frac{1}{T}\int_0^T \big(\big|\log(t/2\pi)\big| + \tau_k\big)^{k^2}\,dt\right)^{1/2k} \tag{32}$$

for some absolute constant $C_2 > 0$. Following the argument in [FGH], we will, at the end, apply the above with k proportionate to $(\log(T)/\log\log(T))^{1/2}$.

The portion of the integral, $t \in (0, 2\pi)$ where $\log(t/2\pi)$ is negative contributes $O((k^2)^{k^2})$, on using: $\int_0^{2\pi} |\log(t/2\pi)|^{k^2}\,dt = 2\pi k^2!$, the binomial expansion, Stirling's formula for $k^2!$, and also $\sum_0^{k^2} \tau_k^r/r! < \exp(\tau_k)$ combined with (17). (We could also slightly modify the argument in [FGH] and ignore this interval outright.)

Next, by (17), we have $\tau_k = O(k\log k)$. Thus, if $k \leqslant C_3\log(T)/\log\log(T)$, for some absolute constant C_3, the contribution to the integral for $t \in [2\pi, T]$ is $O(T(C_4\log(T))^{k^2})$, for some absolute constant C_4.

Therefore, if $k = O(\log(T)^{1/2})$, we can ignore the portion of the integral from 0 to 2π, and get:

$$m_T \ll 2\big(c_0(k)C_5 T\log T\big)^{1/2k}(C_4\log T)^{k/2} \tag{33}$$

for some absolute constant C_5, i.e.

$$\log m_T \ll \frac{\log c_0(k)}{2k} + \frac{\log(T) + \log\log(T)}{2k} + \frac{k}{2}\log\log T + O(k). \tag{34}$$

Combining Conrey and Gonek's estimate [CG]:

$$\log a_k \sim -k^2\log\big(2e^{\gamma}\log k\big) + o\big(k^2\big) \quad \text{for } k \to \infty, \tag{35}$$

Table 2
A comparison of three estimates for the moments of zeta, with $T = 100000000.643$, and $k \leqslant 13$. The second and third columns are taken from [RY].

k	$\int_0^T \lvert\zeta(1/2+it)\rvert^{2k}\,dt$	$\int_0^T P_k(\log(t/2\pi))\,dt$	$c_0(k)T\log(T)^{k^2}$	$c_0(k)\int_0^T (\lvert\log(t/2\pi)\rvert+\tau_k)^{k^2}\,dt$
1	1.6737236904e+09	1.6737234985e+09	1.8420680869e+09	1.6737235247e+09
2	6.3738834341e+11	6.3738992350e+11	5.8330132790e+11	6.7489927655e+11
3	8.0458531434e+14	8.0458140334e+14	1.3940397179e+14	1.2999952534e+15
4	1.7376480696e+18	1.7374512576e+18	4.3322247610e+15	8.5349032584e+18
5	5.0837678819e+21	5.0816645028e+21	6.0772270922e+15	1.8070544717e+23
6	1.8153019937e+25	1.8136396872e+25	1.8242195930e+14	1.2033327456e+28
7	7.4805129691e+28	7.4688841259e+28	6.5819531631e+10	2.4552753344e+33
8	3.4385117285e+32	3.4309032713e+32	1.7844629682e+05	1.4940783176e+39
9	1.7238857795e+36	1.7191846566e+36	2.4462083265e−03	2.6420504382e+45
10	9.2785048601e+39	9.2517330046e+39	1.2040915381e−13	1.3256809885e+52
11	5.2991086420e+43	5.28630715e+43	1.5747149879e−26	1.8471999998e+59
12	3.1825481927e+47	3.17945e+47	4.1820123844e−42	7.0111752824e+66
13	1.9956246380e+51	2.00e+51	1.7694787451e−60	7.1249837060e+74

with the asymptotics of the Barnes G-function, see (3.17) and (3.18) of [FGH], gives:

$$\frac{\log c_0(k)}{2k} = -\frac{k\log k}{2} + O(k\log\log k). \tag{36}$$

Hence,

$$\log m_T \ll \frac{\log(T)+\log\log(T)}{2k} + \frac{k}{2}\log\log T - \frac{k\log k}{2} + O(k\log\log k), \tag{37}$$

i.e. bound (3.20) of [FGH] continues to hold even when we use our upper bound for the moment polynomials, rather than the much smaller and less precise (as k grows) leading term.

Taking, as in [FGH], $k \sim c(\log(T)/\log\log(T))^{1/2}$, and choosing the optimal $c = 2^{1/2}$, thus gives the identical upper bound (3.9) of [FGH]:

$$m_T \ll \exp\left(\left(\frac{1}{2}\log T\log\log T\right)^{1/2} + O\left(\frac{(\log T)^{1/2}\log\log\log T}{(\log\log T)^{1/2}}\right)\right). \tag{38}$$

Table 2 compares values of $\int_0^T \lvert\zeta(1/2+it)\rvert^{2k}\,dt$, for $T = 100000000.643$, $k \leqslant 13$, to: the Keating and Snaith leading term $c_0(k)T\log(T)^{k^2}$ prediction, the full asymptotics $\int_0^T P_k(\log(t/2\pi))\,dt$, and, finally, using our upper bound for $P_k(x)$, i.e. to $c_0(k)\int_0^T (\lvert\log(t/2\pi)\rvert+\tau_k)^{k^2}\,dt$.

The values for the third column in Table 2 come from [RY], and the lower accuracy for $k = 11, 12, 13$ reflects the precision to which we computed, in [RY], the coefficients of the moment polynomials. The numerical integration of the moments of zeta was carried out in [RY] using tanh–sinh quadrature, integrating the humps between successive zeros of zeta on the critical line, hence we stopped at 100000000.643 rather than 10^8.

The values in the 4-th and 5-th columns are given with more precision as they only rely on $c_0(k)$ and $c_1(k)$ which have been computed to higher accuracy. The table shows, first, that the full moment conjecture successfully captures, here, the moments well beyond $k = 4 \approx (2\log(T)/\log\log(T))^{1/2}$. It also shows that the leading term alone quickly (for example, at $k = 4$) fails to capture the true size of the moments, whereas, our upper bound for the moment polynomials seems to give an upper bound for the moments of zeta valid for a large range of k, hence justifying its use in bounding the maximum size of zeta, m_T.

2. Proof of the main theorem

In the remainder of the paper, asymptotic constants are always absolute, and are taken as $k \to \infty$, unless otherwise is stated.

Proof of the main theorem. Let $\alpha := (\alpha_1, \dots, \alpha_{2k})$ be a $2k$-tuple in $\mathbb{Z}^{2k}_{\geqslant 0}$, and let $|\alpha| := \alpha_1 + \cdots + \alpha_{2k}$ denote its weight. Write

$$\log A(z_1, \dots, z_{2k}) =: \log a_k + B_k \sum_{i=1}^{k} z_i - z_{k+i} + \sum_{|\alpha|>1} a_\alpha z_1^{\alpha_1} \cdots z_{2k}^{\alpha_{2k}}, \tag{39}$$

the second sum being over tuples with weight greater than 1. Also, write

$$\log\left(\prod_{i,j=1}^{k} (z_i - z_{k+j})\zeta(1 + z_i - z_{k+j}) \right) =: \gamma k \sum_{i=1}^{k} z_i - z_{k+i} + \sum_{|\alpha|>1} b_\alpha z_1^{\alpha_1} \cdots z_{2k}^{\alpha_{2k}}. \tag{40}$$

The linear term in the Taylor expansion (40) is γk, which is an easy consequence of the expansion $z\zeta(1+z) = 1 + \gamma z + \cdots$. Lastly, define

$$p_k(x, \alpha) := \frac{(-1)^k}{k!^2} \frac{1}{(2\pi i)^{2k}} \oint \cdots \oint \frac{\Delta^2(z_1, \dots, z_{2k}) e^{\frac{x}{2}\sum_{i=1}^{k} z_i - z_{k+i}}}{\prod_{i,j=1}^{k} (z_i - z_{k+j}) \prod_{i=1}^{2k} z_i^{2k}} z_1^{\alpha_1} \cdots z_{2k}^{\alpha_{2k}} \, dz_1 \cdots dz_{2k}, \tag{41}$$

and let c_α be the Taylor coefficients determined by

$$e^{\sum_{|\alpha|>1} (a_\alpha + b_\alpha) z_1^{\alpha_1} \cdots z_{2k}^{\alpha_{2k}}} =: 1 + \sum_{|\alpha|>1} c_\alpha z_1^{\alpha_1} \cdots z_{2k}^{\alpha_{2k}}. \tag{42}$$

So, on recalling $\tau_k = 2B_k + 2\gamma k$, the c_α's satisfy:

$$A(z_1, \dots, z_{2k}) \prod_{i,j=1}^{k} (z_i - z_{k+j})\zeta(1 + z_i - z_{k+j}) = a_k e^{\frac{\tau_k}{2}\sum_{i=1}^{k} z_i - z_{k+i}} \left(1 + \sum_{|\alpha|>1} c_\alpha z_1^{\alpha_1} \cdots z_{2k}^{\alpha_{2k}} \right), \tag{43}$$

where, as before, $\tau_k \sim 4k \log k$ as $k \to \infty$. Therefore, we have

$$P_k(x) = a_k p_k(x + \tau_k, 0) + a_k \sum_{|\alpha|>1} c_\alpha p_k(x + \tau_k, \alpha), \tag{44}$$

where the second argument in $p_k(x + \tau_k, 0)$ stands for the zero $2k$-tuple.

Notice the sum in (44) is actually finite, because if $|\alpha| > k^2$ (or if $\alpha_j \geqslant 2k$ for some j), then $p_k(x, \alpha) = 0$, because by degree considerations the integrand in the residue (41) defining $p_k(x, \alpha)$ will have no poles. Also, by the change of variables, $z_j \leftarrow x z_j$, we have

$$p(x, \alpha) = x^{k^2 - |\alpha|} p(1, \alpha), \tag{45}$$

which, along with the formulas in Section 2.7 of [CFKRS1], yields

$$p_k(x, 0) = x^{k^2} p_k(1, 0) = x^{k^2} \frac{g_k}{k^2!}. \tag{46}$$

(We used formulas (45) and (46) to evaluate $p_k(x+\tau_k,0)$ in (27) earlier.) In light of property (45), it is convenient to set

$$p_k(\alpha) := p_k(1,\alpha). \tag{47}$$

Combining (44), the observation made thereafter, and (45), we arrive at

$$P_k(x) = a_k(x+\tau_k)^{k^2} p_k(0) + a_k \sum_{n=2}^{k^2} (x+\tau_k)^{k^2-n} \sum_{|\alpha|=n} c_\alpha p_k(\alpha). \tag{48}$$

In particular, observing $a_k p_k(0) = c_0(k)$, and equating the coefficient of x^{k^2-r} on both sides of (48), we obtain

$$\begin{aligned} c_r(k) &= \tau_k^r \binom{k^2}{r} c_k(0) + a_k \sum_{n=2}^{r} \tau_k^{r-n} \binom{k^2-n}{r-n} \sum_{|\alpha|=n} c_\alpha p_k(\alpha) \\ &= \tau_k^r \binom{k^2}{r} c_k(0) \left[1 + \sum_{n=2}^{r} \frac{r!(k^2-n)!}{(r-n)!k^2!} \frac{1}{\tau_k^n} \sum_{|\alpha|=n} c_\alpha \frac{p_k(\alpha)}{p_k(0)} \right]. \end{aligned} \tag{49}$$

The above is an identity, valid for any $0 \leqslant r \leqslant k^2$. Also, notice the double sum in (49) is empty if $r = 0, 1$, so $c_r(k) = \tau_k^r \binom{k^2}{r} c_0(k)$ for $r = 0, 1$.

Our plan is to show, for $0 \leqslant r \leqslant k^\beta$, $c_r(k) \approx \tau_k^r \binom{k^2}{r} c_0(k)$. To do so, we will show that the term 1 preceding the double sum in (49) dominates. This will follow from the following three bounds, as we soon explain:

- First bound: By Theorem 5.2, as $k \to \infty$ and uniformly in $|\alpha| < k/2$, we have

$$\frac{p_k(\alpha)}{p_k(0)} \ll \big(\lambda_1 k \log\big(|\alpha|+10\big)\big)^{|\alpha|}, \tag{50}$$

  where λ_1 is some absolute constant. This is proved in Section 5 as a by-product of the "symmetrization algorithm" (see Section 3), and the algorithm to compute a certain "symmetrized version" of $p_k(\alpha)$, which we denote by $N_k^0(\alpha)$ (see Section 4.1). The notation $N_k^0(\alpha)$ is chosen to distinguish it from the related function $N_k(\alpha)$, defined in [CFKRS2]. The said algorithms are essentially combinatorial recursions. In the case of $N_k^0(\alpha)$, the recursion stops much earlier than what is obvious, due to a certain anti-symmetry relation, which is the reason algorithm is able to produce a non-trivial bound on $N_k^0(\alpha)$, essentially by counting the number of terms involved in it. We remark the bound (50) is sharp in the power of k, as the second example in Section 4.2 illustrates.
- Second bound: By Theorem 6.1, the coefficients a_α in the Taylor expansion of $\log A(z_1,\dots,z_{2k})$, which were defined in (39), satisfy:

$$a_\alpha \ll \lambda_2^{|\alpha|} (\log k)^{|\alpha|} \big[m(\alpha)^{|\alpha|} k^{2-\min\{m(\alpha),2\}} + |\alpha|! k^{2-m(\alpha)} \big], \tag{51}$$

  where $m(\alpha)$ denotes the number of non-zero entries in α, and λ_2 is some absolute constant. This is proved in Section 6 by an elementary, though lengthy, counting of the terms that contribute. It will transpire that, for $0 \leqslant r \leqslant k^\beta$, most of the contribution to $c_r(k)$ comes from "the combinatorial sum for the small primes", see Section 6.1.1.

- Third bound: By Lemma 7.1, the Taylor coefficients b_α of the product of zetas, which were defined in (40), satisfy:

$$b_\alpha \ll \lambda_3^{|\alpha|} k^{2-m(\alpha)}. \tag{52}$$

This is proved in Section 7 by means of Cauchy's estimate.

We now appeal to the auxiliary lemma stated later in this section. Specifically, by (51) and (52), the coefficients $a_\alpha + b_\alpha$ still satisfy the conditions of that lemma. So on applying the lemma we obtain the following bound on the Taylor coefficients c_α, which were defined in (42): As $k \to \infty$, and uniformly in $n < k/e$,

$$\sum_{|\alpha|=n} |c_\alpha| \ll (\lambda_4 k \log k)^n. \tag{53}$$

Notice the number of summands on the lhs above is not far off from the upper bound, so, on average, the $|c_\alpha|$'s are not large when $|\alpha| < k/e$.

Substituting (50) and (53) directly into identity (49), and recalling $r \leqslant k^\beta$, yields

$$\begin{aligned}\sum_{n=2}^{r} \frac{r!(k^2-n)!}{(r-n)!k^2!}\frac{1}{\tau_k^n} \sum_{|\alpha|=n}\left|c_\alpha \frac{p_k(\alpha)}{p_k(0)}\right| &\ll \sum_{n=2}^{r} \frac{r^n}{k^{2n}\tau_k^n}(\lambda_1 k\log k)^n\big(\lambda_4 k\log(n+10)\big)^n \\ &\ll \sum_{n=2}^{r} \frac{(\lambda r\log n)^n}{k^n},\end{aligned} \tag{54}$$

for some absolute constant λ. Here, we used the following elementary bound

$$\frac{r!(k^2-n)!}{(r-n)!k^2!} \leqslant \frac{r^n}{k^{2n}}, \tag{55}$$

which follows from $(r-j)/(k^2-j) = (r/k^2)(1-j/r)/(1-j/k^2) \leqslant r/k^2$ with $j \leqslant (n-1) < r$, and $r < k^2$ (in fact, $r < k$ in this proof).

Finally, summing the series in (54), and using the assumed bound on r, shows that the lhs of (54) is bounded by $O_\beta((r/k)^2)$, completing the proof. □

Auxiliary lemma. *Let f be a multi-variate series in $2k$ variables*

$$f(x_1,\dots,x_{2k}) := \sum_{n=2}^{\infty} \sum_{\substack{\alpha\in\mathbb{Z}_{\geqslant 0}^{2k} \\ |\alpha|=n}} a_\alpha x_1^{\alpha_1}\cdots x_{2k}^{\alpha_{2k}}. \tag{56}$$

Assume the coefficients a_α satisfy bounds (51)*. Then the coefficients c_α in the Taylor expansion*

$$e^{f(x_1,\dots,x_{2k})} =: 1 + \sum_{n=2}^{\infty}\sum_{|\alpha|=n} c_\alpha x_1^{\alpha_1}\cdots x_{2k}^{\alpha_{2k}} \tag{57}$$

satisfy

$$\sum_{|\alpha|=n} |c_\alpha| \ll (\lambda_5 \log k)^n k^n, \quad \text{for } n < k/e, \tag{58}$$

for some absolute constant λ_5.

Remarks. i) This lemma applies as well if we replace a_α by $a_\alpha + b_\alpha$, with b_α satisfying (52), because $a_\alpha + b_\alpha$ together satisfy a bound of the same form as (51), but with λ_2 replaced by the maximum of λ_2 and λ_3. ii) We are using this lemma in (53).

Proof of the auxiliary lemma. Define

$$C(n) := \sum_{|\alpha|=n} |c_\alpha|, \qquad A(q) := \sum_{|\alpha|=q} |a_\alpha|. \tag{59}$$

We plan to obtain a bound on $C(n)$ in terms of an expression involving $A(q)$, then we will bound $A(q)$ with the aid of estimate (51) for the a_α's, which is assumed in the statement of the lemma.

To this end, exponentiate (56), turning the outer sum into a product, and writing, for the inner sum,

$$\exp\Big(\sum_{|\alpha|=n} a_\alpha x_1^{\alpha_1} \cdots x_{2k}^{\alpha_{2k}} \Big) = \sum_{d=0}^{\infty} \frac{1}{d!} \Big(\sum_{|\alpha|=n} a_\alpha x_1^{\alpha_1} \cdots x_{2k}^{\alpha_{2k}} \Big)^d, \tag{60}$$

we get, on multiplying out the product, that

$$1 + \sum_{n=2}^{\infty} \sum_{|\alpha|=n} c_\alpha x_1^{\alpha_1} \cdots x_{2k}^{\alpha_{2k}} = \prod_{n=2}^{\infty} \sum_{d_n=0}^{\infty} \frac{1}{d_n!} \Big(\sum_{|\alpha|=n} a_\alpha x_1^{\alpha_1} \cdots x_{2k}^{\alpha_{2k}} \Big)^{d_n}. \tag{61}$$

By choosing which of the sums in the above infinite product contribute (i.e., which of the sums has a term chosen from it different from 1), we obtain

$$C(n) \leqslant \sum_{\substack{q_1 d_1 + \cdots + q_r d_r = n, r \geqslant 1 \\ q_r > \cdots > q_2 > q_1 \geqslant 2, d_i \geqslant 1}} \frac{1}{d_1! d_2! \cdots d_r!} A(q_1)^{d_1} A(q_2)^{d_2} \cdots A(q_r)^{d_r}. \tag{62}$$

We now derive a bound on the $A(q_j)$'s. Given an integer $2 \leqslant q \leqslant n$, write

$$A(q) = \sum_{j=1}^{q} \sum_{\substack{|\alpha|=q \\ m(\alpha)=j}} |a_\alpha| = \sum_{\substack{|\alpha|=q \\ m(\alpha)=1}} |a_\alpha| + \sum_{j=2}^{q} \sum_{\substack{|\alpha|=q \\ m(\alpha)=j}} |a_\alpha|, \tag{63}$$

where, recall, $m(\alpha)$ is equal to the number of non-zero α_i's. Substituting the bounds (51) for the $|a_\alpha|$'s, we get

$$A(q) \ll \sum_{\substack{|\alpha|=q \\ m(\alpha)=1}} (\lambda_2)^q q! (\log k)^q k + \sum_{j=2}^{q} \sum_{\substack{|\alpha|=q \\ m(\alpha)=j}} (\lambda_2)^q j^q (\log k)^q + \sum_{j=2}^{q} \sum_{\substack{|\alpha|=q \\ m(\alpha)=j}} \frac{(\lambda_2)^q q! (\log k)^q}{k^{j-2}}. \tag{64}$$

But

$$\sum_{\substack{|\alpha|=q\\ m(\alpha)=j}} 1 = \binom{2k}{j}\binom{q-1}{j-1}, \tag{65}$$

as there are $\binom{2k}{j}$ ways to select j of the z_i's and $\binom{q-1}{j-1}$ ways to sum to q using precisely j positive (ordered) integers. The latter fact can be seen by arranging q 'dots' in a row and breaking them into j summands by selecting $j-1$ out of $q-1$ barriers between the dots.

Therefore, for $q < k/2$ (for later purposes, we actually assume $q \leqslant n < k/e$ in this proof), we have generously,

$$\sum_{j=2}^{q}\binom{2k}{j}\binom{q-1}{j-1}j^q \leqslant \sum_{j=2}^{q}\frac{(2k)^j j^q q^j}{(j!)^2} \leqslant k^q(100)^q. \tag{66}$$

The first inequality follows by expanding the binomial coefficients as ratios of factorials, and noting that: i) $(2k)!/(2k-j)! \leqslant (2k)^j$. ii) $j(q-1)!/(q-j)! \leqslant jq^{j-1} \leqslant q^j$. The second inequality in (66) follows by noticing that the terms of the sum are, in our range, increasing (consider the ratio of two successive terms), hence an upper bound for sum is q times the last term, which can be estimated by Stirling's formula. Similarly,

$$\sum_{j=2}^{q}\binom{2k}{j}\binom{q-1}{j-1}q!k^{2-j} \leqslant 2^q k^2 \sum_{j=2}^{q}\frac{q^j q^q}{(j!)^2} \leqslant k^2 q^q (100)^q. \tag{67}$$

Using (66) to bound the second sum in (64), using (67) to bound the third sum, and noting that the number of terms in the first sum there is

$$\sum_{\substack{|\alpha|=q\\ m(\alpha)=1}} 1 = 2k, \tag{68}$$

which follows since there are $2k$ choices for the z_j's, together yields

$$A(q) \ll k^2(\lambda_2 q\log k)^q + k^q(100\lambda_2\log k)^q + k^2(100\lambda_2 q\log k)^q \tag{69}$$

$$\ll k^q(100\lambda_2\log k)^q\left[1 + k^2\left(\frac{q}{k}\right)^q\right]. \tag{70}$$

Substituting the above into (62), we obtain for some absolute constant λ_6,

$$C(n) \ll k^n(\lambda_6\log k)^n \sum_{\substack{q_1d_1+\cdots+q_rd_r=n, r\geqslant 1\\ q_r>\cdots>q_2>q_1\geqslant 2, d_i\geqslant 1}} \frac{1}{d_1!d_2!\cdots d_r!}\prod_{i=1}^{r}\left[1 + k^2\left(\frac{q_i}{k}\right)^{q_i}\right]^{d_i}. \tag{71}$$

Since the function $(x/k)^x$ is monotonically decreasing for $x \in [1, k/e)$, it follows

$$k^2\left(\frac{q_i}{k}\right)^{q_i} \leqslant 4, \quad \text{if } 2 \leqslant q_i < k/e. \tag{72}$$

Thus,

$$\prod_{i=1}^{r}\left[1+k^2\left(\frac{q_i}{k}\right)^{q_i}\right]^{d_i} \leqslant 5^n, \quad \text{if } 2\leqslant q_i < k/e. \tag{73}$$

Here we have used $\sum d_i \leqslant n$. Also,

$$\sum_{\substack{q_1d_1+\cdots+q_rd_r=n, r\geqslant 1\\ q_r>\cdots>q_2>q_1\geqslant 1, d_i\geqslant 1}} \frac{1}{d_1!d_2!\cdots d_r!} < e^n, \tag{74}$$

because the lhs is the coefficient of x^n in $\prod_{m=1}^{n}\sum_{d=1}^{\infty} x^{md}/d!$ (we truncate the product at $m=n$ since each $q_i \leqslant n$). But that coefficient is less than the sum total of all the coefficients, i.e. $< \prod_{m=1}^{n}\sum_{d=1}^{\infty} 1/d! < e^n$.

Substitute (73) and (74) into (71), we have, for $n < k/e$,

$$C(n) \ll (5\lambda_6 \log k)^n k^n \sum_{\substack{q_1d_1+\cdots+q_rd_r=n\\ q_i\geqslant 2, d_i\geqslant 1, r\geqslant 1}} \frac{1}{d_1!d_2!\cdots d_r!} \tag{75}$$

$$\ll (15\lambda_6 \log k)^n k^n, \tag{76}$$

as claimed. □

3. An algorithm to reduce to the first half

We show that the residue expression for $p_k(\alpha)$, given by (41) and (45), can be reduced to variables in the first half only; i.e., involving $z_1,\ldots,z_k$ only. To do so, we will need the following two lemmas.

Lemma 3.1. *Suppose $H(z_1,\ldots,z_{2n})$ is regular in $\mathcal{D} := \{|(z_1,\ldots,z_{2n})| < \delta\}$. For $(\alpha_1,\ldots,\alpha_{2n}) \in \mathcal{D}$, such that the α_i's are distinct, define*

$$\mathcal{K}(\alpha_1,\ldots,\alpha_{2n}) := \sum_{\sigma\in S_{2n}} \frac{H(\alpha_{\sigma(1)},\ldots,\alpha_{\sigma(2n)})}{\prod_{i,j=1}^{n}(\alpha_{\sigma(i)}-\alpha_{\sigma(n+j)})}, \tag{77}$$

where S_{2n} is the permutation group of $2n$ elements. Then, it holds

$$\mathcal{K}(\alpha_1,\ldots,\alpha_{2n}) = \frac{(-1)^n}{(2\pi i)^{2n}}\oint\cdots\oint \frac{H(z_1,\ldots,z_{2n})\Delta^2(z_1,\ldots,z_{2n})}{\prod_{i,j=1}^{n}(z_i-z_{n+j})\prod_{i,j=1}^{2n}(z_i-\alpha_j)}\,dz_1\cdots dz_{2k}, \tag{78}$$

where the integration contour consists of circles contained in $\mathcal{D}$ around the α_i's. In particular, if the integration contour is chosen so each circle encloses 0 as well, then the limit

$$\lim_{\substack{\alpha_i\to 0\\ 1\leqslant i\leqslant 2n}} \mathcal{K}(\alpha_1,\ldots,\alpha_{2n}) = \frac{(-1)^n}{(2\pi i)^{2n}}\oint\cdots\oint \frac{H(z_1,\ldots,z_{2n})\Delta^2(z_1,\ldots,z_{2n})}{\prod_{i,j=1}^{n}(z_i-z_{n+j})\prod_{i=1}^{2n}z_i^{2n}}\,dz_1\cdots dz_{2k} \tag{79}$$

exists, and is finite.

Proof. This lemma is a slight variant of Lemmas 2.5.1 and 2.5.3 in [CFKRS1]. □

Lemma 3.2. *Let $H(z_1,\dots,z_{2n})$ and $f(z_1,\dots,z_{2n})$ be two regular functions in $\mathcal{D}$. Suppose also f is symmetric with respect to all its arguments (so f is invariant under the action of S_{2n}). Define*

$$I(f) := \frac{(-1)^n}{(2\pi i)^{2n}} \oint \cdots \oint \frac{H(z_1,\dots,z_{2n}) f(z_1,\dots,z_{2n}) \Delta^2(z_1,\dots,z_{2n})}{\prod_{i,j=1}^{n}(z_i - z_{n+j}) \prod_{i=1}^{2n} z_i^{2n}}\, dz_1 \cdots dz_{2k}, \tag{80}$$

where the integration contour consists of circles in $\mathcal{D}$ around 0. Then

$$I(f) = f(0,\dots,0) I(1). \tag{81}$$

Proof. Define

$$\mathcal{K}_f(\alpha_1,\dots,\alpha_{2n}) := \sum_{\sigma\in S_{2n}} \frac{H(\alpha_{\sigma(1)},\dots,\alpha_{\sigma(2n)}) f(\alpha_{\sigma(1)},\dots,\alpha_{\sigma(2n)})}{\prod_{i,j=1}^{n}(\alpha_{\sigma(i)} - \alpha_{\sigma(n+j)})}. \tag{82}$$

Then,

$$\begin{aligned} I(f) &= \lim_{\substack{\alpha_i\to 0\\ 1\leqslant i\leqslant 2n}} \mathcal{K}_f(\alpha_1,\dots,\alpha_{2n}) && (83)\\ &= \lim_{\substack{\alpha_i\to 0\\ 1\leqslant i\leqslant 2n}} f(\alpha_1,\dots,\alpha_{2n}) \lim_{\substack{\alpha_i\to 0\\ 1\leqslant i\leqslant 2n}} \mathcal{K}_1(\alpha_1,\dots,\alpha_{2n}) && (84)\\ &= f(0,\dots,0) I(1). \qquad \square && (85) \end{aligned}$$

3.1. The first step: from $p_k(\alpha)$ to $p_k(\lambda;0)$

Recall, for a tuple $\alpha = (\alpha_1,\dots,\alpha_{2k}) \in \mathbb{Z}^{2k}_{\geqslant 0}$ we defined

$$p_k(\alpha) := \frac{(-1)^k}{k!^2} \frac{1}{(2\pi i)^{2k}} \oint \cdots \oint \frac{\Delta^2(z_1,\dots,z_{2k}) e^{\frac{1}{2}\sum_{i=1}^{k} z_i - z_{k+i}}}{\prod_{i,j=1}^{k}(z_i - z_{k+j}) \prod_{i=1}^{2k} z_i^{2k}} z_1^{\alpha_1} \cdots z_{2k}^{\alpha_{2k}}\, dz_1 \cdots dz_{2k}. \tag{86}$$

In this subsection we show that $p_k(\alpha)$ can always be written as a relatively short (for purposes of our analysis) linear combination of functions of the form $p_k(\beta_1,\dots,\beta_k,0,\dots,0)$, where $\beta_i \in \mathbb{Z}_{\geqslant 0}$ for all $1\leqslant i\leqslant k$. So consider a $2k$-tuple $\alpha = (\alpha_1,\dots,\alpha_{k+d},0,\dots,0)$ where $1\leqslant d\leqslant k$, and such that $\alpha_{k+i} > 0$ for $1\leqslant i\leqslant d$. Since the integral (86) is then symmetric in $z_{k+d},\dots,z_{2k}$, it follows

$$\begin{aligned} p_k(\alpha) = \frac{(-1)^k}{k!^2} \frac{1}{(2\pi i)^{2k}} \oint \cdots \oint & \frac{\Delta^2(z_1,\dots,z_{2k}) e^{\frac{1}{2}\sum_{i=1}^{k} z_i - z_{k+i}}}{\prod_{i,j=1}^{k}(z_i - z_{k+j}) \prod_{i=1}^{2k} z_i^{2k}} z_1^{\alpha_1} \cdots z_{k+d-1}^{\alpha_{k+d-1}} \\ & \times \frac{1}{k-d+1} \left(\sum_{j=k+d}^{2k} z_j^{\alpha_{k+d}} \right) dz_1 \cdots dz_{2k}, \end{aligned} \tag{87}$$

and by Lemma 3.2,

$$p_k(\alpha) = \frac{(-1)^k}{k!^2}\frac{1}{(2\pi i)^{2k}}\oint\cdots\oint \frac{\Delta^2(z_1,\dots,z_{2k})e^{\frac{1}{2}\sum_{i=1}^k z_i - z_{k+i}}}{\prod_{i,j=1}^k (z_i - z_{k+j})\prod_{i=1}^{2k} z_i^{2k}} z_1^{\alpha_1}\cdots z_{k+d-1}^{\alpha_{k+d-1}} \times \frac{1}{k-d+1}\left(\sum_{j=k+d}^{2k} z_j^{\alpha_{k+d}} - \sum_{j=1}^{2k} z_j^{\alpha_{k+d}}\right) dz_1\cdots dz_{2k}. \tag{88}$$

This can be seen from Lemma 3.2 by pulling out the second sum in brackets in front of the integral, evaluated at all $z_j = 0$, to give 0. For $1 \leqslant j \leqslant 2k$, let us thus define

$$\eta^{(j)} := \big(\overbrace{0,\dots,0}^{j-1 \text{ zeros}}, \alpha_{k+d}, 0,\dots,0\big), \tag{89}$$

$$\alpha^{(j)} := \alpha - \eta^{(k+d)} + \eta^{(j)}, \tag{90}$$

where the addition and subtraction in the definition of $\alpha^{(j)}$ is done component-wise. Then we have

$$p_k(\alpha) = \frac{-1}{k-d+1}\sum_{j=1}^{k+d-1} p_k\big(\alpha^{(j)}\big). \tag{91}$$

In particular, we have expressed $p_k(\alpha)$ as the sum of $k+d-1$ functions of the form $p_k(\beta)$, where each tuple β has its last possibly non-zero entry in position $k+d-1$ (instead of position $k+d$, as was the case for α itself), and each β satisfies $|\beta| = |\alpha|$. By iterating this procedure several times, we obtain the following lemma.

Lemma 3.3. *Let $\alpha = (\alpha_1,\dots,\alpha_{2k}) \in \mathbb{Z}^{2k}_{\geqslant 0}$, and let d be the number of non-zero entries in the second half of α (i.e. among the entries $\alpha_{k+1},\dots,\alpha_{2k}$). Further, given $\lambda = (\lambda_1,\dots,\lambda_k) \in \mathbb{Z}^k_{\geqslant 0}$, define $p_k(\lambda;0) := p_k(\lambda_1,\dots,\lambda_k,0,\dots,0)$. Then the function $p_k(\alpha)$ can be written in the form*

$$p_k(\alpha) = \frac{(-1)^d}{\prod_{j=1}^d (k-d+j)}\sum_{\lambda\in\mathcal{S}_\alpha} p_k(\lambda;0), \tag{92}$$

where $\mathcal{S}_\alpha$ is a certain set of tuples $\lambda \in \mathbb{Z}^k_{\geqslant 0}$, with $|\lambda| = |\alpha|$, of cardinality $|\mathcal{S}_\alpha| = \prod_{j=1}^d (k+d-j)$.

3.2. An example

Given a tuple of the form

$$(\alpha_1,\dots,\alpha_l,0,\dots,0,\alpha_{k+1},\dots,\alpha_{k+d},0,\dots,0) \in \mathbb{Z}^{2k}_{\geqslant 0}, \tag{93}$$

where the α_j's are possibly non-zero, let us write it, for notational convenience, in the form $(\alpha_1,\dots,\alpha_l;\ \alpha_{k+1},\dots,\alpha_{k+d})$. Now suppose we wish to symmetrize $p_k(2,2,1;2,1)$. By independent means, using the determinantal identities in [CFKRS2] for specific values of k and polynomial interpolation, one can compute

$$p_k(2,2,1;2,1) = 6(k+2)\big(k^2-10\big)(k+1)^2 p_k(0). \tag{94}$$

On the other hand, the first iteration of the symmetrization algorithm applied to $p_k(2,2,1;2,1)$ produces

$$p_k(2,2,1;2,1) = \frac{1}{k-1}\Bigg[-p_k(3,2,1;2) - p_k(2,3,1;2) - p_k(2,2,2;2) - \sum_{r=1}^{k-3} p_k\big(2,2,1,\overbrace{0,\dots,0}^{r-1 \text{ zeros}},1;2\big) - p_k(2,2,1;3)\Bigg]. \tag{95}$$

Therefore, by routine symmetry considerations,

$$p_k(2,2,1;2,1) = \frac{1}{1-k}\big[2p_k(3,2,1;2) + p_k(2,2,2;2) + (k-3)p_k(2,2,1,1;2) + p_k(2,2,1;3)\big]. \tag{96}$$

We verify the two sides of the above equality are equal. By independent means,

$$p_k(3,2,1;2) = 2(k+2)(k+1)\big(k^4 - 58k^2 + 417\big)p_k(0), \tag{97}$$

$$p_k(2,2,2;2) = -72(k+2)(k+1)\big(k^2 - 11\big)p_k(0), \tag{98}$$

$$p_k(2,2,1;3) = 6(k-3)(k-4)(k+4)(k+3)(k+2)(k+1)p_k(0), \tag{99}$$

$$p_k(2,2,1,1;2) = -8(k+3)(k+2)(k+1)\big(2k^2 - 47\big)p_k(0). \tag{100}$$

Using some algebraic manipulations, we thus obtain

$$2p_k(3,2,1;2) + p_k(2,2,2;2) + p_k(2,2,1;3) + (k-3)p_k(2,2,1,1;2) = -6(k-1)(k+2)\big(k^2-10\big)(k+1)^2 p_k(0). \tag{101}$$

Upon dividing the above by $1-k$, we arrive at $p_k(2,2,1;2,1)$, as claimed.

3.3. The second step: from $p_k(\lambda;0)$ to $N_k^0(\lambda)$

According to Lemma 3.3, the function $p_k(\alpha)$, where $\alpha \in \mathbb{Z}^{2k}_{\geqslant 0}$, can be written in terms of functions of the form

$$p_k(\lambda;0) := \frac{(-1)^k}{k!^2}\frac{1}{(2\pi i)^{2k}}\oint\cdots\oint \frac{\Delta^2(z_1,\dots,z_{2k})e^{\frac{1}{2}\sum_{i=1}^k z_i - z_{k+i}}}{\prod_{i,j=1}^k (z_i - z_{k+j})\prod_{i=1}^{2k} z_i^{2k}} z_1^{\lambda_1}\cdots z_k^{\lambda_k}\,dz_1\cdots dz_{2k}, \tag{102}$$

where $\lambda = (\lambda_1,\dots,\lambda_k) \in \mathbb{Z}^k_{\geqslant 0}$, and $p_k(\lambda;0) = p_k(\lambda_1,\dots,\lambda_k,0,\dots,0)$. We now show that the variables $z_{k+1},\dots,z_{2k}$, can be completely eliminated from the above expression for $p_k(\lambda;0)$. That is, the integral (102) can be made to involve variables in the first half only (so the "cross-terms" are eliminated).

Lemma 3.4. *Let $\lambda = (\lambda_1,\dots,\lambda_k) \in \mathbb{Z}^k_{\geqslant 0}$, $k \geqslant 2$, and define*

$$N_k^0(\lambda) := \frac{(-1)^{\binom{k}{2}}}{k!}\frac{1}{(2\pi i)^k}\oint\cdots\oint \frac{\Delta^2(z_1,\dots,z_k)e^{\sum_{i=1}^k z_i}}{\prod_{i=1}^k z_i^{2k}} z_1^{\lambda_1}\cdots z_k^{\lambda_k}\,dz_1\cdots dz_k. \tag{103}$$

Then $p_k(\lambda;0) = N_k^0(\lambda)$.

Proof. Applying Lemma 3.2 to (102) with $f(z_1,\dots,z_{2k}) = \exp(\frac{1}{2}\sum_1^{2k} z_i)$, so that $f(0,\dots,0)=1$,

$$p_k(\lambda;0) = \frac{(-1)^k}{k!^2}\frac{1}{(2\pi i)^{2k}}\oint\cdots\oint \frac{\Delta^2(z_1,\dots,z_{2k})e^{\sum_{i=1}^k z_i}}{\prod_{i,j=1}^k (z_i - z_{k+j})\prod_{i=1}^{2k} z_i^{2k}} z_1^{\lambda_1}\cdots z_k^{\lambda_k}\,dz_1\cdots dz_{2k}. \tag{104}$$

Also,

$$\Delta^2(z_1,\dots,z_{2k}) = \Delta^2(z_1,\dots,z_k)\Delta^2(z_{k+1},\dots,z_{2k})\prod_{i,j=1}^{k}(z_i - z_{k+j})^2. \tag{105}$$

Therefore,

$$\begin{aligned} p_k(\lambda;0) = \frac{(-1)^k}{k!^2}\frac{1}{(2\pi i)^{2k}}&\oint\cdots\oint \frac{\Delta^2(z_1,\dots,z_k)e^{\sum_{i=1}^k z_i}}{\prod_{i=1}^k z_i^{2k}} z_1^{\lambda_1}\cdots z_k^{\lambda_k} \\ &\times\oint\cdots\oint \frac{\Delta^2(z_{k+1},\dots,z_{2k})\prod_{i,j=1}^k (z_i - z_{k+j})}{\prod_{i=1}^k z_{k+i}^{2k}}\,dz_{k+1}\cdots dz_{2k}\,dz_1\cdots dz_k. \end{aligned} \tag{106}$$

The polynomial $\Delta^2(z_{k+1},\dots,z_{2k})$ is homogeneous of degree $2\binom{k}{2} = k^2 - k$. Also, the polynomial $\prod_{i,j=1}^k (z_i - z_{k+j})$ is homogeneous of degree k^2. Note that the coefficient of $z_{k+1}^{k-1}\cdots z_{2k}^{k-1}$ in $\Delta^2(z_{k+1},\dots,z_{2k})$ is $(-1)^{\binom{k}{2}}k!$, and the coefficient of $z_{k+1}^k\cdots z_{2k}^k$ in $\prod_{i,j=1}^k (z_i - z_{k+j})$ is $(-1)^{k^2} = (-1)^k$. So, computing the residue at $z_{k+1} = \cdots = z_{2k} = 0$ gives

$$\frac{(-1)^k}{(2\pi i)^k}\oint\cdots\oint \frac{\Delta^2(z_{k+1},\dots,z_{2k})\prod_{i,j=1}^k (z_i - z_{k+j})}{\prod_{i=1}^k z_{k+i}^{2k}}\,dz_{k+1}\cdots dz_{2k} = (-1)^{\binom{k}{2}}k!. \tag{107}$$

The lemma follows. □

4. An algorithm to compute $N_k^0(\lambda)$

Given a multivariate formal power series $Q(z_1,\dots,z_k)$, define

$$[\lambda_1,\dots,\lambda_k]_Q := \text{Coefficient of } \prod_{j=1}^k z_j^{2k-\lambda_j-1} \text{ in } Q(z_1,\dots,z_k). \tag{108}$$

Let

$$F(z_1,\dots,z_k) := \Delta^2(z_1,\dots,z_k)e^{\sum_{i=1}^k z_i}. \tag{109}$$

Then,

$$\frac{1}{(2\pi i)^k}\oint\cdots\oint \frac{F(z_1,\dots,z_k)}{\prod_{i=1}^k z_i^{2k}} z_1^{\lambda_1}\cdots z_k^{\lambda_k}\,dz_1\cdots dz_k = [\lambda_1,\dots,\lambda_k]_F. \tag{110}$$

Also, by Lemma 3.4, and the definition (110),

$$p_k(\lambda_1,\dots,\lambda_k,0,\dots,0) = N_k^0(\lambda) = \frac{(-1)^{\binom{k}{2}}}{k!}[\lambda_1,\dots,\lambda_k]_F. \tag{111}$$

The purpose of this section is to derive an algorithm to compute the coefficients $[\lambda_1,\dots,\lambda_k]_F$. As an easy by-product of the algorithm, sharp enough upper bounds on the magnitude of these coefficients are obtained. The algorithm comes in the form of a recursion that dissipates the entries of a given tuple λ, while also decreasing its weight.

Notice since F is symmetric with respect to the all of the z_j's, then $[\lambda_1,\dots,\lambda_k]_F$ and $N_k^0(\lambda)$ are symmetric with respect to all of the λ_j's.

To help get used to the notation, note for instance, for $k \geqslant 2$,

$$\begin{aligned}\frac{(-1)^{\binom{k}{2}}}{k!}[0,\dots,0]_F &= \frac{(-1)^{\binom{k}{2}}}{k!} \times \text{Coefficient of } z_1^{2k-1}\cdots z_k^{2k-1} \text{ in } F(z_1,\dots,z_k)\\ &= N_k^0(0) = p_k(0) = \frac{g_k}{k^2!}.\end{aligned} \tag{112}$$

The last step is Eq. (46).

We will need several lemmas, and we will make use of the function

$$G_j(z_1,\dots,z_k) := \frac{F(z_1,\dots,z_k)}{z_1 - z_j}. \tag{113}$$

Notice $z_1 - z_j$ divides the Vandermonde determinant in F, so $G_j(z_1,\dots,z_k)$ is a polynomial. In the lemmas to follow, we consider tuples $(\lambda_1,\dots,\lambda_k) \in \mathbb{Z}^k_{\geqslant 0}$. Although the restriction $\lambda_j \geqslant 0$ is what is relevant to our problem, it is often not necessary.

Lemma 4.1. *Let $(\lambda_1,\dots,\lambda_k) \in \mathbb{Z}^k_{\geqslant 0}$. Then,*

$$[\lambda_1,\lambda_2,\dots,\lambda_k]_F = (2k-\lambda_1)[\lambda_1-1,\lambda_2,\dots,\lambda_k]_F - 2\sum_{j=2}^{k}[\lambda_1,\lambda_2,\dots,\lambda_k]_{G_j}. \tag{114}$$

Proof. By logarithmic differentiation, we have

$$\frac{\frac{\partial}{\partial z_1}F(z_1,\dots,z_k)}{F(z_1,\dots,z_k)} = 1 + 2\sum_{j=2}^{k}\frac{1}{z_1 - z_j}. \tag{115}$$

So

$$\begin{aligned}\frac{\partial}{\partial z_1}F(z_1,\dots,z_k) &= F(z_1,\dots,z_k) + 2\sum_{j=2}^{k}\frac{F(z_1,\dots,z_k)}{z_1 - z_j}\\ &= F(z_1,\dots,z_k) + 2\sum_{j=2}^{k} G_j(z_1,\dots,z_k).\end{aligned} \tag{116}$$

Equating the coefficient of $\prod_{j=1}^{k} z_j^{2k-\lambda_j-1}$ on both sides above, we have

$$[\lambda_1,\dots,\lambda_k]_{\frac{\partial}{\partial z_1}F} = [\lambda_1,\dots,\lambda_k]_F + 2\sum_{j=2}^{k}[\lambda_1,\dots,\lambda_k]_{G_j}. \tag{117}$$

By differentiating the power series of F with respect to z_1, the lhs also equals

$$[\lambda_1,\dots,\lambda_k]_{\frac{\partial}{\partial z_1}F} = (2k-\lambda_1)[\lambda_1-1,\lambda_2,\dots,\lambda_k]_F. \tag{118}$$

By substituting (118) into (117), the lemma follows. □

It is actually more convenient to rewrite the recursion (114) in the form

$$[\lambda_1+1,\lambda_2,\dots,\lambda_k]_F = (2k-\lambda_1-1)[\lambda_1,\lambda_2,\dots,\lambda_k]_F - 2\sum_{j=2}^{k}[\lambda_1+1,\lambda_2,\dots,\lambda_k]_{G_j}. \tag{119}$$

Also, for better readability, let us drop entries λ_j unaltered from their "original values" in a *reference tuple* $\lambda=(\lambda_1,\dots,\lambda_k)$, except for the first entry λ_1, which will always be displayed. For example, if $\lambda=(\lambda_1,\dots,\lambda_k)$ is the reference tuple, then the expressions

$$[\lambda_1,\lambda_j+1] \quad\text{and}\quad [\lambda_1+3,\lambda_k+9], \tag{120}$$

will now stand for

$$[\lambda_1,\dots,\lambda_{j-1},\lambda_j+1,\lambda_{j+1},\dots,\lambda_k] \quad\text{and}\quad [\lambda_1+3,\lambda_2,\dots,\lambda_{k-1},\lambda_k+9]. \tag{121}$$

So now the recursion (119) can be expressed more simply as

$$[\lambda_1+1]_F = (2k-\lambda_1-1)[\lambda_1]_F - 2\sum_{j=2}^{k}[\lambda_1+1]_{G_j}. \tag{122}$$

Lemma 4.2. *Let $(\lambda_1,\dots,\lambda_k)\in\mathbb{Z}^k_{\geqslant 0}$ be the reference tuple. Then*

$$[\lambda_1+1]_{G_j} = [\lambda_1]_F + [\lambda_1,\lambda_j+1]_{G_j}. \tag{123}$$

In particular, for any integer $\Delta\geqslant -1$, and $2\leqslant j\leqslant k$, we have

$$[\lambda_1+1]_{G_j} = \sum_{l=0}^{\Delta}[\lambda_1-l,\lambda_j+l]_F + [\lambda_1-\Delta,\lambda_j+\Delta+1]_{G_j}. \tag{124}$$

Proof. The relation (123) is symmetric in the z_j's, $j\geqslant 2$. So we may as well take $j=2$. Write

$$\begin{aligned} G_2(z_1,\dots,z_k) = {} & c_1 z_1^{2k-\lambda_1-2} z_2^{2k-\lambda_2-1} z_3^{2k-\lambda_3-1}\cdots z_k^{2k-\lambda_k-1} \\ & + c_2 z_1^{2k-\lambda_1-1} z_2^{2k-\lambda_2-2} z_3^{2k-\lambda_3-1}\cdots z_k^{2k-\lambda_k-1} + \cdots. \end{aligned} \tag{125}$$

Thus, $c_1=[\lambda_1+1]_{G_2}$, and $c_2=[\lambda_1,\lambda_2+1]_{G_2}$. Notice

$$(z_1 - z_2)G_2(z_1, \dots, z_k) = (c_1 - c_2)z_1^{2k-\lambda_1-1}z_2^{2k-\lambda_2-1}\cdots z_k^{2k-\lambda_k-1} + \cdots. \tag{126}$$

Since, by definition, $F(z_1, \dots, z_k) = (z_1 - z_2)G_2(z_1, \dots, z_k)$, it follows from (126) that

$$[\lambda_1]_F = c_1 - c_2 = [\lambda_1 + 1]_{G_2} - [\lambda_1, \lambda_2 + 1]_{G_2}. \tag{127}$$

Equivalently, $[\lambda_1 + 1]_{G_2} = [\lambda_1]_F + [\lambda_1, \lambda_2 + 1]_{G_2}$. The last part of the lemma follows by applying the recursion (123) a total of $\Delta + 1$ times. □

Lemma 4.3. *Let $(\lambda_1, \dots, \lambda_k) \in \mathbb{Z}^k_{\geqslant 0}$ be the reference tuple. Assume $\lambda_1 \geqslant \lambda_j$ for $j \leqslant k$, and define*

$$\Delta_j := \left\lfloor \frac{\lambda_1 - \lambda_j}{2} \right\rfloor. \tag{128}$$

Then,

$$[\lambda_1 - \Delta_j, \lambda_j + \Delta_j + 1]_{G_j} = \begin{cases} -\frac{1}{2}[\lambda_1 - \Delta_j, \lambda_j + \Delta_j]_F, & \text{if } \lambda_1 - \lambda_j \text{ is even}, \\ 0, & \text{if } \lambda_1 - \lambda_j \text{ is odd}. \end{cases}$$

Proof. Since $F(z_1, \dots, z_k)$ is symmetric with respect to all of the z_j's, it follows that $G_j(z_1, \dots, z_k) = F(z_1, \dots, z_k)/(z_1 - z_j)$ is anti-symmetric with respect to z_1 and z_j; i.e.:

$$G_j(z_1, \dots, z_j, \dots) = -G_j(z_j, \dots, z_1, \dots). \tag{129}$$

In particular, if we view G_j as a polynomial in z_1 and z_j, and write

$$G_j(z_1, \dots, z_k) = \sum_{m,n \in \mathbb{Z}_{\geqslant 0}} c_{m,n} z_1^m z_j^n, \tag{130}$$

so the coefficients $c_{m,n}$ are now polynomials in $\{z_i\colon\ i \neq 1, j\}$, then by the anti-symmetry of G_j, in (129), we have $c_{m,n} = -c_{n,m}$, and so

$$c_{m,m} = 0, \qquad c_{m+1,m} = -c_{m,m+1}. \tag{131}$$

Next, note

$$(\lambda_1 - \Delta_j) - (\lambda_j + \Delta_j + 1) = \begin{cases} -1, & \text{if } \lambda_1 - \lambda_j \text{ is even}, \\ 0, & \text{if } \lambda_1 - \lambda_j \text{ is odd}. \end{cases}$$

If $\lambda_1 - \lambda_j$ is odd, so $\lambda_1 - \Delta_j = \lambda_j + \Delta_j + 1$, it follows from the first relation in (131), with $m = 2k - (\lambda_1 - \Delta_j) - 1 = 2k - (\lambda_j + \Delta_j + 1) - 1$, that

$$[\lambda_1 - \Delta_j, \lambda_j + \Delta_j + 1]_{G_j} = 0. \tag{132}$$

On the other hand, if $\lambda_1 - \lambda_j$ is even, so $\lambda_1 - \Delta_j = \lambda_j + \Delta_j$, then the identity

$$[\lambda_1 - \Delta_j + 1, \lambda_j + \Delta_j]_{G_j} = [\lambda_1 - \Delta_j, \lambda_j + \Delta_j]_F + [\lambda_1 - \Delta_j, \lambda_j + \Delta_j + 1]_{G_j}, \tag{133}$$

readily deducible from the recursion $[\lambda_1+1]_{G_j} = [\lambda_1]_F + [\lambda_1, \lambda_j+1]_{G_j}$ of Lemma 4.2, together with the second relation in (131) applied with $m+1 = 2k-(\lambda_1-\Delta_j)-1$ and $m = 2k-(\lambda_j+\Delta_j+1)-1$, imply

$$[\lambda_1-\Delta_j, \lambda_j+\Delta_j+1]_{G_j} = -\frac{1}{2}[\lambda_1-\Delta_j, \lambda_j+\Delta_j]_F, \tag{134}$$

as required. □

4.1. An algorithm to compute $N_k^0(\lambda)$

We show how to compute $[\lambda_1, \dots, \lambda_k]_F$ via a recursion. Since by relation (111) we have $N_k^0(\lambda) = \frac{(-1)^{\binom{k}{2}}}{k!}[\lambda_1, \dots, \lambda_k]_F$, then the said recursion can be directly used to compute $N_k^0(\lambda)$ as well. We will employ this recursion in Section 5 to bound $N_k^0(\lambda)$.

Lemma 4.4. *Let $(\lambda_1+1, \lambda_2, \dots, \lambda_k) \in \mathbb{Z}^k_{\geqslant 0}$. Assume $\lambda_1+1 \geqslant \lambda_j$ for $j \leqslant k$. Define*

$$\Delta_j := \left\lfloor \frac{\lambda_1-\lambda_j}{2} \right\rfloor, \qquad \delta_j := \begin{cases} -\frac{1}{2}, & \text{if } \lambda_1-\lambda_j \text{ is even}, \\ 0, & \text{if } \lambda_1-\lambda_j \text{ is odd}. \end{cases} \tag{135}$$

Then, with $\lambda = (\lambda_1, \dots, \lambda_k)$ as the reference tuple, we have

$$[\lambda_1+1]_F = (2k-\lambda_1-1)[\lambda_1]_F - 2\sum_{j=2}^{k}\left[\delta_j[\lambda_1-\Delta_j, \lambda_j+\Delta_j]_F + \sum_{l=0}^{\Delta_j}[\lambda_1-l, \lambda_j+l]_F\right]. \tag{136}$$

In other words, the coefficient corresponding to the tuple $(\lambda_1+1, \lambda_2, \dots, \lambda_k)$, which has weight $|\lambda|+1$, can be expressed as a linear combination involving tuples of weight $|\lambda|$ only.

Remark. If $\lambda_1 = \lambda_j - 1$, so $\Delta_j = -1$, then the sum over k in (136) vanishes, since $\delta_j = 0$ in that case.

Proof of Lemma 4.4. By Lemma 4.1,

$$[\lambda_1+1]_F = (2k-\lambda_1-1)[\lambda_1]_F - 2\sum_{j=2}^{k}[\lambda_1+1]_{G_j}. \tag{137}$$

And by Lemma 4.2, applied with $\Delta = \Delta_j$, we have

$$[\lambda_1+1]_{G_j} = \sum_{l=0}^{\Delta_j}[\lambda_1-l, \lambda_j+l]_F + [\lambda_1-\Delta_j, \lambda_j+\Delta_j+1]_{G_j}. \tag{138}$$

Therefore,

$$[\lambda_1+1]_F = (2k-\lambda_1-1)[\lambda_1]_F - 2\sum_{j=2}^{k}\left[\sum_{l=0}^{\Delta_j}[\lambda_1-l, \lambda_j+l]_F + [\lambda_1-\Delta_j, \lambda_j+\Delta_j+1]_{G_j}\right]. \tag{139}$$

The result now follows from Lemma 4.3. □

4.2. Examples

Say we wish to compute $N_k^0(4,2,1,0,\dots,0)$. For notational convenience, given a tuple $(\lambda_1,\dots,\lambda_l, 0,\dots,0)\in\mathbb{Z}^k_{\geqslant 0}$, let us define

$$N_k^0(\lambda_1,\dots,\lambda_l,0,\dots,0) =: N_k^0(\lambda_1,\dots,\lambda_l). \tag{140}$$

Using this notation, the function to be computed is $N_k^0(4,2,1)$. Lemma 4.4 and (111) provides, on collecting terms,

$$\begin{aligned} N_k^0(4,2,1) &= (2k-4)N_k^0(3,2,1) - 2(k-1)N_k^0(3,2,1) - N_k^0(2,2,2) \\ &\quad - 2(k-3)N_k^0(2,2,1,1) \\ &= -2N_k^0(3,2,1) - N_k^0(2,2,2) - 2(k-3)N_k^0(2,2,1,1). \end{aligned} \tag{141}$$

Note the lhs involves a tuple of weight 7, whereas the rhs involves tuples of weight 6 only, as should be. By independent means, using determinantal identities in [CFKRS2] for specific values of k and polynomial interpolation, we computed

$$N_k^0(3,2,1) = -3k(k-3)(k+3)(k+2)(k+1)N_k^0(0), \tag{142}$$

$$N_k^0(2,2,2) = 24k(k+2)(k+1)N_k^0(0), \tag{143}$$

$$N_k^0(2,2,1,1) = 12k(k+3)(k+2)(k+1)N_k^0(0), \tag{144}$$

$$N_k^0(4,2,1) = -6k(k+2)(k+1)\big(3k^2-23\big)N_k^0(0). \tag{145}$$

Let us check that Lemma 4.4 does in fact yield the correct $N_k^0(4,2,1)$. The rhs is

$$\begin{aligned} &\big[6k(k-3)(k+3)(k+2)(k+1) - 24k(k+2)(k+1) \\ &\quad - 24k(k-3)(k+3)(k+2)(k+1)\big]N_k^0(0). \end{aligned} \tag{146}$$

The above can be simplified to

$$\begin{aligned} &6k(k+2)(k+1)\big[(k-3)(k+3) - 4 - 4(k-3)(k+3)\big] \\ &= 6k(k+2)(k+1)\big(-3k^2+23\big), \end{aligned} \tag{147}$$

which agrees with (145).

As another example, let

$$1_n := \big(\overbrace{1,\dots,1}^{n \text{ entries}},0,\dots,0\big). \tag{148}$$

Then one computes, by directly using (136) and the symmetry of $N_k^0(1_n)$ with respect to the λ_j's with $j>n$,

$$N_k^0(1_n) = (2k-1)N_k^0(1_{n-1}) - \sum_{j=n+1}^{k} N_k^0(1_{n-1})$$
$$= (k+n-1)N_k^0(1_{n-1}). \tag{149}$$

From which it follows

$$N_k^0(1_n) = N_k^0(0)\prod_{j=0}^{n-1}(k+j). \tag{150}$$

One can obtain similar simple expressions for other special choices of λ.

5. Applications of the algorithms

As a consequence of the recursions in Section 3 and Section 4.1, we show that $p_k(\alpha)/p_k(0)$ grows at most polynomially in k, and at most exponentially in $|\alpha|$, for $|\alpha| < k/2$. We need the following lemma.

Lemma 5.1. *Let $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{Z}^k_{\geqslant 0}$, such that $|\lambda| < k$. Then,*

$$\frac{N_k^0(\lambda)}{N_k^0(0)} \leqslant \frac{16^{|\lambda|}(\log(|\lambda|+10))^{|\lambda|}k^{|\lambda|}}{\lambda_1\lambda_2\cdots\lambda_{m(\lambda)}}. \tag{151}$$

Proof. Consider a tuple $(\lambda_1+1, \lambda_2, \dots, \lambda_k)$, which has weight $|\lambda|+1$. By the symmetry of $N_k^0(\lambda)$ with respect to all of the λ_j's (see the remark at the beginning of Section 4.1), we may assume $\lambda_1 + 1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k$. Without loss of generality, we may make a similar assumption on the ordering of all the tuples that occur in the present proof.

Maintaining the convention whereby entries unchanged from their values in the reference tuple $\lambda = (\lambda_1, \dots, \lambda_k)$ are dropped, we have by Lemma 4.4, after some simple manipulations, that

$$\left|N_k^0(\lambda_1+1)\right| \leqslant (2k-1)\left|N_k^0(\lambda_1)\right| + 2\sum_{j=2}^{k}\sum_{l=0}^{\Delta_j}\left|N_k^0(\lambda_1 - l, \lambda_j + l)\right|, \tag{152}$$

where $\Delta_j = \lfloor(\lambda_1 - \lambda_j)/2\rfloor$. Note the term $\delta_j[\lambda_1 - \Delta_j, \lambda_j + \Delta_j]_F$ that appears in the lemma is dropped because in the event $\delta_j = -1/2$ it simply reduces the $l = \Delta_j$ term of the inner sum in the lemma by a factor of $1/2$, which is smaller than the stated bound.

The rhs in (152) involves tuples of weight $|\lambda|$ only, while the lhs involves a tuple of weight $|\lambda|+1$. This suggests inducting on $|\lambda|$. So assume we have verified the following induction hypothesis for all tuples λ' of weight $\leqslant |\lambda|$:

$$\frac{|N_k^0(\lambda')|}{N_k^0(0)} \leqslant \frac{16^{|\lambda'|}(\log(|\lambda'|+10))^{|\lambda'|}k^{|\lambda'|}}{\lambda'_1\lambda'_2\cdots\lambda'_{m(\lambda')}}. \tag{153}$$

We now wish to show it holds for $N_k^0(\lambda_1+1)$; that is, we wish to show it for tuples of weight $|\lambda|+1$.

By identity (150), and the assumption $|\lambda| < k$, the induction hypothesis holds for all k-tuples $\lambda' = (1, \dots, 1, 0, \dots, 0)$. So we may take tuples of this form as the base cases for the induction. Also, notice if $\lambda_1 = 0$, then given our assumption $\lambda_1 + 1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k$, the tuple $(\lambda_1+1, \lambda_2, \dots, \lambda_k)$ must be of

the form $(1,\dots,1,0,\dots,0)$, and this falls within the base cases of the induction. Therefore, we may assume $\lambda_1 > 0$, so that $m(\lambda_1+1,\lambda_2,\dots,\lambda_k) = m(\lambda_1,\dots,\lambda_k)$. What we wish to show then is

$$\frac{|N_k^0(\lambda_1+1)|}{N_k^0(0)} \leqslant \frac{16^{|\lambda|+1}(\log(|\lambda|+4))^{|\lambda|+1}k^{|\lambda|+1}}{(\lambda_1+1)\lambda_2\cdots\lambda_{m(\lambda)}}. \tag{154}$$

Consider the first term on the rhs of (152), as well as the terms with $l=0$ in the inner sum there. By the induction hypothesis,

$$\frac{|(2k-1)N_k^0(\lambda_1)|}{N_k^0(0)} + 2\sum_{j=2}^{k}\frac{|N_k^0(\lambda_1,\lambda_j)|}{N_k^0(0)} \leqslant 4\frac{16^{|\lambda|}(\log(|\lambda|+10))^{|\lambda|+1}k^{|\lambda|+1}}{(\lambda_1+1)\lambda_2\cdots\lambda_{m(\lambda)}}, \tag{155}$$

where we used that the above sum involves $\leqslant 4k$ tuples of weight $|\lambda|$, and $(\lambda_1+1)/\lambda_1 \leqslant 2 \leqslant \log(|\lambda|+10)$, which is valid since $\lambda_1 > 0$. Also by the induction hypothesis,

$$2\sum_{j=2}^{m(\lambda)}\sum_{l=1}^{\Delta_j}\frac{|N_k^0(\lambda_1-l,\lambda_j+l)|}{N_k^0(0)} \leqslant \frac{16^{|\lambda|}(\log(|\lambda|+10))^{|\lambda|}k^{|\lambda|}}{(\lambda_1+1)\lambda_2\cdots\lambda_{m(\lambda)}} 2\sum_{j=2}^{m(\lambda)}\sum_{l=1}^{\Delta_j}\frac{(\lambda_1+1)\lambda_j}{(\lambda_1-l)(\lambda_j+l)}. \tag{156}$$

Therefore, since $\lambda_1 - l \geqslant (\lambda_1+1)/2$ for $1 \leqslant l \leqslant \Delta_j$ and $j \leqslant m(\lambda)$, we have

$$2\sum_{j=2}^{m(\lambda)}\sum_{l=1}^{\Delta_j}\frac{(\lambda_1+1)\lambda_j}{(\lambda_1-l)(\lambda_j+l)} \leqslant 4\sum_{j=2}^{m(\lambda)}\sum_{l=1}^{\Delta_j}\frac{\lambda_j}{\lambda_j+l} \leqslant 4|\lambda|\log\big(|\lambda|+10\big), \tag{157}$$

where we used $\sum_{l=1}^{\Delta_j} 1/(\lambda_j+l) \leqslant \log(|\lambda|+10)$, and $\sum_{j=2}^{m(\lambda)}\lambda_j \leqslant |\lambda|$. Combined with $|\lambda| < k$, we obtain

$$2\sum_{j=2}^{m(\lambda)}\sum_{l=1}^{\Delta_j}\frac{|N_k^0(\lambda_1-l,\lambda_j+l)|}{N_k^0(0)} \leqslant 4\frac{16^{|\lambda|}(\log(|\lambda|+10))^{|\lambda|+1}k^{|\lambda|+1}}{(\lambda_1+1)\lambda_2\cdots\lambda_{m(\lambda)}}. \tag{158}$$

Last, since by definition $\lambda_j = 0$ for $j > m(\lambda)$, and since $N(\lambda_1-l,\lambda_j+l)$ is symmetric with respect to the λ_j's, we have

$$\begin{aligned}
2\sum_{j=m(\lambda)+1}^{k}\sum_{l=1}^{\Delta_j}\frac{|N_k^0(\lambda_1-l,\lambda_j+l)|}{N_k^0(0)} &= 2\big(k-m(\lambda)\big)\sum_{1\leqslant l\leqslant\lambda_1/2}\frac{|N_k^0(\lambda_1-l,l)|}{N_k^0(0)} \\
&\leqslant 2\frac{16^{|\lambda|}(\log(|\lambda|+10))^{|\lambda|}k^{|\lambda|+1}}{(\lambda_1+1)\lambda_2\cdots\lambda_{m(\lambda)}}\sum_{1\leqslant l\leqslant\lambda_1/2}\frac{\lambda_1+1}{(\lambda_1-l)l} \\
&\leqslant 8\frac{16^{|\lambda|}(\log(|\lambda|+10))^{|\lambda|+1}k^{|\lambda|+1}}{(\lambda_1+1)\lambda_2\cdots\lambda_{m(\lambda)}},
\end{aligned} \tag{159}$$

where we used $(\lambda_1+1)/(\lambda_1-l) \leqslant 4$ for $l \leqslant \lambda_1/2$, and $\sum_{1\leqslant l\leqslant\lambda_1/2} 1/l \leqslant \log(|\lambda|+10)$. Assembling the bounds (155), (158), and (159), the claim follows. □

Theorem 5.2. *Let $\alpha = (\alpha_1, \dots, \alpha_{2k}) \in \mathbb{Z}^{2k}_{\geqslant 0}$. Then, there exists an absolute constant η such that as $k \to \infty$, and uniformly in $|\alpha| < k/2$,*

$$\frac{p_k(\alpha)}{p_k(0)} \ll \eta^{|\alpha|} \big(k \log(|\alpha| + 10)\big)^{|\alpha|}. \tag{160}$$

Note, from the residue (41) *defining $p_k(\alpha)$, if $\alpha_j \geqslant 2k$ for any $1 \leqslant j \leqslant k$, then $p_k(\alpha) = 0$.*

Proof. By Lemma 3.3,

$$\big|p_k(\alpha)\big| \leqslant \frac{1}{\prod_{j=1}^{d}(k-d+j)} \sum_{\lambda \in \mathcal{S}_\alpha} \big|N_k^0(\lambda)\big|, \tag{161}$$

where d is the number of non-zero entries in the second half of α (i.e. among $\alpha_{k+1}, \dots, \alpha_k$), and S_α is a set of tuples $\lambda \in \mathbb{Z}^k_{\geqslant 0}$ satisfying $|\lambda| = |\alpha|$, of size $|\mathcal{S}_\alpha| = \prod_{j=1}^{d}(k+d-j)$. Since $|\lambda| = |\alpha| < k/2$, we can apply Lemma 5.1 to the $N_k^0(\lambda)$'s, which yields

$$\begin{aligned} \big|p_k(\alpha)\big| &\leqslant \frac{|S_\alpha|}{\prod_{j=1}^{d}(k-d+j)} 16^{|\alpha|} \big(k \log(k+10)\big)^{|\alpha|} N_k^0(0) \\ &\ll (48)^{|\alpha|} \big(k \log(|\alpha|+10)\big)^{|\alpha|} p_k(0), \end{aligned} \tag{162}$$

where we used $N_k^0(0) = p_k(0)$ and the estimate

$$\frac{\prod_{j=1}^{d}(k+d-j)}{\prod_{j=1}^{d}(k-d+j)} = \prod_{j=0}^{d-1} \frac{1+j/k}{1-j/k} \leqslant 3^{|\alpha|}, \tag{163}$$

which holds since $d \leqslant |\alpha| < k/2$ and so $(1+j/k)/(1-j/k) \leqslant 3$ for $j < d$. □

Another, more precise, consequence is that $p_k(\lambda; 0)/p_k(0)$ is a polynomial in k of degree at most $|\lambda|$. This is not specifically used in the proof of the main theorem in this paper, but it is an important fact that the ideas developed so far can prove fairly straightforwardly.

Theorem 5.3. *Fix a positive integer m. Fix $\lambda = (\lambda_1, \dots, \lambda_m, 0, \dots, 0) \in \mathbb{Z}^k_{\geqslant 0}$. Then, $p_k(\lambda; 0)/p_k(0)$ is a polynomial in k of degree $\leqslant |\lambda|$.*

Proof. We induct on $|\lambda|$. The base case is trivial. Assume that we have verified the theorem for all tuples of weight $\leqslant |\lambda|$ and consider the case of $|\lambda| + 1$. By symmetry, we may assume that

$$\lambda_1 + 1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m. \tag{164}$$

And by the recursion in Lemma 4.4, applied with $(\lambda_1, \dots, \lambda_m, 0, \dots, 0)$ as the reference tuple, we have

$$\begin{aligned} p_k(\lambda_1 + 1) = {} & (2k - \lambda_1 - 1) p_k(\lambda_1) \\ & - 2 \sum_{j=2}^{k} \left[\delta_j p_k(\lambda_1 - \Delta_j, \lambda_j + \Delta_j) + \sum_{l=0}^{\Delta_j} p_k(\lambda_1 - l, \lambda_j + l) \right]. \end{aligned} \tag{165}$$

First, observe, by the induction hypothesis, $p_k(\lambda_1)/p_k(0)$ is a polynomial in k of degree at most $|\lambda|$. Therefore, $(2k-\lambda_1-1)p_k(\lambda_1)/p_k(0)$ is a polynomial in k of degree at most $|\lambda|+1$.

Second, since $\lambda_{m(\alpha)+1}=\cdots=\lambda_k=0$, we can collect the terms $j=m(\alpha)+1,\dots,k$ together in the above sum over j, and using $\Delta_{m(\alpha)+1}=\cdots=\Delta_k$, we obtain

$$\sum_{j=2}^{k}\sum_{l=0}^{\Delta_j} p_k(\lambda_1-l,\lambda_j+l)=\sum_{j=2}^{m(\alpha)}\sum_{l=0}^{\Delta_j} p_k(\lambda_1-l,\lambda_j+l)+\sum_{j=m(\alpha)+1}^{k}\sum_{l=0}^{\Delta_j} p_k(\lambda_1-l,\lambda_j+l)$$
$$=\sum_{j=2}^{m(\alpha)}\sum_{l=0}^{\Delta_j} p_k(\lambda_1-l,\lambda_j+l)+\big(k-m(\alpha)\big)\sum_{l=0}^{\Delta_{m(\alpha)+1}} p_k(\lambda_1-l,l). \quad (166)$$

Again, by the induction hypothesis, $p_k(\lambda_1-l,\lambda_j+l)/p_k(0)$ is a polynomial in k of degree at most $|\lambda|$, for all $2\leqslant j\leqslant m(\alpha)$. Also, $m(\alpha)$ and Δ_j are independent of k. Hence, the rhs above, divided by $p_k(0)$, is a polynomial in k of degree at most $|\lambda|+1$.

Last, since δ_j is also independent of k, and since

$$\sum_{j=2}^{k}\delta_j p_k(\lambda_1-\Delta_j,\lambda_j+\Delta_j)=\sum_{j=2}^{m(\alpha)}\delta_j p_k(\lambda_1-\Delta_j,\lambda_j+\Delta_j)$$
$$+\big(k-m(\alpha)\big)\delta_{m(\alpha)+1}p_k(\lambda_1-\Delta_{m(\alpha)+1},\Delta_{m(\alpha)+1}), \quad (167)$$

it follows by another application of the induction hypothesis that the rhs above is a polynomial in k of degree at most $|\lambda|$, completing the proof. □

6. The arithmetic factor

The function $A(z_1,\dots,z_{2k})$ is analytic and does not vanish in a neighborhood of the origin (where it is equal to a_k). So, one may consider the Taylor expansion,

$$\log A(z_1,\dots,z_{2k})=:\log a_k+B_k\sum_{i=1}^{k}z_i-z_{k+i}+\sum_{\substack{\alpha\in\mathbb{Z}^{2k}_{\geqslant 0}\\ |\alpha|>1}} a_\alpha z_1^{\alpha_1}\cdots z_{2k}^{\alpha_{2k}}. \quad (168)$$

The goal of this section is to produce upper bounds on the coefficients a_α (in fact, we give an asymptotic when $m(\alpha)=1$).

Before doing so, let us introduce some notation. Let $\lambda:=(\lambda_1,\dots,\lambda_k)$ and $\rho:=(\rho_1,\dots,\rho_k)$ denote tuples in $\mathbb{Z}^k_{\geqslant 0}$. Further, for primes p, define

$$S_{n,p}:=\sum_{|\lambda|=|\rho|=n} p^{\sum_{i=1}^{k}\rho_i z_{k+i}-\lambda_i z_i}, \qquad A_p:=\prod_{i,j=1}^{k}\left(1-\frac{p^{z_{k+j}-z_i}}{p}\right)\sum_{n=0}^{\infty}\frac{S_{n,p}}{p^n}, \quad (169)$$

where dependencies of $S_{n,p}$ and A_p on $(z_1,\dots,z_{2k})$ are suppressed to avoid notational clutter.

With the above notation, the arithmetic factor can be expressed as

$$A(z_1,\dots,z_{2k}):=\prod_p A_p. \quad (170)$$

For any absolute constant $c > 1$ say, one may write

$$\log A(z_1, \dots, z_{2k}) = \overbrace{\sum_{p \leqslant ck^2} \log A_p}^{\text{“Small primes”}} + \overbrace{\sum_{p > ck^2} \log A_p}^{\text{“Large primes”}}. \tag{171}$$

We will bound the contributions of “the small primes” and “the large primes” to a coefficient a_α, separately. To this end, split “the small primes” sum into

$$\overbrace{\sum_{p \leqslant ck^2} \sum_{i,j=1}^{k} \log\left(1 - \frac{p^{z_{k+j}-z_i}}{p}\right)}^{\text{Convergence factor sum}} + \overbrace{\sum_{p \leqslant ck^2} \log\left(1 + \sum_{n=1}^{\infty} \frac{S_{n,p}}{p^n}\right)}^{\text{Combinatorial sum}}. \tag{172}$$

(Here, we used the fact $S_{0,p} = 1$.) Similarly, split “the large primes” sum into

$$\overbrace{\sum_{p > ck^2} \left[\frac{S_{1,p}}{p} + \sum_{i,j=1}^{k} \log\left(1 - \frac{p^{z_{k+j}-z_i}}{p}\right)\right]}^{\text{Convergence factor sum}} + \overbrace{\sum_{p > ck^2} \left[\log\left(1 + \sum_{n=1}^{\infty} \frac{S_{n,p}}{p^n}\right) - \frac{S_{1,p}}{p}\right]}^{\text{Combinatorial sum}}. \tag{173}$$

So, the sum (over primes) has been separated into four pieces. In the next few subsections, the contribution to a_α of each of piece is bounded, or, in some cases, an asymptotic is provided. In the last subsection, the various bounds are collected, then presented as a theorem.

Before we proceed, let us make two remarks. First, the symmetry

$$\log A(z_1, \dots, z_{2k}) = \log A(-z_{k+1}, \dots, -z_{2k}, -z_1, \dots, -z_k) \tag{174}$$

implies

$$a_{(\alpha_1,\dots,\alpha_k,\alpha_{k+1},\dots,\alpha_{2k})} = (-1)^{|\alpha|} a_{(\alpha_{k+1},\dots,\alpha_{2k},\alpha_1,\dots,\alpha_k)}. \tag{175}$$

Second, the symmetry

$$\log A(z_1, \dots, z_{2k}) = \log A(z_{\sigma(1)}, \dots, z_{\sigma(k)}, z_{k+\tau(1)}, \dots, z_{k+\tau(k)}), \tag{176}$$

where σ and τ are any members of the permutation group of $\{1, \dots, k\}$, implies

$$a_{(\alpha_1,\dots,\alpha_{2k})} = a_{(\alpha_{\sigma(1)},\dots,\alpha_{\sigma(k)},\alpha_{k+\tau(1)},\dots,\alpha_{k+\tau(k)})}. \tag{177}$$

In particular, to understand the Taylor coefficients of $\log A(z_1, \dots, z_{2k})$, it is enough to understand a_α for tuples α of the form

$$\alpha = (\alpha_1, \dots, \alpha_l, 0, \dots, 0, \alpha_{k+1}, \dots, \alpha_{k+d}, 0, \dots, 0), \quad 0 \leqslant d \leqslant l \leqslant k,\ \alpha_i > 0. \tag{178}$$

We will use the convention where if $d = 0$, then $\alpha_{k+1} = \cdots = \alpha_{2k} = 0$.

Throughout this section, it is assumed k and c (in (171)) are large enough. For the sake of definiteness, let us require

$$k > 1000, \quad \text{and} \quad 10 < c < 1000, \tag{179}$$

which will suffice.

6.1. Contribution of "the small primes": via Cauchy's estimate

6.1.1. The combinatorial sum

We wish to estimate the Taylor coefficients (about zero) of

$$\sum_{p \leqslant ck^2} \log\left(1 + \sum_{n=1}^{\infty} \frac{S_{n,p}}{p^n}\right) =: \sum_{p \leqslant ck^2} C_p. \tag{180}$$

Fix a prime p. We consider the coefficient of $z_1^{\alpha_1} \cdots z_{2k}^{\alpha_{2k}}$ in the Taylor expansion of a local factor C_p, and denote it by $a_{\alpha,p}$. Since p is fixed, we may drop the dependency on it in $S_{n,p}$. So, let us write

$$C_p = \log\left(1 + \sum_{n=1}^{\infty} \frac{S_n}{p^n}\right). \tag{181}$$

We consider two possibilities: $m(\alpha) = 1$ or $m(\alpha) > 1$. Let us first handle the case $m(\alpha) > 1$.

As explained earlier, it may be assumed α is of the form

$$\alpha = (\alpha_1, \ldots, \alpha_l, 0, \ldots, 0, \alpha_{k+1}, \ldots, \alpha_{k+d}, 0, \ldots, 0), \quad 0 \leqslant d \leqslant l \leqslant k, \ \alpha_i > 0. \tag{182}$$

By symmetry, it may be further assumed $\alpha_1 \geqslant \cdots \geqslant \alpha_l$ and $\alpha_{k+1} \geqslant \cdots \geqslant \alpha_{k+d}$.

There are two possibilities, either $\alpha_2 = 0$ or not. Assume $\alpha_2 \neq 0$. A quick review of the argument to follow should show that the case $\alpha_2 = 0$ is completely analogous (one will need to differentiate with respect to z_{k+1} instead of z_2, noting the fact that since $m(\alpha) > 1$ then if $\alpha_2 = 0$, then $\alpha_{k+1} \neq 0$). Given the assumption $\alpha_2 \neq 0$, define

$$C_p'' := \frac{\partial^2}{\partial z_1 \partial z_2} C_p \Big|_{\substack{z_i = 0, z_{k+j} = 0 \\ l < i \leqslant k, d < j \leqslant k}}. \tag{183}$$

Then

$$a_{\alpha,p} = \frac{1}{\alpha_1 \alpha_2} \times \text{Coefficient of } z_1^{\alpha_1 - 1} z_2^{\alpha_2 - 1} z_3^{\alpha_3} \cdots z_l^{\alpha_l} z_{k+1}^{\alpha_{k+1}} \cdots z_{k+d}^{\alpha_{k+d}} \text{ in } C_p''. \tag{184}$$

Define

$$Q := 1 + \sum_{n=1}^{\infty} \frac{S_n}{p^n} \Big|_{\substack{z_i = 0, z_{k+j} = 0 \\ l < i \leqslant k, d < j \leqslant k}}, \qquad Q_1 := \sum_{n=1}^{\infty} \frac{1}{p^n} \frac{\partial}{\partial z_1} S_n \Big|_{\substack{z_i = 0, z_{k+j} = 0 \\ l < i \leqslant k, d < j \leqslant k}}, \tag{185}$$

$$Q_2 := \sum_{n=1}^{\infty} \frac{1}{p^n} \frac{\partial}{\partial z_2} S_n \Big|_{\substack{z_i = 0, z_{k+j} = 0 \\ l < i \leqslant k, d < j \leqslant k}}, \qquad Q_{12} := \sum_{n=1}^{\infty} \frac{1}{p^n} \frac{\partial^2}{\partial z_1 \partial z_2} S_n \Big|_{\substack{z_i = 0, z_{k+j} = 0 \\ l < i \leqslant k, d < j \leqslant k}}. \tag{186}$$

By a straightforward calculation,

$$C_p'' = \frac{Q_{12}}{Q} - \frac{Q_1 Q_2}{Q^2}. \tag{187}$$

Letting

$$\Omega := \left\{ |z_1| = \frac{\delta}{10^6 l}, \ldots, |z_l| = \frac{\delta}{10^6 l}, |z_{k+1}| = \frac{\delta}{10^6 l}, \ldots, |z_{k+d}| = \frac{\delta}{10^6 l} \right\}, \tag{188}$$

with $\delta > 0$ chosen so that $Q \neq 0$ on or inside Ω (such a δ exists), it follows from (184) and Cauchy's estimate that

$$|a_{\alpha,p}| \leqslant \left(\frac{\delta}{10^6 l}\right)^{2-|\alpha|} \left[\frac{\max_\Omega |Q_{12}|}{\min_\Omega |Q|} + \frac{\max_\Omega |Q_1|^2}{\min_\Omega |Q|^2} \right]. \tag{189}$$

Now, set

$$\delta = \frac{1}{1000 \log(ck^2)}. \tag{190}$$

We do not know this is a valid choice of δ a priori, but we will know this a posteriori.

The Denominator. We first estimate $\min_\Omega |Q|$. So, let

$$\mu := (\mu_1, \ldots, \mu_l), \qquad \tau := (\tau_1, \ldots, \tau_d), \quad \mu \in \mathbb{Z}^l_{\geqslant 0},\ \tau \in \mathbb{Z}^d_{\geqslant 0}. \tag{191}$$

Then, define

$$Q^{(\mu,\tau)} := \frac{\partial^{|\mu|+|\tau|} Q}{\partial z_1^{\mu_1} \cdots \partial z_l^{\mu_l} \partial z_{k+1}^{\tau_1} \cdots \partial z_{k+d}^{\tau_d}} \bigg|_{\substack{z_i = 0, z_{k+j} = 0 \\ 1 \leqslant i \leqslant l, 1 \leqslant j \leqslant d}}. \tag{192}$$

It follows

$$Q = Q^{(0)} + \sum_{|\mu|+|\tau| \geqslant 1} \frac{Q^{(\mu,\tau)}}{\mu_1! \cdots \mu_l! \tau_1! \cdots \tau_d!} z_1^{\mu_1} \cdots z_l^{\mu_l} z_{k+1}^{\tau_1} \cdots z_{k+d}^{\tau_d}, \tag{193}$$

where by definition,

$$Q^{(0)} = \sum_{n=1}^{\infty} \frac{1}{p^n} \binom{k+n-1}{n}^2. \tag{194}$$

Let

$$\mathcal{D} := \sum_{|\mu|+|\tau| \geqslant 1} \frac{|Q^{(\mu,\tau)}|}{\mu_1! \cdots \mu_l! \tau_1! \cdots \tau_d!} |z_1^{\mu_1} \cdots z_l^{\mu_l} z_{k+1}^{\tau_1} \cdots z_{k+d}^{\tau_d}|. \tag{195}$$

We shall show that there exists an absolute constant $\eta_1 \in (0,1)$ such that

$$\mathcal{D} \leqslant \eta_1 Q^{(0)} \tag{196}$$

for

$$(z_1, \dots, z_l, z_{k+1}, \dots, z_{k+d}) \in \Omega. \tag{197}$$

From that it follows

$$\min_{\Omega} |Q| \geqslant (1-\eta_1) Q^{(0)} = (1-\eta_1)\left[1 + \sum_{n=1}^{\infty} \frac{1}{p^n} \binom{k+n-1}{n}^2\right], \tag{198}$$

because by setting all $z_j = 0$ in (169) we have

$$\sum_{|\lambda|=|\rho|=n} 1 = \binom{k+n-1}{n}^2. \tag{199}$$

The latter can be seen by arranging $k+n-1$ 'dots' in a row and breaking them into k non-negative summands by selecting $k-1$ of the dots as barriers.

Now, bounding the rhs of (195) on Ω gives

$$\mathcal{D} \leqslant \sum_{\substack{h+g \geqslant 1 \\ h \leqslant l, g \leqslant d}} \frac{1}{(10^6 l)^{h+g}} \sum_{\substack{m(\mu)=h \\ m(\tau)=g}} \frac{|Q^{(\mu,\tau)}| \delta^{|\mu|+|\tau|}}{\mu_{i_1}! \cdots \mu_{i_h}! \tau_{j_1}! \cdots \tau_{j_g}!}. \tag{200}$$

Here we have used $h \leqslant |\mu|$ and $g \leqslant |\tau|$ so that $(10^6 l)^{h+g} \leqslant (10^6 l)^{|\mu|+|\tau|}$.

Let us examine the inner sum above. For h and g any non-negative integers satisfying $h+g \geqslant 1$, $h \leqslant l$, $g \leqslant d$, we have

$$\begin{aligned} Q|_{\substack{z_i=0, z_{k+j}=0 \\ h<i\leqslant l, g<j\leqslant d}} &= 1 + \sum_{n=1}^{\infty} \frac{1}{p^n} \sum_{a=0}^{n} \sum_{b=0}^{n} \binom{k+n-h-a-1}{n-a} \binom{k+n-g-b-1}{n-b} \\ &\quad \times \sum_{\substack{\lambda=(\lambda_1,\dots,\lambda_h), \lambda_i \geqslant 0 \\ \rho=(\rho_1,\dots,\rho_g), \rho_i \geqslant 0 \\ |\lambda|=a, |\rho|=b}} p^{\rho_1 z_{k+1} + \cdots + \rho_g z_{k+g} - \lambda_1 z_1 - \cdots - \lambda_h z_h}. \end{aligned} \tag{201}$$

In the above, the binomial coefficient $\binom{k+n-h-a-1}{n-a}$, for example, represents the number of ways to write $n-a$ as the sum of $k-h$ non-negative summands. Notice if $h=0$ then the inner-most sum vanishes unless $a=0$, and if $h=k$ then $\binom{k+n-h-a-1}{n-a}$ is 0 unless $a=n$, in which case it is 1; analogously if $g=0, k$.

So, for $\mu = (\mu_1, \dots, \mu_h, 0, \dots, 0) \in \mathbb{Z}^l_{\geqslant 0}$, and $\tau = (\tau_1, \dots, \tau_g, 0, \dots, 0) \in \mathbb{Z}^d_{\geqslant 0}$, such that $|\mu| + |\tau| \geqslant 1$,

$$\begin{aligned} |Q^{(\mu,\tau)}| &\leqslant \sum_{n=h}^{\infty} \frac{1}{p^n} \sum_{a=h}^{n} \sum_{b=g}^{n} \binom{k+n-h-a-1}{n-a} \binom{k+n-g-b-1}{n-b} \\ &\quad \times \sum_{\substack{\lambda=(\lambda_1,\dots,\lambda_h), \lambda_i \geqslant 1 \\ \rho=(\rho_1,\dots,\rho_g), \rho_i \geqslant 1 \\ |\lambda|=a, |\rho|=b}} (\lambda_1 \log p)^{\mu_1} \cdots (\lambda_h \log p)^{\mu_h} (\rho_1 \log p)^{\tau_1} \cdots (\rho_g \log p)^{\tau_g}. \end{aligned} \tag{202}$$

The sums over a, b start at h, g respectively because the partial derivatives of (201) vanish if the exponent in the innermost sum has fewer than h of $z_1, \dots, z_h$ or fewer than g of $z_{k+1}, \dots, z_{k+g}$. For the same reason, we can start the sum over n at $\max(h, g)$, and choose h, and we can assume $\lambda_i \geqslant 1$ and $\rho_i \geqslant 1$.

Therefore, by symmetry of Q with respect to $z_1,\dots,z_l$, and, separately, with respect to $z_{k+1},\dots,z_{k+d}$,

$$\sum_{\substack{m(\mu)=h\\ m(\tau)=g}} \frac{|Q^{(\mu,\tau)}|\delta^{|\mu|+|\tau|}}{\mu_{i_1}!\cdots\mu_{i_h}!\tau_{j_1}!\cdots\tau_{j_g}!}$$
$$\leqslant \sum_{n=h}^{\infty}\frac{1}{p^n}\sum_{a=h}^{n}\sum_{b=g}^{n}\binom{k+n-h-a-1}{n-a}\binom{k+n-g-b-1}{n-b}$$
$$\times \sum_{\substack{\lambda=(\lambda_1,\dots,\lambda_h),\lambda_i\geqslant 1\\ \rho=(\rho_1,\dots,\rho_g),\rho_i\geqslant 1\\ |\lambda|=a,|\rho|=b}}\ \sum_{\substack{m(\mu)=h\\ m(\tau)=g}} \frac{(\delta\lambda_1\log p)^{\mu_{i_1}}\cdots(\delta\lambda_h\log p)^{\mu_{i_h}}(\delta\rho_1\log p)^{\tau_{j_1}}\cdots(\delta\rho_g\log p)^{\tau_{j_g}}}{\mu_{i_1}!\cdots\mu_{i_h}!\tau_{j_1}!\cdots\tau_{j_g}!}. \tag{203}$$

Summing over $h+g\geqslant 1$, $h\leqslant l$, $g\leqslant d$, we obtain

$$\mathcal{D}\leqslant \sum_{\substack{h+g\geqslant 1\\ h\leqslant l,g\leqslant d}}\frac{1}{(10^6 l)^{h+g}}\binom{l}{h}\binom{d}{g}\sum_{n=h}^{\infty}\frac{1}{p^n}$$
$$\times\sum_{a=h}^{n}\sum_{b=g}^{n}\binom{k+n-h-a-1}{n-a}\binom{k+n-g-b-1}{n-b}\binom{a-1}{h-1}\binom{b-1}{g-1}p^{\delta(a+b)}. \tag{204}$$

In the above sum, the binomial coefficients $\binom{l}{h}$ and $\binom{d}{g}$ represent the number of ways to select the μ_i's and τ_i's so that $m(\mu)=h$ and $m(\tau)=g$. Also, the factor $p^{\delta(a+b)}$ arises from $\exp(\log(p)(\lambda_1+\cdots+\lambda_h+\rho_1+\cdots+\rho_g))$, writing this as a product of exp's and using the Taylor series about 0 for $\exp(x)$ to produce the terms in the innermost sum of (204). There are two special cases: When $g=0$, the quantity $\binom{b-1}{g-1}$ is defined to be zero unless $b=0$, where it is defined to be 1, and when $g=k$, the quantity $\binom{k+n-g-b-1}{n-b}$ is 0, unless $b=n$, in which case it is 1. Similar considerations apply to special values of h.

For $n<8k$ say, use the following estimates. First, notice that $\binom{k+n-h-a-1}{n-a}$ is the number of ways to write $n-a$ as the sum of exactly $k-h$ non-negative integers, and $\binom{a-1}{h-1}$ is equal to the number of ways to write a as the sum of exactly h positive integers. Therefore, $\binom{k+n-h-a-1}{n-a}\binom{a-1}{h-1}$ is at most the number of ways to write n as the sum of exactly k non-negative integers, where the first $k-h$ parts sum to $n-a$ and the last h parts sum to a. So by summing over a, we see

$$\sum_{a=h}^{n}\binom{k+n-h-a-1}{n-a}\binom{a-1}{h-1}\leqslant\binom{k+n-1}{n}, \tag{205}$$

where $\binom{k+n-1}{n}$ is the number of ways to write n as the sum of exactly k non-negative integers. In the range $100h\leqslant n$, we thus obtain

$$\sum_{a=h}^{100h-1}\binom{k+n-h-a-1}{n-a}\binom{a-1}{h-1}p^{\delta a}\leqslant\binom{k+n-1}{n}p^{100\delta h}. \tag{206}$$

In the range $100h \leqslant a \leqslant n$, estimate (205) is no longer good enough for our purposes. Instead, we note

$$\frac{\binom{k+n-h-a-1}{n-a}}{\binom{k+n-1}{n}} = \frac{\prod_{j=0}^{a-1}(n-j)\prod_{j=1}^{h}(k-j)}{\prod_{j=1}^{a+h}(k+n-j)} \leqslant (1+k/n)^{-a}(1+n/k)^{-h}, \tag{207}$$

it follows

$$\sum_{a=100h}^{n}\binom{k+n-h-a-1}{n-a}\binom{a-1}{h-1}p^{\delta a} \leqslant \binom{k+n-1}{n}\sum_{a=100h}^{n}\frac{\binom{a-1}{h-1}p^{\delta a}}{(1+\frac{k}{n})^{a}(1+\frac{n}{k})^{h}}. \tag{208}$$

Recalling $\delta = \frac{1}{1000\log(ck^2)}$ and $p \leqslant ck^2$, we have $p^{\delta} \leqslant 1.001$. Writing $a = 100h+m$, one deduces

$$\frac{\binom{100h+m-1}{h-1}}{\binom{100h-1}{h-1}} = \frac{\prod_{j=0}^{m-1}(100h+j)}{\prod_{j=0}^{m-1}(99h+j+1)} \leqslant (1+1/99)^{m}. \tag{209}$$

Also, for $n < 8k$, it holds $1+k/n \geqslant 9/8$. So it is seen that the sum (208) is bounded by

$$\leqslant 100\binom{k+n-1}{n}\frac{\binom{100h-1}{h-1}p^{100\delta h}}{(\frac{9}{8})^{100h}} \leqslant 100\binom{k+n-1}{n}, \tag{210}$$

where, in the last inequality, we used $\binom{100h-1}{h-1} \leqslant (100h)^h/h! \leqslant 300^h$, $p^{100\delta h} \leqslant (1.2)^h$, and $(9/8)^{100h} \geqslant 1000^h$. Putting together, we have

$$\sum_{n=h}^{8k-1}\frac{1}{p^n}\sum_{a=h}^{n}\sum_{b=g}^{n}\binom{k+n-h-a-1}{n-a}\binom{k+n-g-b-1}{n-b} \times \binom{a-1}{h-1}\binom{b-1}{g-1}p^{\delta(a+b)} \leqslant 10000p^{100\delta(h+g)}Q^{(0)}. \tag{211}$$

For $n \geqslant 8k$, use the estimate

$$\sum_{a=h}^{n}\binom{k+n-h-a-1}{n-a}\binom{a-1}{h-1}p^{\delta a} \leqslant \binom{k+n-1}{n}p^{\delta n}, \tag{212}$$

which, again, is deducible via a combinatorial interpretation of the sum. This estimate yields

$$\sum_{n=8k}^{\infty}\frac{1}{p^n}\sum_{a=h}^{n}\sum_{b=g}^{n}\binom{k+n-h-a-1}{n-a}\binom{k+n-g-b-1}{n-b} \times \binom{a-1}{h-1}\binom{b-1}{g-1}p^{\delta(a+b)} \leqslant \sum_{n=8k}^{\infty}\frac{p^{2\delta n}}{p^n}\binom{k+n-1}{n}^2. \tag{213}$$

Collecting the bounds so far, and using some straightforward manipulations, we have by (204) that $\mathcal{D}$ is bounded by

$$\sum_{\substack{h+g\geqslant 1\\ h\leqslant l,g\leqslant d}} \frac{1}{(10^6 l)^{h+g}}\binom{l}{h}\binom{d}{g}\left[10000p^{\delta 100(h+g)}Q^{(0)}+\sum_{n=8k}^{\infty}\frac{p^{2\delta n}}{p^n}\binom{k+n-1}{n}^2\right]$$

$$\leqslant \sum_{\substack{h+g\geqslant 1\\ h\leqslant l,g\leqslant d}} \frac{10000p^{100\delta(h+g)}l^{h+g}}{(10^6 l)^{h+g}}\left[Q^{(0)}+\frac{p^{16\delta k}}{p^{8k}}\binom{9k-1}{8k}^2\sum_{j=0}^{\infty}\frac{p^{2\delta j}}{p^j}\left(\frac{9}{8}\right)^{2j}\right]\leqslant \frac{Q^{(0)}}{2}. \tag{214}$$

Here we have used the assumption that $d\leqslant l$ in the inequality $\binom{l}{h}\binom{d}{g}\leqslant l^{h+g}$. Also, note in the last inequality we used the following observation: since $Q^{(0)}$ contains the term $\frac{1}{p^k}\binom{2k-1}{k}^2$, and since

$$\frac{\frac{1}{p^{8k}}\binom{9k-1}{8k}^2}{\frac{1}{p^k}\binom{2k-1}{k}^2}=\frac{1}{p^{7k}}\prod_{l=1}^{k-1}\left(1+\frac{7k}{k+l}\right)^2\leqslant\frac{8^{2k}}{2^{7k}}=\frac{1}{2^k} \tag{215}$$

(the above uses $(1+7k/(k+l))<8$ and $p\geqslant 2$), it follows

$$\frac{p^{16\delta k}}{p^{8k}}\binom{9k-1}{8k}^2\leqslant\left(\frac{p^{16\delta}}{2}\right)^k Q^{(0)}\leqslant\frac{Q^{(0)}}{10}. \tag{216}$$

In sum, we have shown

$$\max_{\Omega}\mathcal{D}\leqslant\frac{1}{2}Q^{(0)}\quad\Rightarrow\quad\min_{\Omega}|Q|\geqslant\frac{1}{2}Q^{(0)}. \tag{217}$$

The Numerator. Having disposed of $\min_{\Omega}|Q|$, we direct our attention to $\max_{\Omega}|Q_{12}|$ and $\max_{\Omega}|Q_1|^2$. We deal with $\max_{\Omega}|Q_{12}|$ first. We will show there exists an absolute constant η_2 such that

$$\max_{\Omega}|Q_{12}|\leqslant\eta_2 l^3\frac{(\log p)^2}{p}Q^{(0)}. \tag{218}$$

First, note over Ω,

$$\frac{|Q_{12}|}{(\log p)^2}\leqslant\sum_{n=2}^{\infty}\frac{1}{p^n}\sum_{a=2}^{n}\sum_{b=0}^{n}\binom{k+n-l-a-1}{n-a}\binom{k+n-d-b-1}{n-b}$$
$$\times p^{\delta\frac{a+b}{l}}\sum_{\substack{\lambda=(\lambda_1,\dots,\lambda_l),\lambda_i\geqslant 0\\ \rho=(\rho_1,\dots,\rho_d),\rho_i\geqslant 0\\ |\lambda|=a-2,|\rho|=b}}(\lambda_1+1)(\lambda_2+1). \tag{219}$$

(Note the sum over a starts at 2 instead of 0 because otherwise, either the derivative with respect to z_1 or z_2 will vanish.) Therefore, since $(\lambda_1+1)(\lambda_2+1)\leqslant a^2$,

$$\frac{|Q_{12}|}{(\log p)^2}\leqslant\sum_{n=2}^{\infty}\frac{1}{p^n}\sum_{a=2}^{n}\sum_{b=0}^{n}\binom{k+n-l-a-1}{n-a}\binom{k+n-d-b-1}{n-b}$$
$$\times p^{\delta\frac{a+b}{l}}a^2\sum_{\substack{\lambda=(\lambda_1,\dots,\lambda_l),\lambda_i\geqslant 0\\ \rho=(\rho_1,\dots,\rho_d),\rho_i\geqslant 0\\ |\lambda|=a-2,|\rho|=b}}1. \tag{220}$$

When $n < 8k$, it follows by considering the ranges $b < 100d$ and $100d \geqslant b \leqslant n$ separately as before, while noting that $d \leqslant l$ by hypothesis, that

$$\begin{aligned}
&\sum_{b=0}^{n} \binom{k+n-d-b-1}{n-b} p^{\frac{\delta b}{l}} \sum_{\substack{\rho=(\rho_1,\dots,\rho_d)\\ \rho_i \geqslant 0, |\rho|=b}} 1 \\
&= \sum_{b=0}^{n} \binom{k+n-d-b-1}{n-b} \binom{d+b-1}{b} p^{\frac{\delta b}{l}} \\
&\leqslant \binom{k+n-1}{n} p^{\frac{100\delta d}{l}} + \binom{k+n-1}{n} \sum_{b=100d}^{n} \frac{\binom{d+b-1}{b} p^{\frac{\delta b}{l}}}{(1+\frac{k}{n})^b (1+\frac{n}{k})^d} \leqslant 100 \binom{k+n-1}{n}. \qquad (221)
\end{aligned}$$

When $n \geqslant 8k$, we have

$$\sum_{b=0}^{n} \binom{k+n-d-b-1}{n-b} p^{\frac{\delta b}{l}} \sum_{\substack{\rho=(\rho_1,\dots,\rho_d)\\ \rho_i \geqslant 0, |\rho|=b}} 1 \leqslant \binom{k+n-1}{n} p^{\frac{\delta n}{l}}. \qquad (222)$$

In the above expressions, when $d = 0$, the quantity $\binom{d+b-1}{b}$ is interpreted as 0 unless $b = 0$. Similar care should be taken in interpreting expressions when l or d equals k. In any case, if we define

$$\mathcal{N} := \sum_{n=2}^{8k} \frac{1}{p^n} \binom{k+n-1}{n} \sum_{a=2}^{n} \binom{k+n-l-a-1}{n-a} p^{\frac{\delta a}{l}} a^2 \sum_{\substack{\lambda=(\lambda_1,\dots,\lambda_l)\\ \lambda_i \geqslant 0, |\lambda|=a-2}} 1, \qquad (223)$$

then, after a little bit of work combining (220), (221), and (222), we have generously

$$\frac{|Q_{12}|}{(\log p)^2} \leqslant 100\mathcal{N} + 100(8k)^2 \frac{p^{16\delta k}}{p^{8k}} \binom{9k-1}{8k}^2 \leqslant 100\mathcal{N} + \frac{1}{p} Q^{(0)}. \qquad (224)$$

So, we just need to bound $\mathcal{N}$. To this end, note

$$\sum_{a=2}^{n} \binom{k+n-l-a-1}{n-a} p^{\frac{\delta a}{l}} a^2 \sum_{\substack{\lambda=(\lambda_1,\dots,\lambda_l)\\ \lambda_i \geqslant 0, |\lambda|=a-2}} 1 = \sum_{a=2}^{n} \binom{k+n-l-a-1}{n-a} \binom{l+a-3}{a-2} p^{\frac{\delta a}{l}} a^2. \qquad (225)$$

Define

$$M := \left\lceil \frac{ck}{\sqrt{p}} \right\rceil. \qquad (226)$$

Further define

$$\Sigma_1 := \sum_{n=2}^{M-1} \frac{1}{p^n} \binom{k+n-1}{n} \sum_{a=2}^{n} \binom{k+n-l-a-1}{n-a} \binom{l+a-3}{a-2} p^{\frac{\delta a}{l}} a^2, \qquad (227)$$

$$\Sigma_2 := \sum_{n=M}^{\infty} \frac{1}{p^n} \binom{k+n-1}{n} \sum_{a=2}^{n} \binom{k+n-l-a-1}{n-a} \binom{l+a-3}{a-2} p^{\frac{\delta a}{l}} a^2. \qquad (228)$$

In particular,

$$\mathcal{N} \leqslant \Sigma_1 + \Sigma_2. \tag{229}$$

We bound Σ_1. Observe that

$$\begin{aligned}\sum_{a=2}^{100l-1} \binom{k+n-a-l-1}{n-a}\binom{l+a-3}{a-2} a^2 p^{\frac{\delta a}{T}} \\ \leqslant \sum_{a=2}^{100l} \binom{k+n-3}{n-2} a^2 p^{\frac{\delta a}{T}} \leqslant \binom{k+n-3}{n-2}(100l)^3.\end{aligned} \tag{230}$$

Also, for $n < M$,

$$\frac{k}{n-2} \geqslant \frac{k}{M} \geqslant \frac{\sqrt{p}}{c}. \tag{231}$$

Therefore,

$$\begin{aligned}\sum_{a=100l}^{n} \binom{k+n-a-l-1}{n-a}\binom{l+a-3}{a-2} a^2 p^{\frac{\delta a}{T}} &\leqslant \binom{k+n-3}{n-2} \sum_{a=100l}^{n} \frac{\binom{l+a-3}{a-2} a^2 p^{\frac{\delta a}{T}}}{(1+\frac{k}{n-2})^{a-2}} \\ &\leqslant 100\binom{k+n-3}{n-2}.\end{aligned} \tag{232}$$

In summary,

$$\Sigma_1 \leqslant (100l)^3 \sum_{n=2}^{M-1} \frac{1}{p^n}\binom{k+n-1}{n}^2 \frac{\binom{k+n-3}{n-2}}{\binom{k+n-1}{n}} \leqslant \frac{(100l)^3}{(1+\frac{k}{n})^2} Q^{(0)} \leqslant \frac{\eta_3 l^3}{p} Q^{(0)}, \tag{233}$$

where η_3 is some absolute constant. As for Σ_2, note

$$\sum_{a=2}^{n} \binom{k+n-a-l-1}{n-a}\binom{l+a-3}{a-2} a^2 p^{\frac{\delta a}{T}} \leqslant \binom{k+n-3}{n-2} n^3 p^{\delta n}. \tag{234}$$

Therefore, using the change of variable $n = M + j$, we have

$$\begin{aligned}\Sigma_2 &\leqslant \sum_{n=M}^{\infty} \frac{n^3 p^{\delta n}}{p^n}\binom{k+n-1}{n}\binom{k+n-3}{n-2} \\ &\leqslant \frac{M^3 p^{\delta M}}{p^M}\binom{k+M-1}{M}\binom{k+M-3}{M-2} \sum_{j=0}^{\infty} \frac{(1+j/M)^3 p^{\delta j}}{p^j}\left(1+\frac{2k}{M}\right)^{2j}.\end{aligned} \tag{235}$$

Since

$$\sum_{j=0}^{\infty} \frac{(1+j/M)^3 p^{\delta j}}{p^j}\left(1+\frac{2k}{M}\right)^{2j} \leqslant \sum_{j=0}^{\infty}\left(1+\frac{j}{M}\right)^3\left(\frac{p^{\delta/2}}{\sqrt{p}}+\frac{2}{c}\right)^{2j} \leqslant \eta_4, \tag{236}$$

where η_4 is some absolute constant, it follows

$$\Sigma_2 \leqslant \eta_4 \frac{M^3 p^{\delta M}}{p^M} \binom{k+M-1}{M} \binom{k+M-3}{M-2}. \tag{237}$$

Now, define

$$M_1 := \left\lfloor \frac{5k}{\sqrt{p}} \right\rfloor. \tag{238}$$

Note $Q^{(0)}$ contains the term $\frac{1}{p^{M_1}} \binom{k+M_1-1}{M_1}^2$. Thus,

$$\frac{\frac{1}{p^M}\binom{k+M-1}{M}\binom{k+M-3}{M-2}}{Q^{(0)}} \leqslant \left(\frac{M}{k}\right)^2 \frac{1}{p^{M-M_1}} \left(1 + \frac{k}{M_1+1}\right)^{2(M-M_1)}. \tag{239}$$

Note,

$$\frac{M}{k} \leqslant \frac{4c}{\sqrt{p}}, \tag{240}$$

and

$$\frac{1}{p^{M-M_1}} \left(1 + \frac{k}{M_1+1}\right)^{2(M-M_1)} \leqslant \left(\frac{1}{\sqrt{p}} + \frac{1}{5}\right)^{2(M-M_1)}. \tag{241}$$

Therefore, for some absolute constant η_5, we have

$$\Sigma_2 \leqslant \frac{\eta_5}{p} M^3 p^{\delta M} \left(\frac{1}{\sqrt{p}} + \frac{1}{5}\right)^{2(M-M_1)} Q^{(0)}. \tag{242}$$

Since $M - M_1 \geqslant \frac{ck}{2\sqrt{p}} - 1$, we have

$$p^{\delta M} \left(\frac{1}{\sqrt{p}} + \frac{1}{5}\right)^{2(M-M_1)} \leqslant 2e^{\frac{ck}{500\sqrt{p}}} (0.91)^{\frac{ck}{\sqrt{p}}} \leqslant 2(0.92)^{\frac{ck}{\sqrt{p}}}. \tag{243}$$

Hence,

$$M^3 p^{\delta M} \left(\frac{1}{\sqrt{p}} + \frac{1}{5}\right)^{2(M-M_1)} \leqslant \left(\frac{ck}{\sqrt{p}}\right)^3 (0.92)^{\frac{ck}{\sqrt{p}}} \leqslant \eta_6, \tag{244}$$

for some absolute constant η_6. So, there exists an absolute constant η_7 such that

$$\Sigma_2 \leqslant \frac{\eta_7}{p} Q^{(0)}. \tag{245}$$

Assembling previous bounds together, we thus obtain

$$\max_{\Omega} |Q_{12}| \ll l^3 \frac{(\log p)^2}{p} Q^{(0)}, \tag{246}$$

as claimed. The case $\max_{\Omega}|Q_1|^2$ is similar. There, we obtain

$$\max_{\Omega}|Q_1|^2 \ll \frac{(\log p)^2}{p}\big[Q^{(0)}\big]^2. \tag{247}$$

Summary. Combining (189), (217), (246), (247), and the fact $l \leqslant m(\alpha)$, we have therefore shown the existence of an absolute constant η_8 such that

$$|a_{\alpha,p}| \ll \big(\eta_8 m(\alpha)\big)^{|\alpha|}(\log k)^{|\alpha|-2}\frac{(\log p)^2}{p} \quad \text{for } m(\alpha) > 1. \tag{248}$$

Thus, when $m(\alpha) > 1$, the contribution to a_α of the combinatorial sum corresponding to "the small primes" is

$$\ll \big(\eta_8 m(\alpha)\big)^{|\alpha|}(\log k)^{|\alpha|-2}\sum_{p\leqslant ck^2}\frac{(\log p)^2}{p} \ll \big(\eta_8 m(\alpha)\log k\big)^{|\alpha|}. \tag{249}$$

Finally, the case $m(\alpha) = 1$ can be handled analogously. In that case, we obtain for some absolute constant η_9,

$$|a_{\alpha,p}| \ll (\eta_9 \log k)^{|\alpha|-2}\frac{(\log p)^2}{\sqrt{p}} \quad \text{for } m(\alpha) = 1. \tag{250}$$

Thus, when $m(\alpha) = 1$, the contribution to a_α of the combinatorial sum corresponding to "the small primes" is

$$\ll (\eta_9 \log k)^{|\alpha|-2}\sum_{p\leqslant ck^2}\frac{(\log p)^2}{\sqrt{p}} \ll (\eta_9 \log k)^{|\alpha|-1}k. \tag{251}$$

6.1.2. The convergence factor sum

In this subsection, we redefine, for convenience, C_p and $a_{\alpha,p}$ of the previous subsection.

We wish to bound the Taylor coefficients (about zero) of

$$\sum_{p\leqslant ck^2}\sum_{i,j=1}^{k}\log\left(1-\frac{p^{z_{k+j}-z_i}}{p}\right) =: \sum_{p\leqslant ck^2} C_p, \tag{252}$$

where, again, we redefined C_p to avoid notational clutter. Because only two z_i's appear in each term of the inner sum on the lhs, the Taylor coefficients $a_{\alpha,p}$ of a local factor C_p are zero except for the coefficients of monomials of the type z_i^u, with $1 \leqslant i \leqslant 2k$ (case $m(\alpha) = 1$), or $z_i^u z_{k+j}^v$, with $1 \leqslant i, j \leqslant k$ (case $m(\alpha) = 2$). Here $u, v \in \mathbb{Z}_{\geqslant 0}$. By symmetry, it is enough to consider the monomials z_1^u and $z_1^u z_{k+1}^v$.

We deal with the case $m(\alpha) = 1$ first. So, let $a_{\alpha,p}$ denote the coefficient of z_1^u in C_p, where

$$\alpha = (u, 0, \ldots, 0), \quad u \in \mathbb{Z}_{\geqslant 0}. \tag{253}$$

Consider the derivative

$$C_p' := \frac{\partial}{\partial z_1}C_p\bigg|_{\substack{z_i=0\\ 2\leqslant i\leqslant 2k}} = \frac{k\log p}{p}\frac{p^{-z_1}}{1-\frac{p^{-z_1}}{p}}. \tag{254}$$

Let $\Omega := \{|z_1| = \delta\}$, where δ is sufficiently small (to be specified shortly). By Cauchy's estimate,

$$|a_{\alpha,p}| \leqslant \delta^{1-u} \max_{\Omega} C'_p \leqslant \delta^{1-u} \frac{k \log p}{p} \frac{p^{\delta}}{1 - \frac{p^{\delta}}{p}}. \tag{255}$$

Choosing $\delta = 1/(10 \log ck^2)$, we obtain

$$|a_{\alpha,p}| \leqslant (50 \log k)^{u-1} \frac{50k \log p}{p}. \tag{256}$$

This uses our assumption that $k \geqslant 1000$, $10 \leqslant c \leqslant 1000$, and, here, $p \leqslant ck^2$, so that, with plenty of room to spare, $10 \log(ck^2) < 50 \log(k)$, and $p^{\delta}/(1 - p^{\delta-1}) < 50$.

Therefore, when $m(\alpha) = 1$, the contribution to a_α of the convergence factor sum corresponding to "the small primes" is

$$\ll (50 \log k)^{|\alpha|-1} 50k \sum_{p \leqslant ck^2} \frac{\log p}{p} \ll (50 \log k)^{|\alpha|} k. \tag{257}$$

The case $m(\alpha) = 2$ can be handled similarly. Let $a_{\alpha,p}$ now denote the coefficient of $z_1^u z_{k+1}^v$, where

$$\alpha = (u, 0, \ldots, 0, v, 0, \ldots, 0), \quad u, v \in \mathbb{Z}_{\geqslant 0}. \tag{258}$$

Consider the derivative

$$C''_p := \frac{\partial^2}{\partial z_1 \partial z_{k+1}} C_p \Big|_{\substack{z_i = 0, z_{k+i} = 0 \\ 2 \leqslant i \leqslant k}} = \frac{(\log p)^2}{p} \frac{p^{z_{k+1} - z_1}}{1 - \frac{p^{z_{k+1} - z_1}}{p}} \left[1 + \frac{1}{p} \frac{p^{z_{k+1} - z_1}}{1 - \frac{p^{z_{k+1} - z_1}}{p}} \right]. \tag{259}$$

Let $\Omega := \{|z_1| = \delta, |z_{k+1}| = \delta\}$, with δ chosen as before. By Cauchy's estimate,

$$|a_{\alpha,p}| \leqslant \delta^{2-|\alpha|} \max_{\Omega} C''_p \leqslant \delta^{2-|\alpha|} \frac{50(\log p)^2}{p} \leqslant (50 \log k)^{|\alpha|-2} \frac{50(\log p)^2}{p}. \tag{260}$$

Therefore, when $m(\alpha) = 2$, the contribution to a_α of the convergence factor sum corresponding to "the small primes" is

$$\ll (50 \log k)^{|\alpha|-2} 50 \sum_{p \leqslant ck^2} \frac{(\log p)^2}{p} \ll (50 \log k)^{|\alpha|}. \tag{261}$$

6.2. Contribution of "the large primes": via Taylor expansions

6.2.1. The combinatorial sum

Next we bound the Taylor coefficients (about zero) of

$$\sum_{p > ck^2} \left[\log\left(1 + \sum_{n=1}^{\infty} \frac{S_{n,p}}{p^n} \right) - \frac{S_{1,p}}{p} \right] =: \sum_{p > ck^2} C_p, \tag{262}$$

again redefining C_p. Fix a prime p. Since p is fixed, we may drop dependency on it in $S_{n,p}$. Applying the Taylor expansions to the local factor C_p, we obtain

$$C_p = \sum_{n=2}^{\infty} \frac{S_n}{p^n} + \sum_{m=2}^{\infty} \frac{(-1)^{m+1}}{m} \left(\sum_{n=1}^{\infty} \frac{S_n}{p^n} \right)^m, \tag{263}$$

again redefining C_p. Next, write

$$\sum_{m=2}^{\infty} \frac{(-1)^{m+1}}{m} \left(\sum_{n=1}^{\infty} \frac{S_n}{p^n} \right)^m = \sum_{m=2}^{\infty} \frac{(-1)^{m+1}}{m} \sum_{n_1,n_2,\ldots,n_m \geqslant 1} \frac{S_{n_1} S_{n_2} \cdots S_{n_m}}{p^{n_1+\cdots+n_m}}, \tag{264}$$

sort the n_i's, and count them according to their multiplicity, i.e. let $S_{n_1} S_{n_2} \cdots S_{n_m} = S_1^{\lambda_1} S_2^{\lambda_2} \cdots S_r^{\lambda_r}$, where each $\lambda_i \geqslant 0$, and $\lambda_r \geqslant 1$ with r the largest integer among $n_1, \ldots, n_m$. Notice that $\lambda_1 + 2\lambda_2 + \cdots + r\lambda_r = n_1 + \cdots + n_m$, and that $m = \lambda_1 + \cdots + \lambda_r$. The above thus equals

$$\sum_{n=2}^{\infty} \frac{1}{p^n} \sum_{\substack{\lambda_1+2\lambda_2+\cdots+r\lambda_r=n \\ \lambda_1+\cdots+\lambda_r \geqslant 2 \\ \lambda_i \geqslant 0, r \geqslant 1}} \frac{(-1)^{\lambda_1+\cdots+\lambda_r+1}}{\lambda_1+\cdots+\lambda_r} \frac{(\lambda_1+\cdots+\lambda_r)!}{\lambda_1!\cdots\lambda_r!} S_1^{\lambda_1} S_2^{\lambda_2} \cdots S_r^{\lambda_r}. \tag{265}$$

Next, we can absorb the first sum in (263) into this by changing the condition $\lambda_1 + \cdots + \lambda_r \geqslant 2$ to include the case $\lambda_1 + \cdots + \lambda_r = 1$. But, because $\lambda_r = 1$ we then have $\lambda_1 = \cdots = \lambda_{r-1} = 0$. And because $\lambda_1 + 2\lambda_2 + \cdots + r\lambda_r = n$, we thus have $r = n$, i.e., if we extend the sum to include $\lambda_1 + \cdots + \lambda_r = 1$, it introduces precisely the terms $\sum_{n=2}^{\infty} \frac{S_n}{p^n}$. Therefore, we have arrived at

$$C_p = \sum_{n=2}^{\infty} \frac{1}{p^n} \sum_{\substack{\lambda_1+2\lambda_2+\cdots+r\lambda_r=n \\ \lambda_i \geqslant 0, r \geqslant 1}} \frac{(-1)^{\lambda_1+\cdots+\lambda_r+1}}{\lambda_1+\cdots+\lambda_r} \frac{(\lambda_1+\cdots+\lambda_r)!}{\lambda_1!\cdots\lambda_r!} S_1^{\lambda_1} S_2^{\lambda_2} \cdots S_r^{\lambda_r}. \tag{266}$$

We consider the coefficient of $z_1^{\alpha_1} \cdots z_{2k}^{\alpha_{2k}}$ in the Taylor expansion of C_p. Let us overload notation again and denote the said coefficient by $a_{\alpha,p}$. As noted at the beginning of the current section, it may be assumed that α is of the form

$$\alpha = (\alpha_1, \ldots, \alpha_l, 0, \ldots, 0, \alpha_{k+1}, \ldots, \alpha_{k+d}, 0, \ldots, 0), \quad 0 \leqslant d \leqslant l \leqslant k, \ \alpha_i > 0. \tag{267}$$

In particular, as far as $a_{\alpha,p}$ is concerned, it is equivalent to consider the series

$$\sum_{n=\max\{l,2\}}^{\infty} \frac{1}{p^n} \sum_{\substack{\lambda_1+2\lambda_2+\cdots+r\lambda_r=n \\ \lambda_i \geqslant 0, r \geqslant 1}} \frac{(-1)^{\lambda_1+\cdots+\lambda_r+1}}{\lambda_1+\cdots+\lambda_r} \frac{(\lambda_1+\cdots+\lambda_r)!}{\lambda_1!\cdots\lambda_r!} S_1^{\lambda_1} S_2^{\lambda_2} \cdots S_r^{\lambda_r}. \tag{268}$$

We restrict the sum over n to $\max\{l, 2\}$ because, in order for a term of the form $z_1^{\alpha_1} \cdots z_{2k}^{\alpha_{2k}}$, with $\alpha_i > 0$ for all $i \leqslant li \leqslant k$, we need to have at least l individual z_i's, with $i \leqslant k$, appearing in $S_1^{\lambda_1} S_2^{\lambda_2} \cdots S_r^{\lambda_r}$. But each term in the sum S_j involves at most j individual z_i's, hence overall we require $\sum_j = 1^r j\lambda_j = n \geqslant l$.

Now, define

$$T := \sum_{i=1}^{2k} p^{z_i}. \tag{269}$$

It is then not too hard to see (e.g. by considering the number of ways in which $z_1^{\alpha_1}\cdots z_l^{\alpha_l} z_{k+1}^{\alpha_{k+1}}\cdots z_{k+d}^{\alpha_{k+d}}$ can be formed) that $a_{\alpha,p}$ is bounded by the coefficient of $z_1^{\alpha_1}\cdots z_l^{\alpha_l} z_{k+1}^{\alpha_{k+1}}\cdots z_{k+d}^{\alpha_{k+d}}$ in

$$\sum_{n=\max\{l,2\}}^{\infty} \frac{T^{2n}}{p^n} \sum_{\substack{\lambda_1+2\lambda_2+\cdots+r\lambda_r=n\\ \lambda_i\geqslant 0, r\geqslant 1}} \frac{(\lambda_1+\cdots+\lambda_r)!}{\lambda_1!\cdots\lambda_r!}. \tag{270}$$

Also,

$$\sum_{\substack{\lambda_1+2\lambda_2+\cdots+r\lambda_r=n\\ \lambda_i\geqslant 0, r\geqslant 1}} \frac{(\lambda_1+\cdots+\lambda_r)!}{\lambda_1!\cdots\lambda_r!} \leqslant 2^n \sum_{\substack{\lambda_1+2\lambda_2+\cdots+r\lambda_r=n\\ \lambda_i\geqslant 0, r\geqslant 1}} 1 \leqslant 2^{2n}. \tag{271}$$

For the first step above use:

$$\frac{(\lambda_1+\cdots+\lambda_r)!}{\lambda_1!\cdots\lambda_r!} = \binom{\lambda_r}{\lambda_r}\binom{\lambda_{r-1}+\lambda_{r-2}}{\lambda_{r-1}}\cdots\binom{\lambda_1+\cdots+\lambda_r}{\lambda_1} \tag{272}$$

and bound each binomial coefficient by: $\binom{m}{j} \leqslant 2^m$. For the second step, the number of terms is bounded by the number of unordered partitions of n, which is easily $\leqslant 2^{n-1}$, since the number of ordered partitions of n equals 2^{n-1}.

Hence, $a_{\alpha,p}$ is more simply bounded by the coefficient of $z_1^{\alpha_1}\cdots z_l^{\alpha_l} z_{k+1}^{\alpha_{k+1}}\cdots z_{k+d}^{\alpha_{k+d}}$ in

$$\sum_{n=\max\{l,2\}}^{\infty} \frac{e^{2n}}{p^n} T^{2n}. \tag{273}$$

Let

$$\left[z_1^{\alpha_1}\cdots z_l^{\alpha_l} z_{k+1}^{\alpha_{k+1}}\cdots z_{k+d}^{\alpha_{k+d}}\right]_n := \text{Coefficient of } z_1^{\alpha_1}\cdots z_l^{\alpha_l} z_{k+1}^{\alpha_{k+1}}\cdots z_{k+d}^{\alpha_{k+d}} \text{ in } T^{2n}. \tag{274}$$

Setting $z_{l+1}=\cdots=z_k=0$, and $z_{k+d+1}=\cdots=z_{2k}=0$ in T^{2n} gives

$$\left(\sum_{i=1}^{l} p^{z_i} + \sum_{i=1}^{d} p^{z_{k+i}} + (2k-l-d)\right)^{2n} = \sum_{j=0}^{2n}\binom{2n}{j}(2k-l-d)^{2n-j}\left(\sum_{i=1}^{l} p^{z_i} + \sum_{i=1}^{d} p^{z_{k+i}}\right)^{j}. \tag{275}$$

Taking the multinomial expansion of the bracketed term, and applying the operator

$$\frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}}\cdots\frac{\partial^{\alpha_l}}{\partial z_l^{\alpha_l}}\frac{\partial^{\alpha_{k+1}}}{\partial z_{k+1}^{\alpha_{k+1}}}\cdots\frac{\partial_{\alpha_{k+d}}}{\partial z_{k+d}^{\alpha_{k+d}}}\bigg|_{(z_1,\ldots,z_{2k})=0} \tag{276}$$

to T^{2n}, thus gives

$$\begin{aligned}&\left[z_1^{\alpha_1}\cdots z_l^{\alpha_l} z_{k+1}^{\alpha_{k+1}}\cdots z_{k+d}^{\alpha_{k+d}}\right]_n\\ &\quad= (\log p)^{|\alpha|}\sum_{\substack{\lambda=(\lambda_1,\ldots,\lambda_{l+d})\\ |\lambda|\leqslant 2n,\lambda_i\geqslant 1}}\binom{2n}{|\lambda|}(2k-l-d)^{2n-|\lambda|}\frac{\lambda_1^{\alpha_1}\cdots\lambda_l^{\alpha_l}\lambda_{l+1}^{\alpha_{k+1}}\cdots\lambda_{l+d}^{\alpha_{k+d}}}{\alpha_1!\cdots\alpha_l!\alpha_{k+1}!\cdots\alpha_{k+d}!}\frac{|\lambda|!}{\lambda_1!\cdots\lambda_{l+d}!}.\end{aligned} \tag{277}$$

Note that 0^0 is defined to be 1 whenever it occurs. Thus,

$$\begin{aligned}&\big[z_1^{\alpha_1}\cdots z_l^{\alpha_l}z_{k+1}^{\alpha_{k+1}}\cdots z_{k+d}^{\alpha_{k+d}}\big]_n\\&\quad\leqslant(\log p)^{|\alpha|}\sum_{j=l+d}^{2n}\binom{2n}{j}(2k)^{2n-j}\left(1-\frac{l+d}{2k}\right)^{2n-j}e^j\sum_{\substack{\lambda=(\lambda_1,\dots,\lambda_{l+d})\\|\lambda|=j,\lambda_i\geqslant1}}\frac{j!}{\lambda_1!\cdots\lambda_{l+d}!}.\end{aligned}\tag{278}$$

The factor e^j is accounted for by $e^{\lambda_1+\cdots+\lambda_{l+d}}=e^j$, and comparing to the terms obtained by multiplying out the Taylor series for each e^{λ_j}.

By the multinomial theorem, interpreting $(l+d)^j$ to be $(1+1+\cdots+1)^j$, we therefore get

$$\big[z_1^{\alpha_1}\cdots z_l^{\alpha_l}z_{k+1}^{\alpha_{k+1}}\cdots z_{k+d}^{\alpha_{k+d}}\big]_n\leqslant(\log p)^{|\alpha|}\sum_{j=l+d}^{2n}\binom{2n}{j}(2k)^{2n-j}\left(1-\frac{l+d}{2k}\right)^{2n-j}e^j(l+d)^j.\tag{279}$$

From this we deduce, using $\binom{2n}{j}\leqslant2^{2n}$, and relabeling the sum to start at $j=0$, that

$$\begin{aligned}&\big[z_1^{\alpha_1}\cdots z_l^{\alpha_l}z_{k+1}^{\alpha_{k+1}}\cdots z_{k+d}^{\alpha_{k+d}}\big]_n\\&\quad\leqslant(\log p)^{|\alpha|}4^n(2k)^{2n-l-d}e^{l+d}(l+d)^{l+d}\sum_{j=0}^{2n-l-d}\left(1-\frac{l+d}{2k}\right)^{2n-l-d-j}\left(\frac{e(l+d)}{2k}\right)^j.\end{aligned}\tag{280}$$

Hence,

$$\big[z_1^{\alpha_1}\cdots z_l^{\alpha_l}z_{k+1}^{\alpha_{k+1}}\cdots z_{k+d}^{\alpha_{k+d}}\big]_n\leqslant(\log p)^{|\alpha|}8^n(2k)^{2n-l-d}e^{l+d}(l+d)^{l+d}.\tag{281}$$

And so

$$|a_{\alpha,p}|\leqslant\sum_{n=\max\{l,2\}}^{\infty}\frac{1}{p^n}\big[z_1^{\alpha_1}\cdots z_l^{\alpha_l}z_{k+1}^{\alpha_{k+1}}\cdots z_{k+d}^{\alpha_{k+d}}\big]_n\tag{282}$$

$$\leqslant(\log p)^{|\alpha|}e^{l+d}(l+d)^{l+d}\sum_{n=\max\{l,2\}}^{\infty}\frac{e^{2n}32^nk^{2n-l-d}}{p^n}.\tag{283}$$

Choose c in $p>ck^2$ to be $c=64e^2$ say, then

$$|a_{\alpha,p}|\leqslant e^{l+d}(l+d)^{l+d}k^{2\max\{l,2\}-l-d}\frac{(\log p)^{|\alpha|}}{p^{\max\{l,2\}}}.\tag{284}$$

Finally,

$$\sum_{p>ck^2}|a_{\alpha,p}|\leqslant e^{l+d}(l+d)^{l+d}k^{2\max\{l,2\}-l-d}\sum_{p>ck^2}\frac{(\log p)^{|\alpha|}}{p^{\max\{l,2\}}}\tag{285}$$

$$\ll e^{l+d}(l+d)^{l+d}k^{2\max\{l,2\}-l-d}\frac{|\alpha|!}{l!}\frac{(\log ck^2)^{|\alpha|-1}}{(ck^2)^{\max\{l,2\}-1}}\tag{286}$$

$$\ll\big(32|\alpha|\big)^{|\alpha|}(\log k)^{|\alpha|-1}k^{2-l-d}.\tag{287}$$

In summary, the contribution to a_α of the combinatorial sum corresponding to "the large primes" is

$$\ll \big(32|\alpha|\big)^{|\alpha|}(\log k)^{|\alpha|-1}k^{2-m(\alpha)}. \tag{288}$$

6.2.2. The convergence factor sum

We wish to bound the Taylor coefficients (about zero) of

$$\sum_{p>ck^2}\left[\frac{S_{1,p}}{p}+\sum_{i,j=1}^{k}\log\left(1-\frac{p^{z_{k+j}-z_i}}{p}\right)\right]=:\sum_{p>ck^2}C_p. \tag{289}$$

Expand $\log(1-w)=-\sum_1^\infty w^m/m$, $w=p^{z_{k+j}-z_i-1}$ and cancel the $S_{1,p}/p$ term with the $m=1$ term to get

$$C_p=-\sum_{m=2}^{\infty}\frac{1}{m}\sum_{i,j=1}^{k}\frac{p^{m(z_{k+j}-z_i)}}{p^m}. \tag{290}$$

The Taylor coefficients of a local factor C_p are zero except for the coefficients of monomials of the type z_i^u, with $1\leqslant i\leqslant 2k$ (case $m(\alpha)=1$), or $z_i^u z_{k+j}^v$, with $1\leqslant i,j\leqslant k$ (case $m(\alpha)=2$). Here $u,v\in\mathbb{Z}_{\geqslant 0}$. So, by symmetry, it is enough to consider the monomials z_1^u and $z_1^u z_{k+1}^v$.

We deal with the case $m(\alpha)=1$ first. So, let $a_{\alpha,p}$ denote the coefficient of z_1^u in C_p, where

$$\alpha=(u,0,\dots,0),\quad u\in\mathbb{Z}_{\geqslant 0}. \tag{291}$$

Then,

$$|a_{\alpha,p}|\leqslant k(\log p)^u\sum_{m=2}^{\infty}\frac{m^u}{u!p^m}\leqslant\frac{10k(\log p)^u}{p^2}. \tag{292}$$

Therefore, when $m(\alpha)=1$, the contribution to a_α of the convergence factor sum corresponding to "the large primes" is

$$\ll k\sum_{p>ck^2}\frac{(\log p)^{|\alpha|}}{p^2}\ll\frac{|\alpha|!(4\log k)^{|\alpha|-1}}{k}. \tag{293}$$

The latter inequality follows by comparing the sum to $\int_{ck^2}^\infty \log(t)^{|\alpha|-1}/t^2\,dt$ (with one less power in the exponent to account for the density of primes), integrating by parts $|\alpha|$ times, and using the assumption that $10\leqslant c\leqslant 1000\leqslant k$:

$$\int_{ck^2}^{\infty}\log(t)^{|\alpha|-1}/t^2\,dt=\big(|\alpha|-1\big)!\sum_{j=0}^{|\alpha|-1}\frac{(\log ck^2)^j}{j!ck^2}\ll|\alpha|!\frac{(4\log k)^{|\alpha|-1}}{k^2}. \tag{294}$$

On the other hand, when $m(\alpha)=2$, the contribution to a_α is

$$\ll\sum_{p>ck^2}\frac{(\log p)^{|\alpha|}}{p^2}\ll\frac{|\alpha|!(4\log k)^{|\alpha|-1}}{k^2}. \tag{295}$$

6.3. Bounding the coefficients of the arithmetic factor

We are now ready to state the main theorem of this section.

Theorem 6.1. *The coefficients a_α in the Taylor expansion*

$$\log A(z_1,\dots,z_{2k}) =: \log a_k + B_k \sum_{i=1}^{k} z_i - z_{k+i} + \sum_{|\alpha|>1} a_\alpha z_1^{\alpha_1}\cdots z_{2k}^{\alpha_{2k}} \tag{296}$$

satisfy

$$a_\alpha \ll \begin{cases} \lambda_2^{|\alpha|}(\log k)^{|\alpha|}k + \lambda_2^{|\alpha|}|\alpha|!(\log k)^{|\alpha|-1}k, & \text{if } m(\alpha)=1, \\ \lambda_2^{|\alpha|}m(\alpha)^{|\alpha|}(\log k)^{|\alpha|} + \lambda_2^{|\alpha|}|\alpha|!(\log k)^{|\alpha|-1}k^{2-m(\alpha)}, & \text{if } m(\alpha)>1 \end{cases} \tag{297}$$

as $k\to\infty$, and uniformly in α, where λ_2 is some absolute constant. More simply, but slightly less precisely,

$$a_\alpha \ll \lambda_2^{|\alpha|}(\log k)^{|\alpha|}\big[m(\alpha)^{|\alpha|}k^{2-\min\{m(\alpha),2\}} + |\alpha|!k^{2-m(\alpha)}\big] \tag{298}$$

as $k\to\infty$. Asymptotic constants are absolute.

Proof. The terms $\lambda_2^{|\alpha|}(\log k)^{|\alpha|}k$ and $\lambda_2^{|\alpha|}m(\alpha)^{|\alpha|}(\log k)^{|\alpha|}$ in (297) come from the small primes, and arise by combining the contributions to a_α of:

- The combinatorial sum for the small primes when $m(\alpha)=1$, (251):

$$\ll \eta_9^{|\alpha|}(\log k)^{|\alpha|-1}k. \tag{299}$$

- The combinatorial sum for the small primes when $m(\alpha)>1$, (249):

$$\ll \eta_8^{|\alpha|}m(\alpha)^{|\alpha|}(\log k)^{|\alpha|}. \tag{300}$$

- The convergence factor sum for the small primes when $m(\alpha)=1$, (257):

$$\ll 50^{|\alpha|}(\log k)^{|\alpha|}k. \tag{301}$$

- The convergence factor sum for the small primes when $m(\alpha)=2$, (261):

$$\ll 50^{|\alpha|}(\log k)^{|\alpha|}. \tag{302}$$

While the terms $\lambda_2^{|\alpha|}|\alpha|!(\log k)^{|\alpha|-1}k$ and $\lambda_2^{|\alpha|}|\alpha|!(\log k)^{|\alpha|-1}k^{2-m(\alpha)}$ in (297) come from the large primes, and arise by combining the contributions to a_α of:

- The combinatorial sum for the large primes when $m(\alpha)\geqslant 1$, (288):

$$\ll 32^{|\alpha|}|\alpha|^{|\alpha|}(\log k)^{|\alpha|-1}k^{2-m(\alpha)}. \tag{303}$$

- The convergence factor sum for the large primes when $m(\alpha)=1$, (293):

$$\ll 4^{|\alpha|}(|\alpha|!)(\log k)^{|\alpha|-1}/k. \tag{304}$$

- The convergence factor sum for the large primes when $m(\alpha) = 2$, (295):

$$\ll 4^{|\alpha|}(|\alpha|!)(\log k)^{|\alpha|-1}/k^2. \tag{305}$$

The $\lambda_2^{|\alpha|}|\alpha|!$ in the statement of the theorem accounts for both the $4^{|\alpha|}|\alpha|!$ in (293) and (295), and, on using Stirling's asymptotic, for the $(32|\alpha|)^{|\alpha|}$ in (288). □

Remark. A review of the previous argument shows the statement of the theorem can be made more precise in the case $m(\alpha) = 1$:

Theorem 6.2. *For α satisfying $m(\alpha) = 1$, define*

$$sgn(\alpha) := \begin{cases} (-1)^{|\alpha|+1}, & \text{if } \alpha_{k+1} = \cdots = \alpha_{2k} = 0, \\ -1, & \text{if } \alpha_1 = \cdots = \alpha_k = 0. \end{cases}$$

Then, with $|\alpha|$ fixed, and as $k \to \infty$, the coefficients a_α satisfy

$$\begin{aligned} a_\alpha &= sgn(\alpha)\frac{k}{|\alpha|!}\left(\sum_{p\leqslant k^2}(\log p)^{|\alpha|}\sum_{n=1}^{\infty}\frac{n^{|\alpha|-1}}{p^n}\right)\left[1+O\left(\frac{1}{\log k}\right)\right] \\ &= sgn(\alpha)\frac{k}{|\alpha|!}\left(\sum_{p\leqslant k^2}\frac{(\log p)^{|\alpha|}}{p}\right)\left[1+O\left(\frac{1}{\log k}\right)\right]. \end{aligned} \tag{306}$$

Asymptotic constants depend only on $|\alpha|$. In particular,

$$B_k = a_{(1,0,\ldots,0)} \sim 2k\log k. \tag{307}$$

Proof. Our plan is to show that, asymptotically as $k \to \infty$ and for $|\alpha|$ fixed, the dominant contribution to the a_α when $m(\alpha) = 1$ comes from the convergence factor sum corresponding to the small primes. Notice this asymptotic is not uniform in α, so it is not of immediate utility in the proof of the main theorem, but it is included here because it might be of independent interest.

To this end, by the symmetry of $A(z_1, \ldots, z_{2k})$ in the first half of the variables $z_1, \ldots, z_k$ and, separately, in the second half $z_{k+1}, \ldots, z_{2k}$, we may assume $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_k$ and $\alpha_{k+1} \geqslant \alpha_{k+2} \geqslant \cdots \geqslant \alpha_{2k}$. Thus, since $m(\alpha) = 1$, then all the α_j's are zero except α_1 or α_{k+1}, but not both.

Consider the case $\alpha_1 \neq 0$ first. Then $\alpha = (|\alpha|, 0, \ldots, 0)$, and a_α is the coefficient of $z_1^{|\alpha|}$ in $A(z_1, \ldots, z_{2k})$. By (252) and (254), the contribution of the convergence factor sum corresponding to the small primes to this coefficient is

$$\frac{k}{|\alpha|}\sum_{p\leqslant ck^2}\frac{\log p}{p} \times \text{Coefficient of } z_1^{|\alpha|-1} \text{ in } \frac{p^{-z_1}}{1-\frac{p^{-z_1}}{p}}, \tag{308}$$

where $10 < c < 1000$. Expanding, we obtain

$$\frac{p^{-z_1}}{1-\frac{p^{-z_1}}{p}} = \sum_{m=1}^{\infty}\frac{p^{-mz_1}}{p^{m-1}} = \sum_{m=1}^{\infty}\frac{1}{p^{m-1}}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}m^r(\log p)^r z_1^r. \tag{309}$$

Singling out the case $r = |\alpha| - 1$ above, we have

$$(308) = (-1)^{|\alpha|-1}\frac{k}{|\alpha|!}\sum_{p\leqslant ck^2}(\log p)^{|\alpha|}\sum_{m=1}^{\infty}\frac{m^{|\alpha|-1}}{p^m}$$
$$= sgn(\alpha)\frac{k}{|\alpha|!}\sum_{p\leqslant ck^2}\frac{(\log p)^{|\alpha|}}{p}\big[1+O(1/\log k)\big], \tag{310}$$

where we used $(-1)^{|\alpha|-1} = (-1)^{|\alpha|+1} = sgn(\alpha)$, $\sum_{p\leqslant ck^2}(\log p)^{|\alpha|}/p \gg \log k$, and (hence)

$$\sum_{p\leqslant ck^2}(\log p)^{|\alpha|}\sum_{m=1}^{\infty}\frac{m^{|\alpha|-1}}{p^m} = \sum_{p\leqslant ck^2}\frac{(\log p)^{|\alpha|}}{p} + O(1)$$
$$= \sum_{p\leqslant ck^2}\frac{(\log p)^{|\alpha|}}{p}\big[1+O(1/\log k)\big]. \tag{311}$$

Also, since c is fixed, we may replace the range of summation $p \leqslant ck^2$ in (310) by $p \leqslant k^2$ without affecting the asymptotic.

The remaining contributions to a_α (which, recall, is the coefficient of $z_1^{|\alpha|}$) come from the combinatorial sum for the small primes, the combinatorial sum for the large primes, and the convergence factor sum for the large primes. But these contributions, which are bounded by (299), (303), and (304), respectively, are asymptotically smaller than (310), as $k\to\infty$ and for $|\alpha|$ fixed, by at least a factor of $1/\log k$. Put together, this yields the asymptotic (306) in the case $\alpha_1 \neq 0$.

Last, the analysis in the case $\alpha_{k+1} \neq 0$ is completely similar except the coefficient of $z_1^{|\alpha|-1}$ in $p^{-z_1}/(1-p^{-z_1}/p)$ in (308) is replaced by the coefficient of $z_{k+1}^{|\alpha|-1}$ in $-p^{z_{k+1}}/(1-p^{z_{k+1}}/p)$, thereby changing $sgn(\alpha)$ to -1. □

7. The product of zetas

Finally, we bound the Taylor coefficients b_α of

$$\log\left(\prod_{i,j=1}^{k}(z_i - z_{k+j})\zeta(1+z_i-z_{k+j})\right) =: \gamma k\sum_{i=1}^{k} z_i - z_{k+i} + \sum_{|\alpha|>1} b_\alpha z_1^{\alpha_1}\cdots z_{2k}^{\alpha_{2k}}. \tag{312}$$

The Taylor coefficients are zero except for those of monomials of the type z_i^u, with $1\leqslant i\leqslant 2k$ (case $m(\alpha)=1$), or $z_i^u z_{k+j}^v$, with $1\leqslant i,j\leqslant k$ (case $m(\alpha)=2$). Here $u,v\in\mathbb{Z}_{\geqslant 0}$. By symmetry, it is enough to consider the monomials z_1^u and $z_1^u z_{k+1}^v$.

We deal with the case $m(\alpha)=1$ first. So, let α be of the form

$$\alpha = (u,0,\ldots,0), \quad u\in\mathbb{Z}_{\geqslant 0}. \tag{313}$$

Setting $z_2 = \cdots = z_{2k} = 0$, the lhs of (312) becomes

$$k\log\big[z_1\zeta(1+z_1)\big] = \gamma k z_1 + \sum_{u=2}^{\infty} b_{(u,0,\ldots,0)} z_1^u. \tag{314}$$

Now, by the well-known Taylor expansion, we have

$$z\zeta(1+z) = 1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\gamma_n z^{n+1}, \tag{315}$$

where the γ_n's are the generalized Euler constants satisfying, $\gamma_0 = \gamma = .577\ldots$, and, see Theorem 2 of [B],

$$|\gamma_n| \leqslant 4\frac{(n-1)!}{\pi^n}, \quad n \geqslant 1. \tag{316}$$

Consider the derivative

$$\frac{d}{dz}\log\big[z\zeta(1+z)\big] = \frac{\sum_{n=0}^{\infty} \frac{(-1)^n(n+1)}{n!}\gamma_n z^n}{1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\gamma_n z^{n+1}}. \tag{317}$$

Note in particular, for $|z| < 1/10$, we have

$$\left|\frac{d}{dz}\log\big[z\zeta(1+z)\big]\right| = \frac{8\sum_{n=0}^{\infty}\frac{1}{(10\pi)^n}}{1 - \frac{4}{10}\sum_{n=0}^{\infty}\frac{1}{(10\pi)^n}} \leqslant 100. \tag{318}$$

So, by Cauchy's estimate, the coefficients d_n in the expansion

$$\log\big[z\zeta(1+z)\big] =: \sum_{m=1}^{\infty} d_n z^n \tag{319}$$

satisfy

$$|d_n| \leqslant 100(10)^n. \tag{320}$$

From which it follows

$$|b_\alpha| \ll k(10)^{|\alpha|}, \quad \text{when } m(\alpha) = 1. \tag{321}$$

Analogous reasoning yields

$$|b_\alpha| \ll (100)^{|\alpha|}, \quad \text{when } m(\alpha) = 2. \tag{322}$$

Put together, we have

Lemma 7.1. *The coefficients b_α in the expansion*

$$\log\left(\prod_{i,j=1}^{k}(z_i - z_{k+j})\zeta(1+z_i-z_{k+j})\right) =: \gamma k \sum_{i=1}^{k} z_i - z_{k+i} + \sum_{|\alpha|>1} b_\alpha z_1^{\alpha_1}\cdots z_{2k}^{\alpha_{2k}} \tag{323}$$

are zero when $m(\alpha) > 2$, otherwise, as $k \to \infty$, and uniformly in α, they satisfy

$$b_\alpha \ll \lambda_3^{|\alpha|} k^{2-m(\alpha)}, \tag{324}$$

where λ_3 is some absolute constant. Asymptotic constants are absolute.

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