Stochastic Processes and their Applications 34 (1990) 171-180
North-Holland

A CHARACTERIZATION OF RANDOM-COEFFICIENT AR(1) MODELS

Klaus PÖTZELBERGER
Institut für Statistik und Ökonometrie, Petersgraben 51, CH-4051 Basel, Switzerland

Received 9 February 1988
Revised 10 May 1989

We give a characterization of random-coefficient autoregressive processes of order 1, using analytical properties of the transition probabilities. As an example we show that these transition probabilities can be used to find solutions of certain integro-differential equations.

random-coefficient AR(1) processes * transition probability

1. Introduction

Following Lawrance and Lewis (1987), there are at least three definitions of an autoregressive process of order one. Let $(X_t)_{t=0,\ldots,T}$ be a stationary, univariate ($\mathbb{R}$-valued) and time-homogenous Markov process. We say that (X_t) is an autoregressive process of order 1 (AR(1)), if a $\lambda \in]-1, 1[$ and a sequence (ε_t) of i.i.d. variables exist, such that

$$X_t = \lambda X_{t-1} + \varepsilon_t. \tag{1}$$

This is the standard definition of an AR(1) process and the definition that will be used in this paper. It is not necessary that for all $\lambda \in]-1, 1[$ a process satisfying (1) can be defined. Two other definitions are given by the expectation

$$E(X_t - \mu_1 | X_{t-1}) = \lambda (X_{t-1} - \mu_1) \tag{2}$$

with $\mu_1 = E(X_t)$ and by the Yule-Walker equation, which is equivalent to

$$\rho_k = \lambda^k, \tag{3}$$

where (ρ_k) is the sequence of autocorrelations. It is a common practice in applications of time-series models to justify model (1) by checking (2) or (3). The hierarchy is, however, the other way round. (1) implies (2) and (2) implies (3).

In this paper, which can be regarded as a successor of Pötzelberger (1987), we shall give a characterization of a class of models that are more general than the class of AR(1) models and satisfy equation (2), called random-coefficient autoregressive processes of order 1. A general reference for random coefficient autoregressive processes is the book by Nicholls and Quinn (1982). Some of these processes have been studied in detail by Dewald and Lewis (1985), Gaver and Lewis (1980), Jacobs and Lewis (1983), Lawrance and Lewis (1981, 1985) and McKenzie (1985).

2. Random coefficient AR(1) processes

In this part of the paper we give the definition of random AR(1) processes and establish some features of AR(1) and random AR(1) processes.

A stationary, univariate and time-homogenous Markov process (X_t) is called a random coefficient AR(1) process of order 1 (RAR(1)), if a sequence of i.i.d. random variables $(\Lambda_t, \varepsilon_t)_t$ exists, with $(\Lambda_t, \varepsilon_t)$ independent of $(X_{t-i})_{i\geq 1}$, such that

$$X_t = \Lambda_t X_{t-1} + \varepsilon_t. \tag{4}$$

We shall assume that $P(\Lambda_t \neq 0) > 0$. For a fixed RAR(1) process (X_t), denote by μ the marginal distribution of X_t and by $T(\cdot)(x)$ the transition probability, given by $E(f(X_t) | X_{t-1} = x) = T(f)(x)$. The domain of T depends on μ. Denote by D and J differentiation and integration, i.e. $Df(x) = f'(x)$ and $Jf(x) = \int_0^x f(t)\, \mathrm{d}t$. If μ has moments of all orders, then the linear operators $D^n T J^n$ are defined on C^∞-functions with compact support. Denote by $H(\mu)$ the subset of $[-1, 1]$ given by $H(\mu) = \{\lambda | \exists \mu^\lambda \colon \mu^\lambda$ is a probability measure and $\check{\mu}(s) = \check{\mu}(\lambda s)\check{\mu}^\lambda(s)\}$, where $\check{\mu}$ denotes the characteristic function of the distribution μ. For a $\lambda \in [-1, 1]$ an AR(1) process satisfying (1) exists, if and only if $\lambda \in H(\mu)$. To see that these conditions are equivalent, we remark that if a stationary process satisfying (1) exists, then $\check{\mu}(s) = \check{\mu}(\lambda s)\check{\mu}^\lambda(s)$, where μ^λ is the distribution of ε_t. On the other hand, let $\check{\mu}(s) = \check{\mu}(\lambda s)\check{\eta}(s)$ for a distribution η. Let $(\varepsilon_t)_{t=1,\ldots}$ be a sequence of independent η-distributed variables, let $X_0 \sim \mu$, independent of (ε_t) and define for $t \geq 1$, X_t recursively through (1). Then $X_t \sim \mu$ for all $t \geq 0$ and (X_t) is thus stationary.

Theorem 1. *Let* (X_t) *be a* RAR(1) *process with marginal nondegenerate distribution* μ. *If* μ *has moments of all orders, then*

(a) *For all* $n \in \mathbb{N}$, *a polynomial* q_n *with* degree$(q_n) = n$, *if* n *is even and* degree$(q_n) = n$ *or* $n-1$, *if* n *is odd, exists, such that*

$$E(X_t^n | X_{t-1}) = q_n(X_{t-1}). \tag{5}$$

(b) *Denote the leading coefficient of* q_n *by* λ_n *if* degree$(q_n) = n$ *and let* $\lambda_n = 0$ *if* degree$(q_n) < n$. *Let the marginal distribution of* Λ_t *be* ν. *Then* supp$(\nu) \subseteq [-1, 1]$ *and*

$$\lambda_n = \int \lambda^n \, \mathrm{d}\nu(\lambda). \tag{6}$$

(X_t) *is an* AR(1) *process if and only if* ν *is a Dirac measure,* $\nu = \delta_{\lambda_1}$.

(c) *For each* $n \in 2\mathbb{N}$ *a transition probability* $T_n(\cdot)\,(x)$ *exists, such that* $\mu \circ T_n = \mu$ *and for* C^∞*-functions* f *with compact support,*

$$T_n(f)(x) = D^n T J^n f(x)/\lambda_n. \tag{7}$$

Furthermore for all $n \in 2\mathbb{N}$, $\lambda_n D^n TJ^n - \lambda_{n+1} D^{n+1} TJ^{n+1}$ *and* $\lambda_n D^n TJ^n + \lambda_{n+1} D^{n+1} TJ^{n+1}$ *are positive on* C^∞*-functions with compact support.*

Proof. Denote the marginal distribution of Λ_t by ν. Let μ^λ be a regular condition distribution of ε_t given $\Lambda_t = \lambda$, so that

$$\int f(x)\,\mathrm{d}\mu^\lambda(x) = E(f(\varepsilon_t)\,|\,\Lambda t = \lambda)$$

and

$$Ef(\varepsilon_t) = \int \int f(x)\,\mathrm{d}\mu^\lambda(x)\,\mathrm{d}\nu(\lambda)$$

for integrable functions f. Then for C^∞-functions f with compact support

$$E(f(X_t)\,|\,X_{t-1} = x) = \int E(f(X_t)\,|\,X_{t-1} = x, \Lambda_t = \lambda)\,\mathrm{d}\nu(\lambda)$$
$$= \int \int f(\lambda + t)\,\mathrm{d}\mu^\lambda(t)\,\mathrm{d}\nu(\lambda). \tag{8}$$

For $f(x) = x^{2n}$ a sequence (f_n) of positive C^∞-functions with compact support exists, such that $f_n \uparrow f$, which implies $E(f_n(X_t)\,|\,X_{t-1}) \uparrow E(f(X_t)\,|\,X_{t-1})$. $\int T(f)(x)\,\mathrm{d}\mu(x) = \int f(x)\,\mathrm{d}\mu(x)$ implies that (8) holds even for polynomials.

To complete the proof of (a) and (b), we show that $\mathrm{supp}(\nu) \subseteq [-1, 1]$. It is easy to see that $\lambda_2 \leqslant 1$: Let $\mu_1 = EX_t$ and $\sigma^2 = \mathrm{Var}\,X_t$. Then

$$\sigma^2 = E((\Lambda_t(X_{t-1} - \mu_1) + \varepsilon_t - \mu_1(1 - \Lambda_t))^2)$$
$$= E(\Lambda_t^2)\sigma^2 + E((\varepsilon_t - \mu_1(1 - \Lambda_t))^2) \geqslant E(\Lambda_t^2)\sigma^2,$$

so that $\lambda_2 = E(\Lambda_t^2) \leqslant 1$. Now suppose that $\Lambda_t^2 > 1$ with probability $\neq 0$. Let $(\tilde{\Lambda}_t, \tilde{\varepsilon}_t)$ be a sequence of i.i.d. random variables with $E(f(\tilde{\Lambda}_t, \tilde{\varepsilon}_t)) = E(f(\Lambda_t, \varepsilon_t)\,|\,\Lambda_t^2 > 1)$. The distribution of $(\tilde{\Lambda}_t, \tilde{\varepsilon}_t)$ is the same as the distribution of $(\Lambda_t, \varepsilon_t)$, conditional on $\Lambda_t^2 > 1$. We then define a RAR(1) process (Y_t) with marginal distribution μ by $Y_0 = X_0$ and $Y_t = \tilde{\Lambda}_t Y_{t-1} + \tilde{\varepsilon}_t$. But $E(\tilde{\Lambda}_t^2) = \int_{\lambda^2 > 1} \lambda^2\,\mathrm{d}\nu(\lambda)/\nu(]-\infty, -1[\,\cup\,]1, \infty[) > 1$, a contradiction.

To prove (c), we observe that for C^∞-functions f with compact support $D^n TJ^n(f)(x) = \int \lambda^n \int f(\lambda x + t)\,\mathrm{d}\mu^\lambda(t)\,\mathrm{d}\nu(\lambda)$. The same argument as above (approximation of positive continuous functions by C^∞-functions with compact support) gives (c). □

3. Characterization of RAR(1) processes

Definition Denote the set of C^{∞}-functions with polynomially bounded derivatives by $\mathcal{O}_{\mathcal{M}}$, i.e. $f \in \mathcal{O}_{\mathcal{M}}$, if for all $n \in \mathbb{N}$ a $C \geq 0$ and a $m \in \mathbb{N}$ exists, such that

$$|D^n f(x)| \leq C(1+x^2)^m.$$

Theorem 2. *Let $(X)_{t=0,\dots}$ be a time homogeneous, stationary and univariate Markov process with nondegenerate marginal distribution μ and transition probability T, given by $T(f)(x) = E(f(X_t) \mid X_{t-1} = x)$. Suppose μ has moments of all orders. If for all $n, m \in \mathbb{N}$, $f \in \mathcal{O}_{\mathcal{M}}$,*

(a) $TJ^n f \in \mathcal{O}_{\mathcal{M}}$,

(b) $$\sum_{k=0}^{m} \binom{m}{k} (-1)^k D^{2n+2k} T J^{2n+2k}$$

is positive on $\mathcal{O}_{\mathcal{M}}$,

(c) *a $\lambda_n \in \mathbb{R}$ exists, such that*

$$\mu \circ D^n T J^n = \lambda_n \mu,$$

then (X_t) is equivalent to a RAR(1) *process, i.e. a* RAR(1) *process (Y_t) exists, such that for all $0 \leq t$, $X_0, \dots, X_t$ and $Y_0, \dots, Y_t$ have the same distribution.*

We shall prove the theorem by establishing a series of lemmas.

Lemma 1. *If $D^m T J^m = 0$ for a $m \in 2\mathbb{N}$, then for $f \in \mathcal{O}_{\mathcal{M}}$,*

$$Tf(x) = \int f(t)\, \mathrm{d}\mu(t). \tag{9}$$

Proof. Let $f_n(x) = x^n$. It is easy to see that Tf_n is a polynomial with degree $(Tf_n) \leq n$. Let m be the smallest positive integer with $D^m T J^m = 0$. For all $f \in \mathcal{O}_{\mathcal{M}}$, reals $a_0(f), \dots, a_{m-1}(f)$ exist, such that $Tf(x) = a_0(f) + a_1(f)x + \dots + a_{m-1}(f)x^{m-1}$. $a_{m-1}(f) = \lim_{x\to\infty} Tf(x)/x^{m-1}$ gives $a_{m-1}(f) \geq 0$ for all $f \in \mathcal{O}_{\mathcal{M}}$, $f \geq 0$ and $a_{m-1}(f) = \lim_{x\to-\infty} Tf(x)/x^{m-1}$ gives $a_{m-1} \leq 0$ for all $f \in \mathcal{O}_{\mathcal{M}}$, $f \geq 0$, so that $a_{m-1}(f) = 0$ for any $f \in \mathcal{O}_{\mathcal{M}}$.

If $m = 2$, then $Tf(x) = a_0(f)$, so that $\mu \circ T = \mu$ implies $Tf(x) = \int f(t)\, \mathrm{d}\mu(t)$.

If $m \geq 4$. Then $a_{m-2}(f) = \lim_{x\to\infty} Tf(x)/x^{m-2}$ gives $a_{m-2}(f) \geq 0$ for all $f \in \mathcal{O}_{\mathcal{M}}, f \geq 0$. A positive measure σ exists, such that $a_{m-2}(f) = \sigma(f)$. degree$(T1) \leq 1$ gives $\sigma(\mathbb{R}) = 0$, so that $\sigma = 0$ on $\mathcal{O}_{\mathcal{M}}$, which implies $a_{m-2}(f) = 0$ for all $f \in \mathcal{O}_{\mathcal{M}}$, which contradicts the choice of m. □

Lemma 2. *Let T not be given by (9). Let $f_n(x) = x^n$. Then for all $n \in \mathbb{N}$ a polynomial q_n exists with $Tf_n = q_n$. If n is even, then q_n has degree n and leading coefficient λ_n. If n is odd, then if $\lambda_n \neq 0$,* degree$(q_n) = n$, *and if $\lambda_n = 0$, then* degree$(q_n) = n - 1$. □

Lemma 3. *A probability density ν exists, such that*

$$\lambda_n = \int \lambda^n \, \mathrm{d}\nu(\lambda).$$

Proof. We have to show that for all sequences $((a_i)_{i=0,\ldots,m})$ of reals

$$\sum_{i,j=0}^{m} a_i a_j \lambda_{i+j} \geq 0.$$

This is the solution of Hamburgers moment problem. See Akhiezer (1965). Let $f_i(x) = x^i$ and $f_i^{(y)}(x) = f_i(yx)$. We have

$$0 \leq T\left(\left(\sum_{i=0}^{m} a_i f_i^{(y)}\right)^2\right)(x) = \sum_{i,j=0}^{m} a_i a_j y^{i+j} q_{i+j}(x).$$

Now let $y = 1/x$. Then, for $x \to \infty$, $y^i q_i(x) \to \lambda_i$, which proves the lemma. □

Lemma 4. *Let ν be given by* (10). *Then* $\operatorname{supp}(\nu) \subseteq [-1, 1]$.

Proof. Let $\mu_1 = EX_t$, $\sigma^2 = \operatorname{Var} X_t$ and $b = E((X_t - \mu_1)^2 | X_{t-1} = \mu_1)$. Then $b \geq 0$. A real a exists, such that $E((X_t - \mu_1)^2 | X_{t-1} = x) = \lambda_2 (x - \mu_1)^2 + a(x - \mu_1) + b$, which implies $\sigma^2 = \lambda_2 \sigma^2 + b \geq \lambda_2 \sigma^2$, so that $\lambda_2 \leq 1$. If $\lambda_2 = 0$, then $\nu = \delta_0$, so that $\operatorname{supp}(\nu) \subseteq [-1, 1]$. If $\lambda_2 > 0$, then $\lambda_n > 0$ for all even n, so that $D^n T J^n / \lambda_n$ is the transition probability of a process that satisfies again (a), (b) and (c) of Theorem 2. For $f_2(x) = x^2$, $D^n T J^n f_2 / \lambda_n$ is a polynomial with leading coefficient λ_{n+2}/λ_n. We conclude that $\lambda_{n+2}/\lambda_n \leq 1$, for all even n, which implies $\operatorname{supp}(\nu) \subseteq [-1, 1]$. □

Lemma 5. *Let for $s \in \mathbb{R}$, $c_s(t) = \mathrm{e}^{\mathrm{i}st}$. Then*

$$Tc_s(x) = \int \mathrm{e}^{\mathrm{i}sx\lambda} \frac{\check{\mu}(s)}{\check{\mu}(\lambda s)} \, \mathrm{d}\nu(\lambda). \tag{10}$$

Proof. If T is given by (9), then $\nu = \delta_0$, so that the lemma holds. Now let T not be given by (9). Let for $\varepsilon > 0$, $f_\varepsilon^{(s)}(x) = \mathrm{e}^{-\varepsilon x^2} Tc_s(x)$. $f_\varepsilon^{(s)}$ is in $\mathcal{S}$, the space of functions with rapid decrease. A $m_{s,\varepsilon}^0(t) \in \mathcal{S}$ exists, such that $f_\varepsilon^{(s)}(x) = \int \mathrm{e}^{itx} m_{s,\varepsilon}^0(t) \, \mathrm{d}t$. The substitution $t = s\lambda$ gives $f_\varepsilon^{(s)}(x) = \int \mathrm{e}^{\mathrm{i}s\lambda x} m_{s,\varepsilon}(\lambda) \, \mathrm{d}\lambda$ for a $m_{s,\varepsilon} \in \mathcal{S}$. Let $k(n,s) = \| m_{s,1/n}^{\mathrm{r},+} \| + \| m_{s,1/n}^{\mathrm{r},-} \| + \| m_{s,1/n}^{\mathrm{i},+} \| + \| m_{s,1/n}^{\mathrm{i},-} \|$, where for example $m_{s,1/n}^{\mathrm{r},+}$ denotes the positive real part of $m_{s,1/n}$. If $\check{\mu}(s) \neq 0$, then for even m,

$$D^m T J^m c_s(x) = \lim_{\varepsilon \to 0} \int \lambda^m \, \mathrm{e}^{\mathrm{i}s\lambda x} m_{s,\varepsilon}(\lambda) \, \mathrm{d}\lambda$$

and

$$\mu \circ D^m T J^m c_s = \lambda_m \check{\mu}(s)$$

imply that $k(n, s) \nrightarrow \infty$. For any countable dense subset $Q \subseteq \mathbb{R}$ with $\check{\mu}(s) \neq 0$ on Q, a complex measure η and a measurable function $h_s(\lambda)$ exist, such that

$$m_{s,1/n}(\lambda)\,\mathrm{d}\lambda \xrightarrow{w^*} h_s(\lambda)\eta\,(\mathrm{d}\lambda).$$

We conclude that $\lambda_n \check{\mu}(s) = \int \lambda^n \check{\mu}(\lambda s) h_s(\lambda)\,\mathrm{d}\eta(\lambda)$, so that for any $s \in Q$, $\check{\mu}(\lambda s) h_s(\lambda)\eta(\mathrm{d}\lambda)$ is a complex measure having the same moments as $\check{\mu}(s)\nu(\mathrm{d}\lambda)$, so that $\operatorname{supp}(\nu) \subseteq [-1, 1]$ implies $\check{\mu}(s)\nu(\mathrm{d}\lambda) = \check{\mu}(\lambda s) h_s(\lambda)\eta(\mathrm{d}\lambda)$, which proves the lemma. □

Lemma 6. *If T is not given by (9), let $K_0 = \{\nu_{n,m} \mid n, m \in \mathbb{N}\}$, where the probability measures $\nu_{n,m}$ are absolutely continuous with respect to ν and the Radon–Nikodym derivative is given by*

$$\frac{\mathrm{d}\nu_{n,m}}{\mathrm{d}\nu}(\lambda) = \lambda^{2n}(1-\lambda^2)^m \Big/ \int \lambda^{2n}(1-\lambda^2)^m\,\mathrm{d}\nu(\lambda).$$

Denote by K the smallest w^*-closed and convex set of Borel measures on $[-1, 1]$ containing K_0. An $\eta \in K$ is called extremal, if for $\alpha \in]0, 1[$, $\eta_1, \eta_2 \in K$, $\eta = \alpha\eta_1 + (1-\alpha)\eta_2$ implies $\eta = \eta_1 = \eta_2$.

Now, if $\eta \in K$ is external, then b, $\alpha \in [0, 1]$ exist, such that

$$\eta = \alpha\delta_{-b} + (1-\alpha)\delta_b.$$

Proof. It is easy to see that $\eta \in K$ with $\operatorname{supp}(\eta) \nsubseteq \{-1, 0, 1\}$ implies η_1, $\eta_2 \in K$, where η_1 and η_2 are absolutely continuous with respect to η and the Radon–Nikodym derivatives are given by

$$\frac{\mathrm{d}\eta_1}{\mathrm{d}\eta}(\lambda) = \lambda^2 \Big/ \int \lambda^2\,\mathrm{d}\eta(\lambda)$$

and

$$\frac{\mathrm{d}\eta_2}{\mathrm{d}\eta}(\lambda) = (1-\lambda^2) \Big/ \int (1-\lambda^2)\,\mathrm{d}\eta(\lambda).$$

Let $\alpha = \int \lambda^2\,\mathrm{d}\eta(\lambda)$. Then $\eta = \alpha\eta_1 + (1-\alpha)\eta_2$, so that the function λ^2 is constant on $\operatorname{supp}(\eta)$, if η is extremal.

It is easy to see that if $\operatorname{supp}(\eta) = \{-1, 0, 1\}$, with $\alpha_1 = \eta(\{-1\})$, $\alpha_2 = \eta(\{0\})$, $\alpha_3 = \eta(\{1\})$, then $\delta_0 \in K$ and $\beta\delta_{-1} + (1-\beta)\delta_1 \in K$, where $\beta = \alpha_1/(\alpha_1 + \alpha_3)$. This proves the lemma. □

Lemma 7. *Let α, $b \in]0, 1[$ and suppose that for all $x \in \mathbb{R}$,*

$$g_x(s) = \alpha\,\mathrm{e}^{-\mathrm{i}sbx}\frac{\check{\mu}(s)}{\check{\mu}(-bs)} + (1-\alpha)\,\mathrm{e}^{\mathrm{i}sbx}\frac{\check{\mu}(s)}{\check{\mu}(bs)}$$

is the characteristic function of a probability measure on $\mathbb{R}$.

Then $\check{\mu}(s)/\check{\mu}(-bs)$ and $\check{\mu}(s)/\check{\mu}(bs)$ are characteristic functions of probability measures on $\mathbb{R}$.

Proof. We shall show that $\check{\mu}(s)/\check{\mu}(bs)$ is a characteristic function. For all $x \in \mathbb{R}$, $h_x(s) = g_x(s)\,\mathrm{e}^{-\mathrm{i}sbx}$ is the characteristic function of a probability measure ρ_x. $2(1-\alpha)\check{\mu}(s)/\check{\mu}(bs) = h_0(s) + h_{\pi/2bs}$ implies that $\check{\mu}(s)/\check{\mu}(bs)$ (and $\check{\mu}(s)/\check{\mu}(-bs)$) is a bounded function of s. Let $f \in \mathcal{C}_{\mathcal{U}}$, $f \geqslant 0$. Then $\check{f}$ is in $\mathcal{C}_{\mathcal{U}}$ and $0 \leqslant \int f(t)\,\mathrm{d}\rho_x(t) = \int \check{f}(-s)\check{\rho}_x(s)\,\mathrm{d}s/2\pi$, so that $\int \check{f}(-s)\,\mathrm{e}^{-2xsb}\check{\mu}(s)/\check{\mu}(-bs)\,\mathrm{d}s \to 0$ for $x \to \infty$ gives $0 \leqslant \int \check{f}(-s)\check{\mu}(s)/\check{\mu}(bs)\,\mathrm{d}s$. A probability measure μ^b exists, such that for $f \in \mathcal{C}_{\mathcal{U}}$, $\int f(t)\,\mathrm{d}\mu^b(t) = \int \check{f}(-s)\check{\mu}(s)/\check{\mu}(bs)\,\mathrm{d}s/2\pi$. We conclude that $\check{\mu}^b(s) = \check{\mu}(s)/\check{\mu}(bs)$. □

Proof of Theorem 2. Lemma 1 to Lemma 7 show that $Tc_s(x) = \int \mathrm{e}^{\mathrm{i}s\lambda x}\check{\mu}^\lambda(s)\,\mathrm{d}\nu(\lambda)$, where $c_s(t) = \mathrm{e}^{\mathrm{i}st}$ and μ^λ is a probability measure on $\mathbb{R}$ with $\check{\mu}^\lambda(s)\check{\mu}(\lambda s) = \check{\mu}(s)$. We define a distribution ξ on $[-1,1]\times\mathbb{R}$ by

$$\xi(B) = \int\int I_B(\lambda, t)\,\mathrm{d}\mu^\lambda(t)\,\mathrm{d}\nu(\lambda) \tag{11}$$

(for any Borel set B). To prove that a RAR(1) process (Y_t) which is equivalent to X_t exists, let $((\Lambda_t, \varepsilon_t)_{t=1,\ldots})$ be a sequence of independent ξ-distributed random variables, independent of X_0 too. Let $Y_0 = X_0$ and for $t \geqslant 1$, $Y_t = \Lambda_t Y_{t-1} + \varepsilon_t$. For any $t \geqslant 0$, $X_0, \ldots, X_t$ and $Y_0, \ldots, Y_t$ have the same distribution. □

4. Example

Section 2 shows how models of AR(1) and RAR(1) processes can be constructed. Let μ be the marginal distribution of the process. Then for any distribution ν with $\operatorname{supp}(\nu) \subseteq H(\mu)$, a RAR(1) process (X_t) exists with $\Lambda_t \sim \nu$ and $(\Lambda_t, \varepsilon_t) \sim \xi$, where Λ_t, ε_t and ξ are given by (4) and (11).

We give an example of a RAR(1) process with uniform marginal distribution. This example is a well-known special case of a stationary Beta distributed process.

Example. Denote by μ the uniform distribution on $[0, 1]$. Then $H(\mu) = \{0\} \cup \{\lambda \mid 1/\lambda$ is an integer$\}$. For $\lambda \in H(\mu)$, $\lambda \neq 0$, the distribution μ^λ, (given by $\check{\mu}^\lambda(s)\check{\mu}(\lambda s) = \check{\mu}(s)$) is discrete. If $\lambda = 1/n$, $n \in \mathbb{N}$, then

$$\mu^\lambda = \frac{1}{n}\sum_{k=0}^{n-1} \delta_{k/n}.$$

if $\lambda = -1/n$, $n \in \mathbb{N}$, then

$$\mu^\lambda = \frac{1}{n}\sum_{k=1}^{n} \delta_{k/n}.$$

For any positive integer p and any $h > 0$, let ν^h be a probability measure on $\{(1/p)^n \mid n \in \mathbb{N}\}$, given by $\nu^h(\{1/p^n\}) = \mathrm{e}^{-h}h^n/n!$ (if $\Lambda_h \sim \nu^h$, then $-\ln \Lambda_h/\ln p$ has a

Poisson distribution with mean h). We define a transition probability T^h by (11), so that for any continuous bounded function f,

$$T^h f(x) = \mathrm{e}^{-h}\left(f(x) + \sum_{n=1}^{\infty}\sum_{k=0}^{p^n-1} f((x+k)/p^n)\frac{h^n}{n!\,p^n}\right). \tag{12}$$

A "continuously indexed" process $(X_t)_{t\in[0,\infty[}$ exists, such that for any continuous bounded function f and $0<h<t$, $E(f(X_t)\,|\,X_{t-h}=x) = T^h f(x)$.

We call a process $(Y_t)_{t\in[0,\infty[}$ a continuously indexed RAR(1) process (CRAR(1)), if for all $0<s<t$, a random variable $(A_{t,s}, \varepsilon_{t,s})$ independent of $(X_r)_{r\leq s}$ exists, such that

$$X_t = A_{t,s}X_s + \varepsilon_{t,s}. \tag{13}$$

We give another method to construct a CRAR(1) process with transition probability given by (12). Let $(P_{t,s})_{0\leq s\leq t}$ be the semigroup of positive operators, defined on bounded continuous functions $f:\mathbb{R}^2\to\mathbb{R}$ by

$$P_{t,s}f(x,y) = \sum_{n=0}^{\infty}\sum_{k=0}^{p^n-1} f(x/p^n, (y+k)/p^n)\frac{(t-s)^n}{n!\,p^n}\mathrm{e}^{-(t-s)}.$$

Then $P_{t,s}\circ P_{s,r} = P_{t,r}$ for any $0<r<s<t$. A process $(A_t, \varepsilon_t)_{t\in[0,\infty[}$ exists, such that for any $0<s<t$, $E(f(A_t,\varepsilon_t)\,|\,A_s=x, \varepsilon_s=y) = P_{t,s}f(x,y)$. Let Y_0 be a random variable, independent of $(A_t, \varepsilon_t)_{t\in[0,\infty[}$ and uniformly distributed on $[0,1]$. For $t>0$, set $Y_t = A_tY_0+\varepsilon_t$. To prove that (Y_t) is a CRAR(1) process, define for $0<s<t$, $A_{t,s}$ and $\varepsilon_{t,s}$ by $A_{t,s}=A_t/A_s$ and $\varepsilon_{t,s}=\varepsilon_t - A_{t,s}\varepsilon_s$. Then $Y_t = A_{t,s}Y_s+\varepsilon_{t,s}$. We have to show that $(A_{t,s}, \varepsilon_{t,s})$ is independent of $(Y_r)_{r\leq s}$. It is sufficient to show that $(A_{t,s}, \varepsilon_{t,s})$ is independent of (A_s, ε_s). We have $P(A_{t,s}\in A, \varepsilon_{t,s}\in B\,|\,A_s=x, \varepsilon_s=y) = P_{t,s}f(x,y)$ (for Borel sets A, B), where $f(u,v) = I_A(u/x)I_B(v-uy/x)$, so that

$$\begin{aligned}P_{t,s}f(x,y) &= \sum_{n=0}^{\infty}\sum_{k=0}^{p^n-1} I_A(p^{-n}x/x)I_B((y+k)/p^n - y/p^n)\frac{(t-s)^n}{n!\,p^n}\mathrm{e}^{-(t-s)}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{p^n-1} I_A(1/p^n)I_B(k/p^n)\frac{(t-s)^n}{n!\,p^n}\mathrm{e}^{-(t-s)}.\end{aligned}$$

Thus $P_{t,s}f(x,y)$ does not depend on (x,y), which implies that $(A_{t,s}, \varepsilon_{t,s})$ is independent of (A_s, ε_s).

For bounded continuous functions f, $(T^hf(x)-f(x))/h$ converges for $h\to 0$ to

$$Gf(x) = -f(x) + \frac{1}{p}\sum_{k=0}^{p-1} f\left(\frac{x+k}{p}\right). \tag{14}$$

G is the infinitesimal generator of the CRAR(1) process. G is an integral operator, in contrast to the infinitesimal generator of an AR(1) process, which has the form

$$\begin{aligned}G_{\mathrm{AR}}f(x) &= -xf'(x) + \alpha f'(x) + \beta^2 f''(x)\\ &\quad + \int (f'(x+t) - f'(x)/(1+t^2))h(t)\,\mathrm{d}t,\end{aligned} \tag{15}$$

for $f \in \mathcal{O}_{\mathcal{M}}$, where α, $\beta \in \mathbb{R}$ and $h(t)$ is a measurable function, nondecreasing in $]0, \infty[$ and in $]-\infty, 0[$.

5. Solution of an integro-differential equation

The example in Section 4 shows that the function $g(t, x) = T^t f(x)$ is a solution of a certain integro-differential equation.

Theorem 3. *Let η be a probability measure on $\mathbb{R}$, $\lambda \in]0, 1[$ and let f be a bounded and continuous function on $\mathbb{R}$. If*

$$\int_{|x|>1} \ln|x| \, \mathrm{d}\eta(x) < \infty, \tag{16}$$

then a solution of the equation

$$\frac{\partial g}{\partial t}(t, x) = -g(t, x) + \int g(t, \lambda x + u) \, \mathrm{d}\eta(u),$$

$$g(0, x) = f(x), \tag{17}$$

is given by $g(t, x) = T^t f(x)$, where T^t is the transition probability of a CRAR(1) *process with marginal distribution μ, given by*

$$\check{\mu}(s) = \prod_{k=0}^{\infty} \check{\eta}(\lambda^k s). \tag{18}$$

Proof. The sequence of probability measures η_n, defined by $\eta_{-1} = \delta_0$ and $\check{\eta}_n(s) = \Pi_{k=0}^{n} \check{\eta}(\lambda^k, s)$, converges to a probability measure μ on $\mathbb{R}$, if and only if (16) holds. A proof can be found in Wolfe (1982). Then for any $k \in \mathbb{N}$, $\lambda^k \in H(\mu)$. Let for any $t > 0$, ν^t be the distribution of λ^{N_t}, where N_t has a Poisson distribution with mean t. Define T^t on bounded continuous functions f by

$$\begin{aligned} T^t f(x) &= \int \int f(\lambda^k x + u) \, \mathrm{d}\eta_{k-1}(u) \, \mathrm{d}N_t(k) \\ &= \sum_{k=0}^{\infty} \int f(\lambda^k x + u) \, \mathrm{d}\eta_{k-1}(u) \frac{t^k}{k!} \mathrm{e}^{-t}. \end{aligned} \tag{19}$$

T^t is the transition probability of a CRAR(1) process. The infinitesimal generator G is then $Gf(x) = -f(x) + \int f(\lambda x + u) \, \mathrm{d}\eta(u)$, so that $g(t, x) = T^t f(x)$ is a solution of (17). □

References

N.I. Akhiezer, The Classical Moment Problem (Oliver and Boyd, Edinburgh, 1965).

L.S. Dewald and P.A.W. Lewis, A new Laplace second order autoregressive time series model-NLAR(2), IEEE Trans. Inform. Theory 31 (1985) 645-651.

D.P. Gaver and P.A.W. Lewis, First-order autoregressive gamma sequences and point processes, Adv. Appl. Probab. 12 (1980) 727–745.

P.A. Jacobs and P.A.W. Lewis, A mixed autoregressive-moving average exponential sequence and point process (EARMA 1, 1). Adv. Appl. Probab. 9 (1977) 87–104.

A.J. Lawrance and P.A.W. Lewis, A new autoregressive time series model in exponential variables (NEAR(1)), Adv. Appl. Probab. 13 (1981) 826–845.

A.J. Lawrance and P.A.W. Lewis, Modelling and residual analysis of nonlinear autoregressive time series in exponential variables, J. Roy. Statist. Soc. Ser. B 47 (1985) 165–202.

A.J. Lawrance and P.A.W. Lewis, Higher-order residual analysis for nonlinear time series with autoregressive correlation structures, Int. Statist. Rev. 55 (1987) 21–35.

E. McKenzie, An autoregressive process for beta random variables, Management Sci. (1985) 988–997.

D.F. Nicholls and B.G. Quinn, Random coefficient autoregressive models: An introduction, in: Lecture Notes on Statistics, Vol. 11 (Springer, New York, 1982).

K. Pötzelberger, Completely positive operators and processes of Ornstein–Uhlenbeck type, Stochastic Process Appl. 25 (1987) 249–254.

S.J. Wolfe, On a continuous analogue of the stochastic difference equation $X_n = \rho X_{n-1} + B_n$, Stochastic Process. Appl. 12 (1982) 301–312.