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ON THE STRUCTURE OF COMPLETE GRAPHS WITHOUT ALTERNATING CYCLES

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The structure of edge-coloured complete graphs K_p which do not contain cycles with adjacent lines differently coloured is completely characterized. This characterization further enables us to prove that every such K_p ($p \geqslant 2$) has exactly one connected monochromatic spanning subgraph. Moreover, assume that edges in K_p are coloured by k colours so that no vertex is on more than Δ edges of each colour and less than δ edges of each colour. If such a K_p exists then $p \leqslant \Delta + 1$ and $p \geqslant k\delta - 1$; whereas, any one of these inequalities will ensure the existence of such a K_p.

1. Introduction

Recently quite a number of articles have been devoted to the study of global or local conditions on the colouring of edges of complete graphs which ensure that a particular kind or class of coloured graphs exist (or do not exist) as subgraphs. For instance, in [1] and [3], the structure of edge-coloured complete graphs with no triangles having edges of different colours is studied; and [1] further provides solutions to some of the problems raised in [2]. In [4], a sufficient condition is given for the existence of cycles of every possible size with adjacent edges differently coloured in an edge-coloured complete graph. Also in [5], some sufficient conditions are obtained to ensure the existence of subgraphs of certain size which contains all the colours in an edge-coloured complete graph.

Throughout this paper, K_p will denote a complete graph on p vertices whose edges are coloured by k colours so that no vertex is on more than Δ edges and less than δ edges of each colour. A cycle in K_p is called an *alternating cycle* if every two adjacent edges in the cycle are differently coloured. The main aim of this paper is to characterize all K_p without alternating cycles. This characterization enables us to prove that every K_p ($p \geqslant 2$) without alternating cycles has exactly one connected monochromatic spanning subgraph. If a K_p without alternating cycles exists then $p \leqslant \Delta + 1$ and $p \geqslant k\delta + 1$; and any one of these inequalities will guarantee the existence of a K_p without alternating cycles.

2. The structure theorem

Let A_i be a complete graph on p_i vertices ($i = 1, \ldots, n$). To each A_i, there is associated a colour c_i such that all edges in A_i are coloured c_i if A_i has more than one element. We shall denote by $A_i(n)$ the edge-coloured complete graph on $p_1+\cdots+p_n$ vertices obtained by first taking the disjoint union of the A_i's and next joining each vertex of A_j to each vertex of A_k by an edge coloured c_j for all $1\leq j<k\leq n$.

Lemma 1. *$A_i(n)$ contains no alternating cycles.*

Proof. Let $a_0a_1\cdots a_n$ be any cycle in $A_i(n$ ih a_j in $A_{i(j)}$ for $j=0,1,\ldots,n$. We assume that $i(1)\leq i(k)$ for all $k=0,1,\ldots$ n. Then from the structure of $A_i(n)$, the colours of a_0a_1 and a_1a_2 are bot . Hence $a_0a_1\cdots a_n$ is not an alternating cycle in $A_i(n)$. This proves Lemma 1.

The main object of this paper is to prove the following:

Theorem 1. *Let K_p be an edge-coloured complete graph on p vertices. Then K_p contains no alternating cycles if and only if there exist complete graphs A_i associated with colours c_i ($i=1,\ldots,n$) such that $c_j\neq c_{j+1}$ for $j=1,\ldots,n-1$ and $K_p=A_i(n)$.*

Proof. The sufficiency follows from Lemma 1.

We shall prove the necessity by induction on p. If $p=1$, then the result is obvious. Assume that the result holds for $p=q\geq 1$ and consider an edge-coloured complete graph K_{q+1} without alternating cycles. Pick any vertex a in K_{q+1} and consider the complete graph K_q obtained from K_{q+1} by deleting a and all edges incident with a. Then K_q contains no alternating cycles and so by induction hypothesis, there exist complete graphs A_i associated with colours c_i ($i=1,\ldots,n$) such that $c_j\neq c_{j+1}$ for $j=1,\ldots,n-1$ and $K_q=A_i(n)$.

If all edges from a to each A_i is coloured c_i, then immediately $K_{q+1}=B_i(n)$ where $B_i=A_i$ for $i=1,\ldots,n-1$, whereas, $B_n=A_n\cup\{a\}$. On the other hand, let j be the smallest number such that a is joined to a vertex a_j in A_j by an edge with colour c different from c_j. Then for any vertex b in A_j, since aba_ja is not an alternating cycle, ab must be coloured either c or c_j. Let B be the set of all vertices y of A_j such that ay is coloured c. We have the following four cases:

Case 1. $j=n$ and $A_j=B$.

In this case, if $c=c_{j-1}$, then $K_{q+1}=B_i(n)$ where $B_i=A_i$ for $i=1,\ldots,j-2$; $B_{j-1}=A_{j-1}\cup\{a\}$; $B_j=A_j$ and the colours associated with B_{j-1} and B_j are respectively c_{j-1} and c_j.

On the other hand, if $c_{j-1}\neq c$, then $K_{q+1}=B_i(n+1)$ where $B_i=A_i$ for $i=1,\ldots,j-1$; $B_j=\{a\}$; $B_{j+1}=A_j$ and the colours associated with B_j and B_{j+1} are respectively c and c_j.

Case 2. $j=n$ and $B\neq A_j$.

In this case, $K_{q+1}=B_i(n+2)$ where $B_i=A_i$ for $i=1,\ldots,j-1$; $B_j=A_j\setminus B$; $B_{j+1}=\{a\}$; $B_{j+2}=B$ and the colours associated with B_j, B_{j+1} and B_{j+2} are respectively c_j, c and c_j.

Case 3. $j<n$ and $B=A_j$.

In this case, for each y in A_{j+1}, the edge ay is coloured either c or c_j since aa_jya is not alternating. Assume first that ay is coloured c_j.

Claim 1. A_{j+1} is a singleton and $j+1=n$.

Indeed, if z is an element of A_{j+1} other than y, then az must have colour c_j or c_{j+1} (otherwise $azya$ is alternating); if az has colour c_j, then aa_jyza is alternating. If az has colour c_{j+1}, then aza_j is alternating unless $c=c_{j+1}$, but in this case aa_jzya is alternating. Hence A_{j+1} is a singleton. Next, if $j+1<n$, then let u be any element in A_n. Then aa_jua and $ayua$ are both non-alternating cycles. So au must be coloured c_j. But then, we would obtain an alternating cycle, viz. aa_jyua, a contradiction. This establishes Claim 1.

From Claim 1, it is clear that $K_{q+1}=B_i(n+1)$ where $B_i=A_i$ for $i=1,\dots,j-1$; $B_j=A_{j+1}$; $B_{j+1}=\{a\}$; $B_{j+2}=B$ and the colours associated with B_j, B_{j+1} and B_{j+2} are respectively c_j, c and c_j.

Next, assume that ay is coloured c.

Claim 2. For each vertex z in A_k, $k=j+1,\dots,n$, the edge az is coloured c.

Indeed, since aa_jza is non-alternating, so az must be coloured either c or c_j. If az is coloured c_j, then aa_jyza is an alternating cycle, a contradiction. Hence az must be coloured c.

With Claim 2, we have $K_{q+1}=B_i(n+1)$ where $B_i=A_i$ for $i=1,\dots,j-1$; $B_j=\{a\}$; $B_{j+1}=A_j$; $B_k=A_{k-1}$ for $k=j+2,\dots,n+1$ and the colours associated with B_j, B_{j+1} are c and c_j respectively (provided that $c_{j-1}\neq c$). However, if $c_{j-1}=c$, then $K_q=B_i(n)$ where $B_i=A_i$ for $i=1,\dots,j-2,j,\dots,n$; $B_{j-1}=A_{j-1}\cup\{a\}$ and the colour associated with B_{j-1} is c.

Case 4. $j<n$ and $B\neq A_j$.

In this case, for each y in A_{j+1}, the edge ay is coloured either c_j or c. If ay is coloured c_j, by Claim 1, A_{j+1} is a singleton and $j+1=n$. Hence $K_{q+1}=B_i(n+1)$ where $B_i=A_i$ for $i=1,\dots,j-1$; $B_j=(A_j\setminus B)\cup A_{j+1}$; $B_{j+1}=\{a\}$ and $B_{j+2}=B$ and the colours associated with B_j, B_{j+1} and B_{j+2} are respectively c_j, c and c_j.

On the other hand if ay is coloured c, then by Claim 2, az is coloured c for all z in A_k, $k=j+1,\dots,n$. Then $K_{q+1}=B_i(n+2)$ where $B_i=A_i$ for $i=1,\dots,j-1$; $B_j=A_j\setminus B$; $B_{j+1}=\{a\}$; $B_{j+2}=B$; $B_k=A_{k-2}$ for $k=j+3,\dots,n+2$ and the colours associated with B_j, B_{j+1}, and B_{j+2} are respectively c_j, c and c_j.

Now, since all four possible cases have been settled, the proof of Theorem 1 is thus complete.

3. A few applications

As claimed in the first section, we shall derive the following consequences out of Theorem 1.

Theorem 2. *Every edge-coloured complete graph* K_p $(p\geq 2)$ *without alternating cycles contains exactly one connected monochromatic spanning subgraph.*

Proof. By Theorem 1, $K_q = A_i(n)$. Consider the subgraph induced by edges coloured c_1. Then from the construction of $A_i(n)$ it follows that this is the only connected monochromatic spanning subgraph of K_p.

Theorem 3. *There exists a complete graph K_p without alternating cycles whose edges are coloured with k colours such that no vertex is on less than δ edges of each colour if and only if $p \geq k\delta + 1$.*

Proof. Assume that $p \geq k\delta + 1$. Let $K_p = A_i(k)$ where $|A_i| = \delta$ for $i = 1, \ldots, k-1$; $|A_k| = p - (k-1)\delta \geq \delta + 1$ and the colours associated with each A_i is c_i $(c_i \neq c_{i+1})$. It then follows from the structure of $A_i(k)$ that each vertex is incident with at least δ edges of each colour.

Next assume that there exists a complete graph K_p without alternating cycles whose edges are coloured with k colours such that no vertex is on less than δ edges of each colour. Then by Theorem 1, $K_p = A_i(n)$ with $n = k$ or $k+1$. Assume first that $n = k$. Let a be any vertex in A_n. Since a must be incident with at least δ edges of colour c_i $(i = 1, \ldots, n)$, we must have $|A_i| \geq \delta$ for $i = 1, \ldots, n-1$ and $|A_k| \geq \delta + 1$. Therefore

$$p = |A_1| + \cdots + |A_k| \geq (k-1)\delta + \delta + 1 = k\delta + 1.$$

On the other hand, if $n = k+1$, then A_n must be a singleton $\{a\}$, or else we would have $k+1$ colours. Again since a is incident with at least δ edges of colour c_i $(i = 1, \ldots, k)$, we must have $|A_i| \geq \delta$ for all $i = 1, \ldots, k$. Therefore, $p = |A_1| + \cdots + |A_k| + |A_n| \geq k\delta + 1$, completing the proof.

Theorem 4. *There exists an edge-coloured complete graph K_p without alternating cycles such that no vertex is on more than Δ edges of each colour if and only if $p \leq \Delta + 1$.*

Proof. Assume that $p \leq \Delta + 1$. Consider the complete graph K_p with all edges coloured c_1. Then K_p has no alternating cycles and every vertex is on $p - 1 \leq \Delta$ edges of each colour.

Next assume that there exists an edge-coloured complete graph K_p without alternating cycles such that no vertex is on more than Δ edges of each colour. Then by Theorem 1, $K_p = A_i(n)$. Let a be any vertex of A_1. Then a is on $p-1$ edges of colour c_1. Hence $p - 1 \leq \Delta$ which implies that $p \leq \Delta + 1$, as required.

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