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Multiobjective Control Problem with Generalized Invexity

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A multiobjective control problem is introduced. The definition of invexity for continuous functions is extended to ρ-invexity, ρ-pseudoinvexity, and ρ-quasi-invexity. Duality results are established for Wolfe as well as Mond–Wier type duals. © 1995 Academic Press, Inc.

1. INTRODUCTION AND PRELIMINARIES

In this paper we introduce the following multiobjective control problem:

$$\text{(VCP)}\quad \text{Minimize}\left[\int_a^b f_1(t, x, y)\,dt,\ \int_a^b f_2(t, x, y)\,dt,\ \ldots,\ \int_a^b f_r(t, x, y)\,dt\right]$$

$$\text{subject to } x(a) = \alpha \qquad x(b) = \beta \tag{1}$$

$$\overset{\circ}{x} = h(t, x, y) \qquad t \in I \tag{2}$$

$$g(t, x, y) \leq 0 \qquad t \in I. \tag{3}$$

Here R^n denotes an n-dimensional euclidean space and $I = [a, b]$ is a real interval. Each f_i: $I \times R^n \times R^m \to R$ for $i = 1, 2, \ldots, r$, g: $I \times R^n \times$

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$R^m \to R^p$, and h: $I \times R^n \times R^m \to R^n$ is a continuously differentiable function.

Let x: $I \to R^n$ be differentiable with its derivative $\mathring{x}$, and let y: $I \to R^m$ be a smooth function. Denote the partial derivatives of f by f_t, f_x, and f_y, where

$$f_t = \frac{\partial f}{\partial t},\ f_x = \left[\frac{\partial f}{\partial x^1}, \frac{\partial f}{\partial x^2}, \ldots, \frac{\partial f}{\partial x^n}\right],\ f_y = \left[\frac{\partial f}{\partial y^1}, \frac{\partial f}{\partial y^2}, \ldots, \frac{\partial f}{\partial y^m}\right],$$

where superscripts denote the vector components.

Similarly we have h_t, h_x, h_y and g_t, g_x, g_y. X is the space of continuously differentiable state functions x: $I \to R^n$ such that $x(a) = \alpha$ and $x(b) = \beta$ and is equipped with the norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, and Y, the space of piecewise continuous control functions y: $I \to R^m$, has the uniform norm $\|\cdot\|_\infty$. The differential equation (2) with initial conditions expressed as $x(t) = x(a) + \int_a^b h(s, x(s), y(s))\, ds$, $t \in I$ may be written as $H_x = H(x, y)$, where H: $X \times Y \to C(I, R^n)$, $C(I, R^n)$ being the space of continuous functions from I to R^n defined as $H(x, y)(t) = h(t, x(t), y(t))$. Two duals for (VCP) are proposed and duality relationships are established under generalized ρ-invexity assumptions:

Wolfe Vector Control Dual

$$\text{(WVCD) Maximize } \left[\int_a^b \{f_1(t, u, v) + w(t)^{\mathrm{T}} g(t, u, v)\}\, dt, \ldots, \int_a^b \{f_r(t, u, v) + w(t)^{\mathrm{T}} g(t, u, v)\}\, dt\right]$$

subject to $$x(a) = \alpha \qquad x(b) = \beta \tag{4}$$

$$\sum_{i=1}^r \lambda_i f_{iu}(t, u, v) + w(t)^{\mathrm{T}} g_u(t, u, v) + z(t)^{\mathrm{T}} h_u(t, u, v) + \mathring{z}(t) = 0 \qquad t \in I \tag{5}$$

$$\sum_{i=1}^r \lambda_i f_{iv}(t, u, v) + w(t)^{\mathrm{T}} g_v(t, u, v) + z(t)^{\mathrm{T}} h_v(t, u, v) = 0 \qquad t \in I \tag{6}$$

$$\int_a^b z(t)^{\mathrm{T}}[h(t, u, v) - \mathring{u}(t)]\, dt \geq 0 \qquad t \in I \tag{7}$$

$$w(t) \geq 0 \qquad t \in I \tag{8}$$

$$\lambda_i \geq 0,\ i = 1, 2, \ldots, r,\ \sum_{i=1}^r \lambda_i = 1; \tag{9}$$

Mond–Wier Vector Control Dual

(MVCD)

$$\text{Maximize} \left[\int_a^b f_1(t, u, v)\, dt, \int_a^b f_2(t, u, v)\, dt, \ldots, \int_a^b f_r(t, u, v)\, dt\right]$$

$$\text{subject to } x(a) = \alpha \qquad x(b) = \beta \tag{10}$$

$$\sum_{i=1}^{r} \lambda_i f_{iu}(t, u, v) + w(t)^{\mathrm{T}} g_u(t, u, v) + z(t)^{\mathrm{T}} h_u(t, u, v) + \mathring{z}(t) = 0 \qquad t \in I \tag{11}$$

$$\sum_{i=1}^{r} \lambda_i f_{iv}(t, u, v) + w(t)^{\mathrm{T}} g_v(t, u, v) + z(t)^{\mathrm{T}} h_v(t, u, v) = 0 \qquad t \in I \tag{12}$$

$$\int_a^b z(t)^{\mathrm{T}}[h(t, u, v) - \mathring{u}(t)]\, dt \geq 0 \qquad t \in I \tag{13}$$

$$\int_a^b w(t)^{\mathrm{T}} g(t, u, v)\, dt \geq 0 \qquad t \in I \tag{14}$$

$$w(t) \geq 0 \qquad t \in I \tag{15}$$

$$\lambda_i \geq 0,\ i = 1, 2, \ldots, r,\ \sum_{i=1}^{r} \lambda_i = 1. \tag{16}$$

Optimization in VCP, WVCD, and MVCD means obtaining efficient solutions for the corresponding programs.

DEFINITION 1. A feasible solution (x^0, y^0) for VCP is efficient for VCP if and only if there is no other feasible (x, y) for VCP such that

$$\int_a^b f_i(t, x, y)\, dt < \int_a^b f_i(t, x^0, y^0)\, dt \qquad \text{for some } i \in \{1, 2, \ldots, r\} \tag{17}$$

$$\int_a^b f_j(t, x, y)\, dt \leq \int_a^b f_j(t, x^0, y^0)\, dt \qquad \text{for all } j \in \{1, 2, \ldots, r\}. \tag{18}$$

In the case of maximization, the signs of inequalities (17) and (18) are reversed. We need the following information for the proofs of strong duality results.

LEMMA 1 (Chankong and Haimes [2]). *(x^0, y^0) is an efficient solution for* VCP *if and only if (x^0, y^0) solves $P_k(x^0, y^0)$ for all $k = 1, 2, \ldots, r$, defined as*

$$P_k(x^0, y^0)\text{: Minimize } \int_a^b f_k(t, x, y)\, dt$$

$$\begin{aligned} \text{subject to } x(a) = \alpha, \qquad & x(b) = \beta \\ & \mathring{x} = h(t, x, y) \\ & g(t, x, y) \leq 0 \\ & f_j(t, x, y) \leq f_j(t, x^0, y^0), \end{aligned}$$

for all $j \in \{1, 2, \ldots, r\}, j \neq k$.

Chandra *et al.* [1] gave the Fritz–John necessary optimization conditions for the existence of an extremal solution for the single objective control problem (CP):

$$\text{(CP) Minimize } \int_a^b f(t, x, y)\, dt$$

$$\begin{aligned} \text{subject to} \qquad & x = h(t, x, y) \\ & g(t, x, y) \leq 0 \end{aligned}$$

where f, g, h are as defined earlier.

Mond and Hanson [3] pointed out that if the optimal solution for the CP is normal, then Fritz–John conditions reduce to Kuhn–Tucker conditions.

THEOREM 1 (Kuhn–Tucker Necessary Optimality Conditions). *If* $(x^0, y^0) \in X \times Y$ *solves* CP, *if the Fréchet derivative* $[D - H_x(x^0, y^0)]$ *is surjective, and if the optimal solution* (x^0, y^0) *is normal, then there exist piecewise smooth* $w^0\colon I \to R^p$ *and* $z^0\colon I \to R^n$, *satisfying the following for all* $t \in I$:

$$f_x(t, \mathring{x}, \mathring{y}) + w^0(t)^{\mathrm{T}} g_x(t, x^0, y^0) + z^0(t)^{\mathrm{T}} h_x(t, x^0, y^0) + \mathring{z}^0(t) = 0 \quad (19)$$

$$f_y(t, x^0, y^0) + w^0(t)^{\mathrm{T}} g_y(t, x^0, y^0) + z^0(t)^{\mathrm{T}} h_y(t, x^0, y^0) = 0 \quad (20)$$

$$w^0(t)^{\mathrm{T}} g(t, x^0, y^0) = 0 \quad (21)$$

$$w^0(t) \geq 0. \quad (22)$$

Let $\phi\colon X \to R$ defined by $\phi(x) = \int_a^b f(t, x, \mathring{x})\, dt$ be Fréchet differentiable. Let there exist functions $\eta\colon I \times X \times X \to R$ with $\eta(t, x, x) = 0$ and $\xi\colon I \times X \times X \to R$ and a real number ρ.

DEFINITION 2. A functional $\phi(x)$ is said to be ρ-pseudoinvex at $\bar{x}$ with respect to functions η and ξ, if for all $x \in X$,

$$\int_a^b \{\eta(t, x, \bar{x}) f_x(t, x, \mathring{x}) + \frac{d}{dt} \eta(t, x, \bar{x}) f_{\mathring{x}}(t, x, \mathring{x})\}\, dt \geq -\rho \|\xi(t, x, \bar{x})\|^2$$

$$\Rightarrow \phi(x) \geq \phi(\bar{x})$$

or equivalently

$$\phi(x) < \phi(\bar{x})$$

$$\Rightarrow \int_a^b \{\eta(t, x, \bar{x}) f_x(t, x, \mathring{x})$$

$$+ \frac{d}{dt} \eta(t, x, \bar{x}) f_{\mathring{x}}(t, x, \mathring{x})\}\, dt < -\rho \|\xi(t, x, \bar{x})\|^2.$$

DEFINITION 3. A functional $\phi(x)$ is said to be ρ-strictly pseudoinvex at $\bar{x}$ with respect to functions η and ξ, if for all $x \in X$,

$$\int_a^b \{\eta(t, x, \bar{x}) f_x(t, x, \mathring{x}) + \frac{d}{dt} \eta(t, x, \bar{x}) f_{\mathring{x}}(t, x, \mathring{x})\}\, dt \geq -\rho \|\xi(t, x, \bar{x})\|^2$$

$$\Rightarrow \phi(x) > \phi(\bar{x})$$

or equivalently

$$\phi(x) \leq \phi(\bar{x})$$

$$\Rightarrow \int_a^b \{\eta(t, x, \bar{x}) f_x(t, x, \mathring{x})$$

$$+ \frac{d}{dt} \eta(t, x, \bar{x}) f_{\mathring{x}}(t, x, \mathring{x})\}\, dt < -\rho \|\xi(t, x, \bar{x})\|^2.$$

DEFINITION 4. A functional $\phi(x)$ is said to be ρ-quasiinvex at $\bar{x}$ with respect to functions η and ξ, if for all $x \in X$,

$$\phi(x) \leq \phi(\bar{x})$$

$$\Rightarrow \int_a^b \{\eta(t, x, \bar{x}) f_x(t, x, \mathring{x})$$

$$+ \frac{d}{dt} \eta(t, x, \bar{x}) f_{\mathring{x}}(t, x, \mathring{x})\}\, dt \leq -\rho \|\xi(t, x, \bar{x})\|^2$$

or equivalently

$$\int_a^b \{\eta(t, x, \bar{x})f_x(t, x, \mathring{x}) + \frac{d}{dt}\eta(t, x, \bar{x})f_{\mathring{x}}(t, x, \mathring{x})\}\, dt > -\rho\|\xi(t, x, \bar{x})\|^2$$

$$\Rightarrow \phi(x) > \phi(\bar{x}).$$

DEFINITION 5. A functional $\phi(x)$ is said to be ρ-strictly quasiinvex at $\bar{x}$ with respect to functions η and ξ, if for all $x \in X$,

$$\phi(x) \leq \phi(\bar{x})$$

$$\Rightarrow \int_a^b \{\eta(t, x, \bar{x})f_x(t, x, \mathring{x})$$

$$+ \frac{d}{dt}\eta(t, x, \bar{x})f_{\mathring{x}}(t, x, \mathring{x})\}\, dt < -\rho\|\xi(t, x, \bar{x})\|^2$$

or equivalently

$$\int_a^b \{\eta(t, x, \bar{x})f_x(t, x, \mathring{x}) + \frac{d}{dt}\eta(t, x, \bar{x})f_{\mathring{x}}(t, x, \mathring{x})\}\, dt \geq -\rho\|\xi(t, x, \bar{x})\|^2$$

$$\Rightarrow \phi(x) > \phi(\bar{x}).$$

We write the functional $\phi(x)$ to be ρ-PIX, ρ-SPIX, ρ-QIX, and ρ-SQIX if $\phi(x)$ is ρ-pseudoinvex, ρ-strictly pseudoinvex, ρ-quasiinvex, and ρ-strictly quasiinvex, respectively, at each point of X.

2. DUALITY BETWEEN VCP AND WVCD

THEOREM 2 (Weak Duality). *Assume that for all feasible* (x, y) *for* VCP *and all feasible* (u, v, w, z, λ) *for* WVCD:

(i) $f_i(t, x, y) + w(t)^T g(t, x, y)$ *is* ρ_i-PIX *with respect to functions* η_1, ξ_1 *and* ρ_i'-SPIY *with respect to functions* η_2, ξ_2 *(or* ρ_i'-SPIX *with respect to functions* η_1, ξ_1 *and* ρ_i'-PIY *with respect to functions* η_2, ξ_2*), for all* $i \in \{1, 2, \ldots, r\}$.

(ii) $z(t)^T[h(t, x, y) - \mathring{x}(t)]$ *is* σ-QIX *with respect to the same functions* η_1, ξ_1, *and* σ'-QIY *with respect to the same functions* η_2, ξ_2.

(iii) $\sum_{i=1}^r \lambda_i\rho_i + \sigma \geq o$ *and* $\sum_{i=1}^r \lambda_i\rho_i' + \sigma' \geq o$.

Then the following cannot hold:

$$\int_a^b f_i(t, x, y)\, dt < \int_a^b \{f_i(t, u, v) + w(t)^{\mathrm{T}} g(t, u, v)\}\, dt \qquad \textit{for some } i \in \{1, 2, \ldots, r\} \quad (23)$$

$$\int_a^b f_j(t, x, y)\, dt \leq \int_a^b \{f_j(t, u, v) + w(t)^{\mathrm{T}} g(t, u, v)\}\, dt \qquad \textit{for all } j \in \{1, 2, \ldots, r\}. \quad (24)$$

Proof. Suppose contrary to the result that (23) and (24) hold. Since (x, y) is feasible for VCP and (u, v, w, x, λ) is feasible for WVCD, it follows from (3), (8), (23), and (24) that

$$\int_a^b \{f_i(t, x, y) + w(t)^{\mathrm{T}} g(t, x, y)\, dt\} < \int_a^b \{f_i(t, u, v) + w(t)^{\mathrm{T}} g(t, u, v)\}\, dt \qquad \text{for some } i \in \{1, 2, \ldots, r\} \quad (25)$$

$$\int_a^b \{f_j(t, x, y) + w(t)^{\mathrm{T}} g(t, x, y)\, dt\} \leq \int_a^b \{f_j(t, u, v) + w(t)^{\mathrm{T}} g(t, u, v)\}\, dt \qquad \text{for all } j \in \{1, 2, \ldots, r\}. \quad (26)$$

Using (i) and relations (25) and (26) we get

$$\begin{aligned}\int_a^b &\{\eta_1(t, x, u)[f_{iu}(t, u, v) + w(t)^{\mathrm{T}} g_u(t, u, v)] \\ &+ \{\eta_2(t, y, u)[f_{iv}(t, u, v) + w(t)^{\mathrm{T}} g_v(t, u, v)]\}\, dt \\ &< -\rho_i \|\xi_1(t, x, u)\|^2 - \rho_i' \|\xi_2(t, y, v)\|^2 \qquad \text{for all } i \in \{1, 2, \ldots, r\}. \quad (27)\end{aligned}$$

Multiply each inequality of (27) by $\lambda_1 \geq 0$, $i = 1, 2, \ldots, r$, and add

$$\begin{aligned}\int_a^b &\left\{\eta_1(t, x, u)\left[\sum_{i=1}^r \lambda_i f_{iu}(t, u, v) + w(t)^{\mathrm{T}} g_u(t, u, v)\right]\right. \\ &\left. + \eta_2(t, y, v)\left[\sum_{i=1}^r \lambda_i f_{iv}(t, u, v) + w(t)^{\mathrm{T}} g_v(t, u, v)\right]\right\} dt \quad (28) \\ &< -\sum_{i=1}^r \lambda_1 \rho_i \|\xi_1(t, x, u)\|^2 - \sum_{i=1}^r \lambda_i \rho_i' \|\xi_2(t, y, v)\|^2.\end{aligned}$$

From (2) and (7) it follows that

$$\int_a^b z(t)^{\mathrm{T}}[h(t, x, y) - \mathring{x}(t)]\, dt \leq \int_a^b z(t)^{\mathrm{T}}[h(t, u, v) - \mathring{u}(t)]\, dt.$$

Now using (ii) we get

$$\begin{aligned}\int_a^b \Big\{\eta_1(t, x, u)z(t)^{\mathrm{T}}h_u(t, u, v) &- \frac{d}{dt}\eta_1(t, x, u)z(t)\\ &+ \eta_2(t, y, v)z(t)^{\mathrm{T}}h_v(t, u, v)\Big\}\, dt \qquad (29)\\ &\leq -\sigma\|\xi_1(t, x, u)\|^2 - \sigma'\|\xi_2(t, y, v)\|^2.\end{aligned}$$

By integrating $(d/dt)\eta_1(t, x, u)z(t)$ from a to b by parts and applying the boundary conditions (1) we have

$$\int_a^b \frac{d}{dt}\eta_1(t, x, u)z(t)\, dt = -\int_a^b \eta_1(t, x, u)\mathring{z}(t)\, dt. \qquad (30)$$

Using (30) in (29) we get

$$\begin{aligned}&\int_a^b \{\eta_1(t, x, u))[z(t)^{\mathrm{T}}h_u(t, u, v) + \mathring{z}(t)] + \eta_2(t, y, v))[z(t)^{\mathrm{T}}h_v(t, u, v)]\}\, dt\\ &\quad \leq -\sigma\|\xi_1(t, x, u)\|^2 - \sigma'\|\xi_2(t, y, v)\|^2. \qquad (31)\end{aligned}$$

Adding (28) and (31),

$$\begin{aligned}\int_a^b \Big\{\eta_1(t, x, u)\Big[\sum_{i=1}^r \lambda_i f_{iu}(t, u, v) &+ w(t)^{\mathrm{T}}g_u(t, u, v)\\ &+ z(t)^{\mathrm{T}}h_u(t, u, v) + \mathring{z}(t)\Big]\\ &+ \eta_2(t, y, v)\Big[\sum_{i=1}^r \lambda_i f_{iv}(t, u, v) + w(t)^{\mathrm{T}}g_v(t, u, v)\\ &+ z(t)^{\mathrm{T}}h_v(t, u, v)\Big]\Big\}\, dt\\ < -\Big(\sum_{i=1}^r \lambda_i\rho_i + \sigma\Big)\|\xi_1(t, x, u)\|^2 &- \Big(\sum_{i=1}^r \lambda_i\rho_i' + \sigma'\Big)\|\xi_2(t, y, v)\|^2.\end{aligned}$$

Hence by using (iii) we have

$$\int_a^b \left\{\eta_1(t, x, u)\left[\sum_{i=1}^r \lambda_i f_{iu}(t, u, v) + w(t)^T g_u(t, u, v) + z(t)^T h_u(t, u, v) + \overset{\circ}{z}(t)\right] + \eta_2(t, y, v)\left[\sum_{i=1}^r \lambda_i f_{iv}(t, u, v) + w(t)^T g_v(t, u, v) + z(t)^T h_v(t, u, v)\right]\right\} dt < 0.$$

This contradicts (5) and (6). The result follows.

Remark. The weak duality theorem also holds good under the following different types of assumption:

(a)(i) $f_i(t, x, y) + w(t)^T g(t, x, y)$ is ρ_i-QIX with respect to the functions η , ξ_1 and ρ_i'-QIY with respect to functions η_2, ξ_2 for all $i \in \{1, 2, \ldots, r\}$.

(ii) $z(t)^T[h(t, x, y) - \overset{\circ}{x}(t)]$ is σ-QIX with respect to the same functions η_1, ξ_1 and σ'-SQIY with respect to the same functions η_2, ξ_2 (or σ-SQIX with respect to the same functions η_1, ξ_1 and σ'-QIY with respect to the same functions η_2, ξ_2).

(iii) $\sum_{i=1}^r \lambda_i \rho_i + \sigma \geq 0$ and $\sum_{i=1}^r \lambda_i \rho_i' + \sigma' \geq 0$.

(b)(i) $f_i(t, x, y) + w(t)^T g(t, x, y)$ is ρ_i-QIX with respect to functions η_1, ξ_1 and ρ_i'-QIY with respect to functions η_2, ξ_2, for all $i \in \{1, 2, \ldots, r\}$.

(ii) $z(t)^T[h(t, x, y) - \overset{\circ}{x}(t)]$ is σ-QIX with respect to same functions η , ξ_1 and σ'-QIY with respect to the same functions η_2, ξ_2,

(iii) $$\sum_{i=1}^r \lambda_i \rho_i + \sigma \geq 0 \quad \text{and} \quad \sum_{i=1}^r \lambda_i \rho_i' + \sigma' > 0$$

or

$$\sum_{i=1}^r \lambda_i \rho_i + \sigma > 0 \quad \text{and} \quad \sum_{i=1}^r \lambda_i \rho_i' + \sigma' \geq 0.$$

(c)(i) $\lambda_i > o$ for all $i \in \{1, 2, \ldots, r\}$.

(ii) $\sum_{i=1}^r \lambda_i f_i(t, x, y) + w(t)^T g(t, x, y) + z(t)^T[h(t, x, y) - \overset{\circ}{x}(t)]$ is ρ-PIX with respect to functions η_1, ξ_1 and ρ'-PIY with respect to function η_2, ξ_2.

(iii) $\rho \geq 0$ and $\rho' \geq 0$.

(d)(i) $\sum_{i=1}^{r} \lambda_i f_i(t, x, y) + w(t)^{\mathrm{T}} g(t, x, y) + z(t)^{\mathrm{T}}[h(t, x, y) - \mathring{x}(t)]$ is ρ-PIX with respect to function η_1, ξ_1 and ρ'-SPIY with respect to functions η_2, ξ_2 (or ρ-SPIX with respect to functions η_1, ξ_1 and ρ'-PIY with respect to functions η_2, ξ_2).

(ii) $\rho \geq 0$ and $\rho' \geq 0$.

(e)(i) $\sum_{i=1}^{r} \lambda_i f_i(t, x, y) + w(t)^{\mathrm{T}} g(t, x, y) + z(t)^{\mathrm{T}}[h(t, x, y) - \mathring{x}(t)]$ is ρ-QIX with respect to functions η_1, ξ_1 and ρ'-QIY with respect to functions η_2, ξ_2.

(ii) $\rho \geq 0$ and $\rho' > 0$ or $\rho > 0$ and $\rho' \geq 0$.

COROLLARY 1. *Assume that weak duality (Theorem 2) holds between* VCP *and* WVCD. *If* (u^0, v^0) *is feasible for* VCP, $(u^0, v^0, w^0, z^0, \lambda)$ *is feasible for* WVCD *with* $w^0(t)^{\mathrm{T}} g(t, u^0, v^0) = o$. *Then* (u^0, v^0) *is efficient for* VCP *and* $(u^0, v^0, w^0, z^0, \lambda)$ *is efficient for* WVCD.

Proof. Suppose (u^0, v^0) is not efficient for VCP. Then there exists some feasible (x, y) for VCP such that

$$\int_a^b f_i(t, x, y)\, dt < \int_a^b f_i(t, u^0, v^0)\, dt, \qquad \text{for some } i \in \{1, 2, \ldots, r\}$$

$$\int_a^b f_j(t, x, y)\, dt \leq \int_a^b f_j(t, u^0, v^0)\, dt, \qquad \text{for all } j \in \{1, 2, \ldots, r\}.$$

Since $w^0(t)^{\mathrm{T}} g(t, u^0, v^0) = o$, we get

$$\int_a^b f_i(t, x, y)\, dt < \int_a^b \{f_i(t, u^0, v^0) + w^0(t)^{\mathrm{T}} g(t, u^0, v^0)\}\, dt \qquad \text{for some } i \in \{1, 2, \ldots, r\}$$

$$\int_a^b f_j(t, x, y)\, dt \leq \int_a^b \{f_j(t, u^0, v^0) + w^0(t)^{\mathrm{T}} g(t, u^0, v^0)\}\, dt \qquad \text{for all } j \in (1, 2, \ldots, r).$$

This contradicts weak duality. Hence (u^0, v^0) is efficient for VCP.

Now suppose $(u^0, v^0, w^0, z^0, \lambda)$ is not efficient for WVCD. Then there exist (u, v, w, z, λ) feasible for WVCD such that

$$\int_a^b \{f_i(t, u, v) + w(t)^{\mathrm{T}} g(t, u, v)\}\, dt < \int_a^b \{f_i(t, u^0, v^0) + w^0(t)^{\mathrm{T}} g(t, u^0, v^0)\}\, dt \qquad \text{for some } i \in \{1, 2, \ldots, r\}$$

$$\int_a^b \{f_j(t, u, v) + w(t)^T g(t, u, v)\}\, dt \le \int_a^b \{f_j(t, u^0, v^0) + w^0(t)^T g(t, u^0, v^0)\}\, dt \qquad \text{for all } j \in \{1, 2, \ldots, r\}.$$

Since $w^0(t)^T g(t, u^0, v^0) = 0$,

$$\int_a^b \{f_i(t, u, v) + w(t)^T g(t, u, v)\}\, dt < \int_a^b df_i(t, u^0, v^0)\, dt$$

$$\text{for some } i \in \{1, 2, \ldots, r\}$$

$$\int_a^b f_j(t, u, v) + w(t)^T g(t, u, v)\}\, dt \le \int_a^b df_j(t, u^0, v^0)\, dt$$

$$\text{for all } j \in \{1, 2, \ldots, r\}.$$

This contradicts weak duality. Hence $(u^0, v^0, w^0, z^0, \lambda)$ is efficient for WVCD.

THEOREM 3 (Strong Duality). *Let (u^0, v^0) be efficient for* VCP *and assume that (u^0, v^0) satisfies the constraint qualification of Theorem* 1 *for $P_k(u_0, v_0)$ for at least one $k \in \{1, 2, \ldots, r\}$. Then there exist $\lambda^0 \in R^r$ and piecewise smooth w^0: $I \to R^p$ and z^0: $I \to R^n$ such that $(u^0, v^0, w^0, z^0, \lambda^0)$ is feasible for* WVCD *and $w^0(t)^T g(t, u^0, v^0) = 0$. If weak duality also holds between* VCP *and* WVCD *then $(u^0, v^0, w^0, z^0, \lambda^0)$ is efficient for* WVCD.

Proof. As (u^0, v^0) satisfy the constraint qualifications of Theorem 1 for at least one $k \in \{1, 2, \ldots, r\}$, it follows from Theorem 1 that there exist $\lambda' \in R^{r-1}$ and piecewise smooth w': $I \to R^p$ and z': $I \to R^n$, satisfying for all $t \in I$ the following:

$$f_{ku}(t, u^0, v^0) + \sum_{\substack{i=1 \\ i \ne k}}^{r} \lambda_i' f_{iu}(t, u^0, v^0) + w'(t)^T g_u(t, u^0, v^0) + z'(t)^T h_u(t, u^0, v^0) + \mathring{z}'(t) = 0 \tag{32}$$

$$f_{kv}(t, u^0, v^0) + \sum_{\substack{i=1 \\ i \ne k}}^{r} \lambda_i' f_{iv}(t, u^0, v^0) + w'(t)^T g_v(t, u^0, v^0) + z'(t)^T h_v(t, u^0, v^0) = 0 \tag{33}$$

$$w'(t)^T g(t, u^0, v^0) = 0 \tag{34}$$

$$w'(t) \ge 0, \qquad \lambda_i' \ge o,\ i = 1, 2, \ldots, r,\ i \ne k. \tag{35}$$

Now let us set, for $i = 1, 2, \ldots, r$, $i \neq k$,

$$\lambda_i^0 = \lambda_i' \Big/ \Big(1 + \sum_{\substack{i=1 \\ i \neq k}}^{r} \lambda_i'\Big), \qquad \lambda_k^0 = 1 \Big/ \Big(1 + \sum_{\substack{i=1 \\ i \neq k}}^{r} \lambda_i'\Big)$$

$$w^0(t) = w'(t) \Big/ \Big(1 + \sum_{\substack{i=1 \\ i \neq k}}^{r} \lambda_i'\Big), \qquad z^0(t) = z'(t) \Big/ \Big(1 + \sum_{\substack{i=1 \\ i \neq k}}^{r} \lambda_i'\Big).$$

Dividing (32) and (33) by $1 + \sum_{i=1, i \neq k}^{r} \lambda_i'$ we get

$$\sum_{i=1}^{r} \lambda_i^0 f_{iu}(t, u^0, v^0) + w^0(t)^{\mathrm{T}} g_u(t, u^0, v^0) + z^0(t)^{\mathrm{T}} h_u(t, u^0, v^0) + \mathring{z}^0(t) = 0 \tag{36}$$

$$\sum_{i=1}^{r} \lambda_i^0 f_{iv}(t, u^0, v^0) + w^0(t)^{\mathrm{T}} g_v(t, u^0, v^0) + z^0(t)^{\mathrm{T}} h_v(t, u^0, v^0) = 0. \tag{37}$$

Also, we get

$$\sum_{i=1}^{r} \lambda_i^0 = 1. \tag{38}$$

As (u^0, v^0) is feasible for VCP, $\mathring{u}^0(t) = h(t, u^0, v^0)$. Hence

$$\int_a^b z^0(t)^{\mathrm{T}} [h(t, u^0, v^0) - \mathring{u}^0(t)]\, dt \geq 0. \tag{39}$$

Now it follows from (35)–(39) that $(u^0, v^0, w^0, z^0, \lambda^0)$ is feasible for WVCD.

Also, $w^0(t)^{\mathrm{T}} g(t, u^0, v^0) = 0$ and weak duality holds between VCP and WVCD. The result now follows from Corollary 1.

3. Duality between VCP and MVCD

THEOREM 4 (Weak Duality). *Assume for all feasible (x, y) for* VCP *and all feasible (u, v, w, z, λ) for* MVCD *that if*

(i) *$f_i(t, x, y)$ is ρ_i-QIX with respect to functions η_1, ξ_1 and ρ_i'-QIY with respect to functions η_2, ξ_2, for all $i \in \{1, 2, \ldots, r\}$,*

(ii) *$w(t)^T g(t, x, y)$ is σ-QIX with respect to the same functions η_1, ξ_1 and σ'-QIY with respect to the same functions η_2, ξ_2,*

(iii) *$x(t)^T[h(t, x, y) - \mathring{x}(f)]$ is μ-SQIX with respect to the same functions η_1, ξ_1 and μ'-QIY with respect to the same functions η_2, ξ_2 (or μ-QIX and μ'-SQIY with respect to the same functions η_1, ξ_1 and η_2, ξ_2, respectively), and*

(iv) *$\sum_{i=1}^r \lambda_i \rho_i + \sigma + \mu \geq 0$ and $\sum_{i=1}^r \lambda_i \rho_i' + \sigma' + \mu' \geq 0$,*

then the following cannot hold:

$$\int_a^b f_i(t, x, y)\,dt < \int_a^b f_i(t, u, v)\,dt, \qquad \text{for some } i \leq \{1, 2, \ldots, r\} \tag{40}$$

$$\int_a^b f_j(t, x, y)\,dt \leq \int_a^b f_j(t, u, v)\,dt, \qquad \text{for all } j \leq \{1, 2, \ldots, r\}. \tag{41}$$

Proof. Suppose, contrary to the result, that (40) and (41) hold. Then (i) yields

$$\int_a^b \{\eta_1(t, x, u) f_{iu}(t, u, v) + \eta_2(t, y, v) f_{iv}(t, u, v)\}\,dt$$
$$\leq -\rho_i \|\xi_1(t, x, u)\|^2 - \rho_i' \|\xi_2(t, y, u)\|^2 \tag{42}$$
$$\text{for all } i \in \{1, 2, \ldots, r\}.$$

Multiplying each inequality of (42) by $\lambda_i \geq 0$, and summing up for all $i = 1, 2, 3, \ldots, r$, we get

$$\int_a^b \left\{\eta_1(t, x, u) \sum_{i=1}^r \lambda_i f_{iu}(t, u, v) + \eta_2(t, y, v) \sum_{i=1}^r \lambda_i f_{iv}(t, u, v)\right\} dt$$
$$\leq -\sum_{i=1}^r \lambda_i \rho_i \|\xi_1(t, x, u)\|^2 - \sum_{i=1}^r \lambda_i \rho_i' \|\xi_2(t, y, v)\|^2. \tag{43}$$

As (x, y) is feasible for VCP and (u, v, w, z, λ) is feasible for MVCD, it follows from (3), (14), and (15) that

$$\int_a^b w(t)^T g(t, x, y)\,dt \leq \int_a^b w(t)^T g(t, u, v)\,dt.$$

It now follows from (ii) that

$$\int_a^b \{\eta_1(t, x, u)w(t)^{\mathrm{T}} g_u(t, u, v) + \eta_2(t, y, v)g_v(t, u, v)\}\, dt \\ \leq -\sigma\|\xi_1(t, x, u)\|^2 - \sigma'\|\xi_2(t, y, v)\|^2. \tag{44}$$

From (2) and (13) we have

$$\int_a^b z(t)^{\mathrm{T}}[h(t, x, y) - \overset{\circ}{x}(t)]\, dt \leq \int_a^b z(t)^{\mathrm{T}}[h(t, u, v) - \overset{\circ}{u}(t)]\, dt.$$

From (iii) it follows that

$$\int_a^b \Big\{\eta_1(t, x, u)z(t)^{\mathrm{T}} h_u(t, u, v) - \frac{d}{dt}\eta_1(t, x, u)z(t) \\ + \eta_2(t, y, v)z(t)^{\mathrm{T}} h_v(t, u, v)\, dt \\ < -\mu\|\xi_1(t, x, u)\|^2 - \mu'\|\xi_2(t, y, v)\|^2. \tag{45}$$

Using (30) in (45) we have

$$\int_a^b \{\eta_1(t, x, u)z(t)^{\mathrm{T}} h_u(t, u, v) + \overset{\circ}{z}(t)] \\ + \eta_2(t, y, v)z(t)^{\mathrm{T}} h_v(t, u, v)\}\, dt \\ < -\mu\|\xi_1(t, x, u)\|^2 - \mu'\|\xi_2(t, y, v)\|^2. \tag{46}$$

Adding (43), (44), and (46),

$$\int_a^b \Big\{\eta_1(t, x, u)\Big[\sum_{i=1}^r \lambda_i f_{iu}(t, u, v) + w(t)^{\mathrm{T}} g_u(t, u, v) \\ + z(t)^{\mathrm{T}} h_u(t, u, v) + \overset{\circ}{z}(t)\Big] \\ + \eta_2(t, y, v)\Big[\sum_{i=1}^r \lambda_i f_{iv}(t, u, v) + w(t)^{\mathrm{T}} g_v(t, u, v) \\ + z(t)^{\mathrm{T}} h_v(t, u, v)\Big]\Big\}\, dt \\ < -\Big(\sum_{i=1}^r \lambda_i\rho_i + \sigma + \mu\Big)\|\xi_1(t, x, u)\|^2 - \\ \Big(\sum_{i=1}^r \lambda_i\rho_i' + \sigma' + \mu'\Big)\|\xi_2(t, y, v)\|^2.$$

It follows from hypothesis (iv) that

$$\int_a^b \Bigg\{\eta_1(t, x, u)\left[\sum_{i=1}^r \lambda_i f_{iu}(t, u, v) + w(t)^T g_u(t, u, v) + z(t)^T h_u(t, u, v) + \overset{\circ}{z}(t)\right] + \eta_2(t, y, v)\left[\sum_{i=1}^r \lambda_i f_{iv}(t, u, v) + w(t)^T g_v(t, u, v) + z(t)^T h_v(t, u, v)\right]\Bigg\} dt < 0. \tag{47}$$

Equation (47) contradicts (11) and (12). Hence we have the result.

Remark. The weak duality theorem also holds good under the following assumptions:

(a)(i) $f_i(t, x, y)$ is ρ_i-QIX with respect to functions η_1, ξ_1 and ρ_i'-SQIY with respect to functions η_2, ξ_2 (or ρ_i-SQIX and ρ_i'-QIY with respect to functions η_1, ξ_1 and η_2, ξ_2, respectively) for all $i \in \{1, 2, \ldots, r\}$.

(ii) $w(t)^T g(t, x, y)$ is σ-QIX with respect to the same functions η_1, ξ_1 and σ'-QIY with respect to the same functions η_2, ξ_2.

(iii) $z(t)[h(t, x, y) - \overset{\circ}{x}(t)]$ is μ-QIX with respect to the same functions η_1, ξ_1 and μ'-QIY with respect to the same functions η_2, ξ_2.

(iv) $\sum_{i=1}^r \lambda_i \rho_i + \sigma + \mu \geq 0$ and $\sum_{i=1}^r \lambda_i \rho_i' + \sigma' + \mu' \geq 0$.

(b)(i) $f_i(t, x, y)$ is ρ_i-QIX with respect to functions η_1, ξ_1 and ρ_i'-QIY with respect to functions η_2, ξ_2 for all $i \in \{1, 2, \ldots, r\}$.

(ii) $w(t)^T g(t, x, y)$ is σ-QIX with respect to the same functions η_1, ξ_1 and σ'-SQIY with respect to the same functions η_2, ξ_2 (or σ-SQIX and σ'-QIY with respect to the same functions η_1, ξ_1 and η_2, ξ_2, respectively).

(iii) $z(t)^T[h(t, x, y) - \overset{\circ}{x}(t)]$ is μ-QIX with respect to the same functions η_1, ξ_1 and μ'-QIY with respect to the same functions η_2, ξ_2.

(iv) $\sum_{i=1}^r \lambda_i \rho_i + \sigma + \mu \geq 0$ and $\sum_{i=1}^r \lambda_i \rho_i' + \sigma' + \mu' \geq 0$.

(c)(i) $f_i(t, x, y)$ is ρ_i-QIX with respect to functions η_1, ξ_1 and ρ_i'-QIY with respect to functions η_2, ξ_2 for all $i \in \{1, 2, \ldots, r\}$.

(ii) $w(t)^T g(t, x, y)$ is σ-QIX with respect to the same functions η_1, ξ_1 and σ'-QIY with respect to the same functions η_2, ξ_2.

(iii) $z(t)^T[h(t, x, y) - \overset{\circ}{x}(t)]$ is μ-QIX with respect to the same functions η_1, ξ_1 and μ'-QIY with respect to the same functions η_2, ξ_2.

(iv) $\sum_{i=1}^{r} \lambda_i\rho_i + \sigma + \mu \geq 0$ and $\sum_{i=1}^{r} \lambda_i\rho_i' + \sigma' + \mu' > 0$ or $\sum_{i=1}^{r} \lambda_i\rho_i + \sigma + \mu > 0$ and $\sum_{i=1}^{r} \lambda_i\rho_i' + \sigma' + \mu' \geq 0$.

It may further be remarked that Theorem 4 also holds good under the assumptions (c), (d), and (e) of Theorem 2.

COROLLARY 2. *Assume that weak duality* (*Theorem* 4) *holds between* VCP *and* MVCD. *If* (u, v) *is feasible for* VCP *and* (u, v, w, z, λ) *is feasible for* MVCD, *then* (u, v) *is efficient for* VCP *and* (u, v, w, z, λ) *is efficient for* MVCD.

Proof. Suppose (u, v) is not efficient for VCP. Then there exists some feasible (x, y) for VCP such that

$$\int_a^b f_i(t, x, y)\, dt < \int_a^b f_i(t, u, v)\, dt, \qquad \text{for some } i \in \{1, 2, \ldots, r\}$$

$$\int_a^b f_j(t, x, y)\, dt \leq \int_a^b f_j(t, u, v)\, dt, \qquad \text{for all } j \in \{1, 2, \ldots, r\}.$$

This contradicts weak duality. Hence (u, v) is efficient for VCP. Now suppose (u, v, w, z, λ) is not efficient for MVCD. Then there exist some feasible $(\bar{u}, \bar{v}, \bar{w}, \bar{z}, \bar{\lambda})$ for MVCD such that

$$\int_a^b f_i(t, \bar{u}, \bar{v})\, dt > \int_a^b f_i(t, u, v)\, dt, \qquad \text{for some } i \in \{1, 2, \ldots, r\}$$

$$\int_a^b f_j(t, \bar{u}, \bar{v})\, dt \geq \int_a^b f_j(t, u, v)\, dt, \qquad \text{for all } j \in \{1, 2, \ldots, r\}.$$

This contradicts weak duality. Hence (u, v, w, z, λ) is efficient for MVCD.

THEOREM 5 (Strong Duality). *Let* (u^0, v^0) *be efficient for* VCP *and assume that* (u^0, v^0) *satisfy the constraint qualification of Theorem* 1 *for at least one* $k \in \{1, 2, \ldots, r\}$. *Then there exists* $\lambda^0 \in R^r$ *and piecewise smooth* $w^0: I \to R^p$ *and* $z^0: I \to R^n$ *such that* $(u^0, v^0, w^0, z^0, \lambda^0)$ *is feasible for* MVCD. *If also weak duality holds between* VCP *and* MVCD *then* $(u^0, v^0, w^0, z^0, \lambda^0)$ *is efficient for* MVCD.

Proof. Proceeding on the same lines as in Theorem 3, it follows that there exist piecewise smooth $w^0: I \to R^p$, $z^0: I \to R^n$, and $\lambda^0 \in R^r$, satisfying for all $t \in I$ the following relations:

$$\sum_{i=1}^{r} \lambda_i^0 f_{iu}(t, u^0, v^0) + w^0(t)^{\mathrm{T}} g_u(t, u^0, v^0) + z^0(t)^{\mathrm{T}} h_u(t, u^0, v^0) + \mathring{z}^0(t) = 0$$

$$\sum_{i=1}^{r} \lambda_i^0 f_{iv}(t, u^0, v^0) + w^0(t)^{\mathrm{T}} g_v(t, u^0, v^0) + z^0(t)^{\mathrm{T}} h_v(t, u^0, v^0) = 0$$

$$w^0(t)^{\mathrm{T}} g(t, u^0, v^0) = 0$$

$$w^0(t) \geq 0$$

$$\lambda_i^0 \geq 0, \qquad \sum_{i=1}^{r} \lambda_i^0 = 1$$

The relations $\int_a^b w^0(t)^{\mathrm{T}} g(t, u^0, v^0)\, dt \geq 0$ and $\int_a^b z^0(t)^{\mathrm{T}} h(t, u^0, v^0) - \mathring{u}^0]$ $dt \geq 0$ are obvious.

The above relations imply that $(u^0, v^0, w^0, z^0, \lambda^0)$ is feasible for MVCD. The result now follows from Corollary 2.

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