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Collectively Compact Sets of Operators and Almost Periodic Functions

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A characterization of collectively compact sets of linear operators on the Banach algebra of almost periodic functions is obtained by using Gelfand–Naimark theory. Properties of collectively compact sets are then used to prove that the Fourier coefficients of an almost periodic function are approximately the eigenvalues of Fredholm type integral operators.

1. INTRODUCTION

Collectively compact sets of linear operators in Banach spaces have been studied and used by Anselone [1] and others in finding approximate solutions of integral and operator equations. An excellent treatment of this development of the theory can be found in Anselone's book [1]. Further, Higgins [6] obtained characterizations of collectively compact sets of operators on several special spaces which frequently occur in analysis.

In this paper we use the characterization obtained by Higgins [6] to derive necessary and sufficient conditions for a family of operators on a commutative B^* algebra with unit to be collectively compact. This characterization is then applied to the Banach algebra of almost periodic functions to obtain approximations for Fourier coefficients of an almost periodic function.

2. COLLECTIVELY COMPACT SETS OF OPERATORS ON THE ALGEBRA OF ALMOST PERIODIC FUNCTIONS

In this section we obtain the characterization of collectively compact sets of operators on the Banach algebra of almost periodic functions. Before we obtain this, it will be convenient to describe the notation which will be used throughout this paper. For Banach spaces X and Y, $BL(X, Y)$ will denote the space of all continuous linear operators from X to Y. Unless otherwise

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stated, the topology on $BL(X, Y)$ will be assumed to be the uniform operator topology. X^* will denote the topological dual of X. $KL(X, Y)$ will be the space of compact linear operators from X to Y. When $X = Y$, we write $BL(X, Y) = BL(X)$ and $KL(X, Y) = KL(X)$. For a compact Hausdorff space S, $C(S)$ will denote the Banach space of continuous complex valued functions on S.

DEFINITION 1. Let X and Y be Banach spaces. Then a set $\mathscr{K} \subset BL(X, Y)$ is said to be collectively compact if $\{K(x): \|x\| \leqslant 1, K \in \mathscr{K}\}$ is relatively compact.

THEOREM 1. *Let S be a compact Hausdorff space and X an arbitrary Banach space. Let $\pi: S \to [C(S)]^*$ be defined by $\pi(s)(f) = f(s)$, $f \in C(S)$. For $T \in BL(X, C(S))$ define $\mu_T: S \to X^*$ by $\mu_T = T^*\pi$. Then $\mathscr{K} \subset BL(X, C(S))$ is collectively compact if and only if* (i) *$\sup\{\|\mu_T(s)\|: T \in \mathscr{K}, s \in S\}$ is finite and* (ii) *$\{\mu_T: T \in \mathscr{K}\}$ is equicontinuous.*

The above theorem is Proposition 1.3.5 from [6]. We use this theorem to obtain a characterization of collectively compact sets in $BL(A)$, where A is a commutative B^* algebra with unit. We recall that for such an algebra, the maximal ideal space can be described as the carrier space Φ_A of multiplicative linear functionals on A.

PROPOSITION 1. *Let A be a commutative B^* algebra with unit and Φ_A its carrier space (or the maximal ideal space) with induced weak star topology. Then $\mathscr{K} \subset BL(A)$ is collectively compact if and only if*

(a) *the family of mappings $\rho_K: \Phi_A \to A^*$ defined by $\rho_K(\phi) = \phi \circ K$, $K \in \mathscr{K}$, is equicontinuous and*

(b) *$\mathscr{K}$ is uniformly bounded.*

Proof. It is well known that by the Gelfand–Naimark theorem, Φ_A is compact Hausdorff and A is isometrically isomorphic to $C(\Phi_A)$. If e denotes the isometric isomorphism from A onto $C(\Phi_A)$, then $e(f) = \hat{f}$ is defined by $\hat{f}(\phi) = \phi(f)$, $\phi \in \Phi_A$. Now it is easy to see that $\mathscr{K}$ is collectively compact if and only if $\mathscr{T} = e\mathscr{K}e^{-1}: C(\Phi_A) \to C(\Phi_A)$ is collectively compact. Let $T \in \mathscr{T}$ be such that $T = eKe^{-1}$, $K \in \mathscr{K}$. Then for any $\phi \in \Phi_A$ and $f \in A$, we have

$$\begin{aligned}[\mu_T(\phi)](\hat{f}) &= [T^*\pi(\phi)](\hat{f}) = [\pi(\phi) \circ T](\hat{f}) \\ &= \pi(\phi)(\widehat{Kf}) = \widehat{Kf}(\phi) = (\phi \circ K)(f) \\ &= [\rho_K(\phi)](f).\end{aligned}$$

Hence we conclude that for any $\phi, \psi \in \Phi_A$ and $T = eKe^{-1}$, $\|\mu_T(\phi) - \mu_T(\psi)\| = \|\rho_K(\phi) - \rho_K(\psi)\|$. Moreover, for any $f \in A$, $\|K(f)\| = \|\widehat{K(f)}\| = \sup\{|\phi(K(f))|:$

$\phi \in \Phi_A\} = \sup\{|[\mu_T(\phi)](\hat{f})|: \phi \in \Phi_A\}$. Hence it follows that $\{\mu_T: T \in \mathscr{F}\}$ is equicontinuous if and only if $\{\rho_K: K \in \mathscr{K}\}$ is equicontinuous and $\sup\{\|\mu_T(\phi)\|: \phi \in \Phi_A,\ T \in \mathscr{F}\}$ is finite if and only if $\mathscr{K}$ is uniformly bounded. Application of Proposition 1 now proves the proposition.

Remark 1. We note that in the above proposition, Φ_A is weak star compact. Hence condition (a) can be equivalently written as $\rho_K: \Phi_A \to A^*$, $K \in \mathscr{K}$, is a uniformly equicontinuous family.

Now we state a purely topological result about an extension of an equicontinuous family to the closure of the domain. It is well known that a uniformly continuous function can be uniquely extended continuously to the closure of the domain [7, Theorem 26, p. 195]. The following lemma constitutes a simple extension of this result to uniformly equicontinuous family of functions. The proof is similar and hence omitted.

LEMMA 1. *Let $\mathscr{F}$ be a family of functions whose domain is subset A of a uniform space $(X, \mathscr{U})$ and range in a complete Hausdorff uniform space $(Y, \mathscr{V})$. Suppose that $\mathscr{F}$ is uniformly equicontinuous on A in the sense that for every V in $\mathscr{V}$, there is U in $\mathscr{U}$ such that $(x_1, x_2) \in U$, $x_1, x_2 \in A$ implies that $(f(x_1), f(x_2)) \in V$ for all $f \in \mathscr{F}$. Then $\bar{\mathscr{F}} = \{\bar{f}: f \in \mathscr{F}\}$, where $\bar{f}$ is a unique uniformly continuous extension of f to closure of A, is uniformly equicontinuous on closure of A.*

Now we come to the principal results of this section. Hereafter A will denote the space of almost periodic functions on $\mathbb{R}$ with supremum norm. Then A with pointwise multiplication becomes a commutative B^* algebra with unit. For any $x \in \mathbb{R}$, let $\hat{x}: A \to \mathbb{C}$ be defined by $\hat{x}(f) = f(x)$, $f \in A$. Then, it is well known that $\hat{\mathbb{R}} = \{\hat{x}: x \in \mathbb{R}\}$ is dense in Φ_A, the carrier space of A. Therefore, applying Proposition 1 to the algebra A of almost periodic functions and using Lemma 1, we obtain the following characterization of collectively compact sets of operators on A.

PROPOSITION 2. *A set $\mathscr{K} \subset BL(A)$ is collectively compact if and only if*

(a) *the family of mappings $\rho_K: \hat{\mathbb{R}} \to A^*$ is uniformly equicontinuous and*

(b) *$\mathscr{K}$ is uniformly bounded.*

For $h \in A$ and $n \geqslant 1$, let the operator $T_n: A \to A$ be defined by $(T_n g)(x) = (1/2n) \int_{-n}^{n} h(x-t)\, g(t)\, dt$, $g \in A$.

PROPOSITION 3. *The sequence $\{T_n\}$ is collectively compact subset of $BL(A)$. If $(Tg)(x) = \lim_{n\to\infty} (1/2n) \int_{-n}^{n} h(x-t)\, g(t)\, dt$, then for each $g \in A$, $\|T_n g - Tg\| \to 0$.*

Proof. It is easy to see that each T_n is continuous. Hence by Proposition 2, it is enough to show that $\{T_n\}$ is uniformly bounded and the family $\rho_n: \hat{\mathbb{R}} \to A^*$ defined by $\rho_n(\hat{x}) = \hat{x} \circ T_n$ is uniformly equicontinuous. Since for each $n \geqslant 1$, $\|T_n\| \leqslant \|h\|$, $\{T_n\}$ is uniformly bounded. Now to prove the uniform equicontinuity of $\{\rho_n\}$, let $\varepsilon > 0$ be given. Since h is an almost periodic function, the set of all translates of h is a totally bounded subset of A. Thus $\{h_t: t \in \mathbb{R}\}$ is totally bounded in A, where $h_t(x) = h(x+t)$, $x \in \mathbb{R}$. Hence there exist $t_1, t_2,..., t_k$ in $\mathbb{R}$ such that for every t in $\mathbb{R}$ there is a t_i, $i \leqslant k$, with $\|h_t - h_{t_i}\| < \varepsilon/3$. Let $\delta = \varepsilon/3$ and $f_i = h_{t_i}$, $i = 1, 2,..., k$. Then for $x_1, x_2 \in \mathbb{R}$ with $|\hat{x}_1(f_i) - \hat{x}_2(f_i)| < \delta$ for all $i = 1, 2,..., k$, we have

$$\begin{aligned}
&\|\rho_n(\hat{x}_1) - \rho_n(\hat{x}_2)\| \\
&\quad = \sup\{|T_n g(x_1) - T_n g(x_2)|: \|g\| \leqslant 1\} \\
&\quad \leqslant \sup\left\{\left|\frac{1}{2n}\int_{-n}^{n} [h(x_1 - t) - h(x_2 - t)]\, g(t)\, dt\right| : \|g\| \leqslant 1\right\} \\
&\quad \leqslant \sup\{|h(x_1 - t) - h(x_2 - t)|: t \in \mathbb{R}\} \\
&\quad \leqslant \sup\{|h_t(x_1) - h_t(x_2)|: t \in \mathbb{R}\}.
\end{aligned}$$

But for each $t \in \mathbb{R}$,

$$\begin{aligned}
|h_t(x_1) - h_t(x_2)| &\leqslant |h_t(x_1) - h_{t_i}(x_1)| + |h_{t_i}(x_1) - h_{t_i}(x_2)| \\
&\quad + |h_{t_i}(x_2) - h_t(x_2)| \\
&\leqslant \|h_t - h_{t_i}\| + |\hat{x}_1(f_i) - \hat{x}_2(f_i)| + \|h_t - h_{t_i}\| \\
&< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\
&= \varepsilon.
\end{aligned}$$

Hence whenever $|\hat{x}_1(f_i) - \hat{x}_2(f_i)| < \delta$ for all $i = 1, 2,..., k$, we have $\|\rho_n(\hat{x}_1) - \rho_n(\hat{x}_2)\| \leqslant \varepsilon$ for all $n \geqslant 1$. This shows that $\{\rho_n: n \geqslant 1\}: \hat{\mathbb{R}} \to A^*$ is uniformly equicontinuous. Thus $\{T_n\}$ is collectively compact subset of $BL(A)$. Now to show $T_n \to T$ pointwise, let $g \in A$ be given. It is enough to show that $\{T_n(g)\}$ is a Cauchy sequence in A. Let $\varepsilon > 0$ be given and $M = 1 + 2\|g\|$. By total boundedness of $\{h_x: x \in \mathbb{R}\}$, there exist $x_1, x_2,..., x_k$ in $\mathbb{R}$ such that for every $x \in \mathbb{R}$, there is x_i with $\|h_x - h_{x_i}\| < \varepsilon/M$. Now we know that for any almost periodic function g, $\lim_{n\to\infty}(1/2n)\int_{-n}^{n} g(t)\, dt$ exists. Hence $\{(1/2n)\int_{-n}^{n} g(t)\, dt\}$ is a Cauchy sequence for every $g \in A$. Thus applying it to finite number of functions g_i, $i = 1, 2,..., k$, defined by $g_i(t) = h(x_i - t)\, g(t)$, we get positive integers $N(i)$, $i = 1, 2,..., k$, satisfying $|(1/2n)\int_{-n}^{n} h(x_i - t)\, g(t)\, dt - (1/2m)\int_{-m}^{m} h(x_i - t)\, g(t)\, dt| < \varepsilon/M$ for all $n, m \geqslant N(i)$, $i = 1, 2,..., k$. Hence if $N = \max\{N(i): i = 1, 2,..., k\}$, for $n, m \geqslant N$, we have

$|(1/2n) \int_{-n}^{n} h(x_i - t)\ g(t)\,dt - (1/2m) \int_{-m}^{m} h(x_i - t)\ g(t)\,dt| < \varepsilon/M$ for each $i \leqslant k$. Now it remains to prove that whenever $n, m \geqslant N$, $\|T_n g - T_m g\| < \varepsilon$. Let $n, m \geqslant N$ and $x \in \mathbb{R}$. Then we have

$$
\begin{aligned}
|T_n g(x) - T_m g(x)| &= \left| \frac{1}{2n} \int_{-n}^{n} h(x-t)\, g(t)\,dt - \frac{1}{2m} \int_{-m}^{m} h(x-t)\, g(t)\,dt \right| \\
&\leqslant \left| \frac{1}{2n} \int_{-n}^{n} [h(x-t) - h(x_i - t)]\, g(t)\,dt \right| \\
&\quad + \left| \frac{1}{2n} \int_{-n}^{n} h(x_i - t)\, g(t)\,dt - \frac{1}{2m} \int_{-m}^{m} h(x_i - t)\, g(t)\,dt \right| \\
&\quad + \left| \frac{1}{2m} \int_{-m}^{m} [h(x_i - t) - h(x-t)]\, g(t)\,dt \right| \\
&< (\varepsilon/m)\,\|g\| + \varepsilon/M + (\varepsilon/M)\,\|g\| \\
&= \varepsilon.
\end{aligned}
$$

This shows that $T_n \to T$ pointwise and completes the proof.

We now give a counterexample to show that T_n does not tend to T in the uniform operator topology. This will mean that in deriving spectral properties of $\{T_n\}$ we shall be using collective compactness in a crucial way.

EXAMPLE 1. We give an example to show that $\{(1/2n) \int_{-n}^{n} g(t)\,dt\}$ is not Cauchy uniformly over $\|g\| \leqslant 1$. Thus it is enough to construct an $\varepsilon > 0$ such that for every $n \geqslant 1$, there exists $m \geqslant n$ and f_n in A with $\|f_n\| \leqslant 1$ satisfying

$$\left| \frac{1}{2n} \int_{-n}^{n} f_n(t)\,dt - \frac{1}{2m} \int_{-m}^{m} f_n(t)\,dt \right| \geqslant \varepsilon.$$

Let $\varepsilon = \sin(1)/4$ and $n \geqslant 1$. Then $(p/n)\sin(n/p) \to 1$ as $p \to \infty$. Hence $|(p/n)\sin(n/p) - 1| < \sin(1)/6$ for $p \geqslant p_1$. Also $(n/q)\sin(q/n) \to 0$ as $q \to \infty$. Hence $|(n/q)\sin(q/n)| < \sin(1)/6$ for $q \geqslant p_2$. Let $m = \max\{n, p_2\}$. Then we have

$$|(n/m)\sin(m/n)| < \sin(1)/6. \tag{1}$$

For the above chosen m, we have $(r/m)\sin(m/r) \to 1$ as $r \to \infty$. Hence $|(r/m)\sin(m/r) - 1| < \sin(1)/6$ for $r \geqslant p_3$. Let $k = \max\{p_1, p_3\}$. Then we have

$$|(k/n)\sin(n/k) - 1| < \sin(1)/6 \tag{2}$$

and

$$|(k/m)\sin(m/k) - 1| < \sin(1)/6. \tag{3}$$

Now define $f_n(t) = (1/2)\{e^{it/n} + e^{it/k}\}$. Then $\|f_n\| = 1$ and $f_n \in A$. Also it is easy to verify that for any $p \geqslant 1$, $(1/2p) \int_{-p}^{p} f_n(t)\,dt = (n/2p)\sin(p/n) + (k/2p)\sin(p/k)$. Hence

$$\left|\frac{1}{2n}\int_{-n}^{n} f_n(t)\,dt - \frac{1}{2m}\int_{-m}^{m} f_n(t)\,dt\right|$$
$$= (1/2)\,|\sin(1) + (k/n)\sin(n/k) - (n/m)\sin(m/n) - (k/m)\sin(m/k)|$$
$$\geqslant (1/2)\left|\sin(1) - |(k/n)\sin(n/k) - (n/m)\sin(m/n) - (k/m)\sin(m/k)|\right|.$$

But from (1), (2) and (3), it is easy to verify that

$$|(k/n)\sin(n/k) - (n/m)\sin(m/n) - (k/m)\sin(m/k)| < \sin(1)/2.$$

Hence

$$\left|\frac{1}{2n}\int_{-n}^{n} f_n(t)\,dt - \frac{1}{2m}\int_{-m}^{m} f_n(t)\,dt\right| \geqslant (1/2)\sin(1)/2 = \sin(1)/4.$$

This proves the required result.

For the rest of the section, T_f, $f \in A$, will denote the operator from A into A defined by $(T_f g)(x) = \lim_{T\to\infty}(1/T) \int_0^T f(x-t)\, g(t)\,dt = \lim_{n\to\infty}(1/2n) \int_{-n}^{n} f(x-t)\, g(t)\,dt$. For $f \in A$ and $\varepsilon > 0$, $\{s \in \mathbb{R}: \|f_s - f\| < \varepsilon\}$ will be denoted by $E(\varepsilon, f)$. For the definitions of relatively dense sets and other ideas related to almost periodic functions, we refer to [2]. In the next proposition, we prove sufficient conditions on $\mathscr{F} \subset A$ to ensure that $\{T_f: f \in \mathscr{F}\}$ is collectively compact subset of $BL(A)$.

PROPOSITION 4. *If $\mathscr{F} \subset A$ satisfies the following conditions, then $\{T_f: f \in \mathscr{F}\}$ is collectively compact.*

(a) *For each $\varepsilon > 0$, there exists a $\delta > 0$ and finite number of functions f_i, $i = 1, 2,..., n$ in A such that $|f_i(x_1) - f_i(x_2)| < \delta$ for all $i \leqslant n$ implies that $x_1 - x_2 \in \bigcap_{f\in\mathscr{F}} E(\varepsilon, f)$.*

(b) *$\{T_f: f \in \mathscr{F}\}$ is uniformly bounded.*

Proof. In the light of Proposition 2, it is enough to show that the condition (a) implies that the family $\rho_f: \hat{\mathbb{R}} \to A^*$ defined by $\rho_f(\hat{x}) = \hat{x} \circ T_f$, $f \in \mathscr{F}$, is uniformly equicontinuous. Let $\varepsilon > 0$ be given. Then by (a), there exist $\delta > 0$ and $f_1, f_2,..., f_m$ in A such that whenever $|f_i(x_1) - f_i(x_2)| < \delta$ for all $i = 1, 2, \ldots, m$, $x_1 - x_2 \in \bigcap_{f \in \mathscr{F}} E(\varepsilon, f)$. Thus whenever $|f_i(x_1) - f_i(x_2)| < \delta$ for all $i \leqslant m$, we have

$$\begin{aligned} &\|\rho_f(\hat{x}_1) - \rho_f(\hat{x}_2)\| \\ &\quad = \sup\{|(T_f g)(x_1) - (T_f g)(x_2)| : \|g\| \leqslant 1\} \\ &\quad = \sup\left\{\left|\lim_{n\to\infty} \frac{1}{2n} \int_{-n}^{n} [f(x_1 - t) - f(x_2 - t)]\, g(t)\, dt\right| : \|g\| \leqslant 1\right\} \\ &\quad \leqslant \|f_{x_1} - f_{x_2}\| \\ &\quad = \|f_{x_1 - x_2} - f\| < \varepsilon \qquad \text{for all} \quad f \in \mathscr{F}. \end{aligned}$$

This proves the proposition.

Remark 2. It is easy to see that $\|T_f\| \leqslant \|f\|$, $f \in A$. Hence if $\mathscr{F}$ is bounded and satisfies condition (a) of the previous proposition, $\{T_f : f \in \mathscr{F}\}$ is collectively compact. But we will prove something more in this situation. For that, we mention one result from [4].

DEFINITION 2. A family $\mathscr{F}$ of almost periodic functions is a uniformly almost periodic family if it is uniformly bounded, and for every $\varepsilon > 0$, $\bigcap_{f \in \mathscr{F}} E(\varepsilon, f)$ is relatively dense and includes an interval about 0.

The above definition is taken from [4, p. 17]. The following theorem is Theorem 2.5 from [4].

THEOREM 2. *If $\mathscr{F}$ is a uniformly almost periodic family, then from every sequence in $\mathscr{F}$, one can extract a subsequence which converges uniformly on* $\mathbb{R}$.

PROPOSITION 5. *Let $\mathscr{F} \subset A$ satisfy the following conditions:*

(a) *$\mathscr{F}$ is uniformly bounded and*

(b) *for every $\varepsilon > 0$, there exist $\delta > 0$ and finite number of functions f_i, $i = 1, 2, \ldots, m$ in A such that whenever $|f_i(x_1) - f_i(x_2)| < \delta$ for all $i = 1, 2, \ldots, m$, we have $x_1 - x_2 \in \bigcap_{f \in \mathscr{F}} E(\varepsilon, f)$. Then $\mathscr{F}$ is totally bounded.*

Proof. In light of the previous theorem, it is enough to show that $\mathscr{F}$ is a uniformly almost periodic family. Since by assumption $\mathscr{F}$ is uniformly bounded, it remains to verify that $\bigcap_{f \in \mathscr{F}} E(\varepsilon, f)$ is relatively dense and contains a neighbourhood around 0 for every $\varepsilon > 0$. Let $\varepsilon > 0$ be given. By (b), there exist $\delta > 0$ and $f_1, f_2, \ldots, f_m$ in A such that whenever $|f_i(x_1) - f_i(x_2)| < \delta$ for all $i \leqslant m$, we have $x_1 - x_2 \in \bigcap_{f \in \mathscr{F}} E(\varepsilon, f)$. Since any finite set of almost periodic functions forms a uniformly almost periodic family, we have that $\{f_i : i \leqslant m\}$ is a uniformly almost periodic family. Hence

$\bigcap_{i\leqslant m} E(\delta, f_i)$ is relatively dense and contains a neighbourhood, say, $(-\delta', \delta')$ around 0. Thus there exists $T>0$ such that for every $a\in\mathbb{R}$, there is $s\in[a, a+T]\cap[\bigcap_{i\leqslant m} E(\delta, f_i)]$ and $(-\delta', \delta')\subset\bigcap_{i\leqslant m} E(\delta, f_i)$. With the same T, a and s, we have $|f_i(s+t)-f_i(t)|<\delta$ for all $i\leqslant m$. Hence $(s+t)-(t)=s\in\bigcap_{f\in\mathcal{F}} E(\varepsilon, f)$ by assumption. Also whenever $|x|<\delta'$, $|f_i(x+t)-f_i(t)|<\delta$ for all $t\in\mathbb{R}$ and $i\leqslant m$. This again implies that $(x+t)-(t)=x\in\bigcap_{f\in\mathcal{F}} E(\varepsilon, f)$. This shows that $\bigcap_{f\in\mathcal{F}} E(\varepsilon, f)$ is relatively dense and contains a neighbourhood around 0. This proves the result.

The converse of the above proposition also holds.

PROPOSITION 6. *Let $\mathcal{F}\subset A$ be totally bounded. Then $\mathcal{F}$ is bounded and for every $\varepsilon>0$, there exist $\delta>0$ and $f_1, f_2, ..., f_n$ in A such that whenever $|f_i(x_1)-f_i(x_2)|<\delta$ for all $i\leqslant n$, $x_1-x_2\in\bigcap_{f\in\mathcal{F}} E(\varepsilon, f)$.*

Proof. Let $\mathcal{F}$ be totally bounded. Then obviously $\mathcal{F}$ is bounded. Hence it remains to prove that for every $\varepsilon>0$, there exists $\delta>0$ and $f_1, f_2, ..., f_n$ in A such that whenever $|f_i(x_1)-f_i(x_2)|<\delta$ for all $i\leqslant n$, we have $\|f_{x_1}-f_{x_2}\|<\varepsilon$ for all $f\in\mathcal{F}$. Let $\varepsilon>0$ be given and $\{g_i: i=1, 2, ..., m\}$ be a finite $\varepsilon/5$-net for $\mathcal{F}$. Thus for every $f\in\mathcal{F}$, there exists g_i such that $\|f-g_i\|<\varepsilon/5$. Consider the set $\{(g_i)_t: t\in\mathbb{R}, i\leqslant m\}$ of translates of $\{g_i: i\leqslant m\}$. Since it is a finite union of totally bounded sets, there exist $f_1, f_2, ..., f_n$ in A such that for every $i\leqslant m$ and $t\in\mathbb{R}$, there is an f_k satisfying $\|(g_i)_t-f_k\|<\varepsilon/5$. Now let $|f_i(x_1)-f_i(x_2)|<\varepsilon/5$ for all $i=1, 2, ..., n$, and $t\in\mathbb{R}$. Then we have for $f\in\mathcal{F}$

$$\begin{aligned}
&|f(x_1+t)-f(x_2+t)|\\
&\quad\leqslant|f(x_1+t)-g_i(x_1+t)|+|g_i(x_1+t)-f_k(x_1)|\\
&\qquad+|f_k(x_1)-f_k(x_2)|+|f_k(x_2)-g_i(x_2+t)|\\
&\qquad+|g_i(x_2+t)-f(x_2+t)|\\
&\quad\leqslant\|f-g_i\|+\|(g_i)_t-f_k\|+\varepsilon/5\\
&\qquad+\|(g_i)_t-f_k\|+\|f-g_i\|\\
&\quad<\varepsilon/5+\varepsilon/5+\varepsilon/5+\varepsilon/5+\varepsilon/5\\
&\quad=\varepsilon.
\end{aligned}$$

Hence

$$\begin{aligned}
\|f_{x_1}-f_{x_2}\|&=\sup\{|f(x_1+t)-f(x_2+t)|: t\in\mathbb{R}\}\\
&\leqslant\varepsilon.
\end{aligned}$$

This proves the result.

Now we construct an example of a family $\mathscr{F}$ of almost periodic functions which is unbounded but $\{T_f : f \in \mathscr{F}\}$ is totally bounded in $KL(A)$. For this. we require the following lemma.

LEMMA 2. *Let* $\{\alpha_k\}$ *be any sequence of positive reals. Then for each* $n \geqslant 1$, $\max\{\sum_1^n \alpha_i a_i : a_i \geqslant 0\}$ *under the constraint* $\sum_1^n a_i^2 = 1$, *is equal to* $(\sum_1^n \alpha_i^2)^{1/2}$.

Proof. Let $n \geqslant 1$ be fixed and $F(a_1, a_2, ..., a_n) = \sum_1^n \alpha_i a_i + \lambda(\sum_1^n a_i^2 - 1)$. Using Lagrange's method of undetermined multipliers for finding extremal values of the function $\sum_1^n \alpha_i a_i$ of n variables $a_1, a_2, ..., a_n$, we equate $\delta F/\delta a_i$ to zero. Hence we get $\alpha_i + 2\lambda a_i = 0$, $i = 1, 2, ..., n$. Using the constraint $\sum_1^n a_i^2 = 1$, we get $\sum_1^n \alpha_i^2 = 4\lambda^2$. Then the extremal value of $\sum_1^n \alpha_i a_i = -(\sum_1^n \alpha_i^2)/2\lambda = -2\lambda$. Solving $\sum_1^n \alpha_i^2 = 4\lambda^2$, two values of λ determine maximum and minimum values of $\sum_1^n \alpha_i a_i$, namely, $\pm(\sum_1^n \alpha_i^2)^{1/2}$. This shows that $\max \sum_{i=1}^n \alpha_i a_i = (\sum_1^n \alpha_i^2)^{1/2}$.

Remark 3. The maximum value of $\sum_{i=1}^n \alpha_i a_i$ under the constraint $\sum_1^n a_i^2 \leqslant 1$ is also equal to $(\sum_1^n \alpha_i^2)^{1/2}$.

EXAMPLE 2. Let $f_n(t) = \sum_1^n (1/k)\, e^{ikt}$, $n \geqslant 1$. Then $\|f_n\| = \sum_1^n (1/k)$. Hence $\{f_n\}$ is unbounded. We now prove that $\{T_{f_n} : n \geqslant 1\}$ is a Cauchy sequence in $KL(A)$. Let $\varepsilon > 0$ be given. Choose a positive integer N such that for $n > m \geqslant N$, $\sum_{m+1}^n (1/k^2) < \varepsilon^2$. Then for $n > m \geqslant N$, we have

$$\begin{aligned}
\|T_{f_n} - T_{f_m}\| &= \sup\{\|(T_{f_n} - T_{f_m})g\| : \|g\| \leqslant 1\} \\
&= \sup\left\{\left\|\sum_{m+1}^n \frac{1}{k} a(k, g)\, e^{ikt}\right\| : \|g\| \leqslant 1\right\} \\
&\leqslant \sup\left\{\left\|\sum_{m+1}^n \frac{1}{k} a(k, g)\, e^{ikt}\right\| : \sum |a(k, g)|^2 \leqslant 1\right\} \\
&\leqslant \sup\left\{\sum_{m+1}^n \frac{1}{k} |a(k, g)| : \sum |a(k, g)|^2 \leqslant 1\right\} \\
&\leqslant \sup\left\{\sum_{m+1}^n \frac{1}{k} |a(k, g)| : \sum_{m+1}^n |a(k, g)|^2 \leqslant 1\right\}.
\end{aligned}$$

Now applying Lemma 2, we get

$$\begin{aligned}
\|T_{f_n} - T_{f_m}\| &\leqslant \left(\sum_{m+1}^n \frac{1}{k^2}\right)^{1/2} \\
&< \varepsilon \qquad \text{for} \quad n > m \geqslant N.
\end{aligned}$$

This shows that $\{T_{f_n}: n \geqslant 1\}$ is a Cauchy sequence in $KL(A)$ and hence is totally bounded.

In the next proposition, we will see that for an almost periodic function f, even if the partial sums f_n of the Fourier series of f do not converge to f, the operators T_{f_n} do converge to T_f in uniform operator topology.

PROPOSITION 7. *For any $f \in A$ with Fourier series $\sum B_k e^{i\lambda_k x}$, let $f_n(x) = \sum_1^n B_k e^{i\lambda_k x}$. Then $\| T_{f_n} - T_f \| \to 0$.*

Proof. By using arguments similar to those in the previous example, it can be proved that $\{T_{f_n}: n \geqslant 1\}$ is a Cauchy sequence in $KL(A)$. Hence it is enough to show that for every trigonometric polynomial P, $T_{f_n}(P) \to T_f(P)$. Equivalently, we show that $T_{f_n}(e^{i\lambda t}) \to T_f(e^{i\lambda t})$ for every $\lambda \in \mathbb{R}$. But it follows easily from the observation that for each $\lambda \in \mathbb{R} - \{\lambda_k\}$, $T_{f_n}(e^{i\lambda t}) = 0 = T_f(e^{i\lambda t})$ and $T_{f_n}(e^{i\lambda_k t}) = B_k e^{i\lambda_k t} = T_f(e^{i\lambda_k t})$ for each $n \geqslant k$, $k = 1, 2, \ldots$. This then proves that $\| T_{f_n} - T_f \| \to 0$.

3. APPROXIMATIONS OF FOURIER COEFFICIENTS

In this section, we investigate the relation between the spectrum of the operator T_f with the Fourier coefficients of an almost periodic function f. This will then be effectively used to find the approximations for the Fourier coefficients of f. For this, we define a convolution type product on the space A of almost periodic functions. For $f, g \in A$, define $f * g$ by $(f * g)(x) = \lim_{n \to \infty} (1/2n) \int_{-n}^{n} f(x-t)\, g(t)\, dt$. Then it is known that with this product, A becomes a commutative Banach algebra without unit [5, 6.2]. In the rest of the section, we will consider A as the Banach algebra with the above-defined product and with the supremum norm. For definitions of completely continuous Banach algebras, minimal idempotents, minimal ideals, etc., we refer to [3].

PROPOSITION 8. *Let A be the Banach algebra of almost periodic functions as defined above. Then the following holds.*

(i) *A is completely continuous.*

(ii) *A is semi-simple.*

(iii) *A contains minimal idempotents and $e \in A$ is a minimal idempotent if and only if e is of the form $e^{i\lambda t}$ for some $\lambda \in \mathbb{R}$.*

Proof. (i) We have already shown that the operator $T_f: A \to A$, defined by $(T_f g)(x) = \lim_{n \to \infty} (1/2n) \int_{-n}^{n} f(x-t)\, g(t)\, dt = (f * g)(x)$, is a compact operator on A. Hence for each $f \in A$, $g \to f * g$ is a compact operator on A.

Now commutativity of the product shows that A is a completely continuous algebra.

(ii) For $h \in A$, let $r(h)$ denote the spectral radius of h and $\mathrm{rad}(A)$ the radical of A. Then we know that $\mathrm{rad}(A) = \{g \in A : r(f * g) = 0$ for all $f \in A\}$. Now to show that A is semi-simple, it is enough to prove that for every nonzero g in A, there exists $f \in A$ such that $r(f * g) \neq 0$. Let g be any nonzero member of A. Then by uniqueness of the Fourier series of an almost periodic function, it follows that there exists $\lambda \in \mathbb{R}$ such that $a(\lambda, g) \neq 0$. Let $f(t) = e^{i\lambda t}$. Then $f * g = a(\lambda, g) f$ and hence $r(f * g) = |a(\lambda, g)|\, r(f) = |a(\lambda, g)| \neq 0$. This proves (ii).

(iii) Since A is a semi-simple completely continuous Banach algebra, A contains minimal idempotents [3, Section 33, Theorem 15]. Let $e(t) = e^{i\lambda t}$ for some $\lambda \varepsilon \mathbb{R}$. Then $(e * e)(x) = \lim_{n\to\infty} (1/2n) \int_{-n}^{n} e^{i\lambda(x-t)} e^{i\lambda t}\, dt = e^{i\lambda x} = e(x)$, $eAe = Ae^2 = Ae$ and for any $f \in A$, $f * e = a(\lambda, f)e$. This shows that e is a minimal idempotent. Now to prove that these are the only minimal idempotents in A, let e be any nonzero minimal idempotent in A. Then there exists $\lambda \in \mathbb{R}$ such that $a(\lambda, e) \neq 0$. Let $f(t) = e^{i\lambda t}$. Then we show that $f = e$. By definition of minimal idempotent $e^2 = e$ and $eAe = Ae^2 = Ae = \mathbb{C}\, e$. Hence $e * f = \alpha e$ for some $\alpha \in \mathbb{C}$. But $e * f = a(\lambda, e) f \neq 0$. Therefore, $f = \beta e$ for $\beta = \alpha / a(\lambda, e) \neq 0$. Since $f^2 = f$, $(\beta e)^2 = \beta^2 e^2 = \beta^2 e = \beta e$ and $\beta = 0$ or 1. But β is nonzero. Thus (iii) is proved.

Let the spectrum of an operator K on A be denoted by $\sigma(K)$ and the spectrum of an f in the algebra A with the convolution type product be denoted by $\sigma(f)$.

PROPOSITION 9. *For any $f \in A$, $\sigma(T_f) = \sigma(f)$.*

Proof. First we observe that 0 is in $\sigma(T_f)$ as well as in $\sigma(f)$. Hence it is enough to show that the two sets coincide for nonzero points. Let $0 \neq \mu$ be in $\sigma(T_f)$. Then μ is an eigenvalue of T_f and hence there exists $g \in A$ such that $T_f g = f * g = \mu g$. Then it is easy to show that $\mu \in \sigma(f)$. Conversely let $0 \neq \mu \in \sigma(f)$. Then the map $g \to f * g$ being a compact operator on A, there exists a minimal idempotent e of A with $f * e = \mu e$ [3, Section 33, Proposition 7]. Hence μ is the eigenvalue of T_f and e is the corresponding eigenvector. This proves the proposition.

PROPOSITION 10. *Let $\sum B_k e^{i\lambda_k t}$ be the Fourier series of $f \in A$. Then $\sigma(f) = \{0\} \cup \{B_k\}$. Moreover, for each $k \geqslant 1$ $(f * e^{i\lambda_k t})(x) = B_k e^{i\lambda_k x}$.*

Proof. By Proposition 9, it is enough to show that $\sigma(T_f) = \{0\} \cup \{B_k\}$. Since $0 \in \sigma(T_f)$ and for every $k \geqslant 1$, $(T_f e^{i\lambda_k t})(x) = a(\lambda_k, f)\, e^{i\lambda_k x} = B_k e^{i\lambda_k x}$, $\{0\} \cup \{B_k\} \subset \sigma(T_f)$. To prove the equality of these sets, let us assume on the contrary that there exists a nonzero μ in $\sigma(T_f)$ such that $\mu \neq B_k$ for any

$k \geqslant 1$. Then there exists a nonzero $g \in A$ such that $T_f g = f * g = \mu g$. Let $\{\alpha_j\}$ be the sequence of Fourier exponents of g. Then for any $j \geqslant 1$, $(f * g) * e^{i\alpha_j x} = \mu g * e^{i\alpha_j x} = \mu a(\alpha_j, g)\, e^{i\alpha_j x}$. Also $(f * g) * e^{i\alpha_j x} = a(\alpha_j, f)\, a(\alpha_j, g)\, e^{i\alpha_j x}$. Hence $a(\alpha_j, g)[a(\alpha_j, f) - \mu] = 0$ for each $j \geqslant 1$. Since $a(\alpha_j, g) \neq 0$ for each $j \geqslant 1$, we get that $a(\alpha_j, f) = \mu$ for each $j \geqslant 1$. Since μ is nonzero, this means that $\{\alpha_j\} \subset \{\lambda_j\}$, the exponents of f. This contradicts the assumption that $\mu \neq B_j = a(\lambda_j, f)$ for any $j \geqslant 1$. This proves that $\sigma(T_f) = \sigma(f) = \{0\} \cup \{B_k\}$.

THEOREM 3. *Let $f \in A$ with Fourier series $\sum B_k e^{i\lambda_k t}$. Then each B_k can be approximated by the eigenvalues of Fredholm operators $S_n\colon C[0, n] \to C[0, n]$ defined by $(S_n g)(x) = (1/n) \int_0^n f(x - t)\, g(t)\, dt$, $0 \leqslant x \leqslant n$.*

Before going to the proof of this theorem, we prove the following lemma.

LEMMA 3. *Let $f \in A$ and $n \geqslant 1$. Define $K\colon C[0, n] \to C[0, n]$ by the equation $(Kg)(x) = (1/n) \int_0^n f(x - t)\, g(t)\, dt$, $0 \leqslant x \leqslant n$. Let μ be an eigenvalue of K with g as corresponding eigenvector. Then μ is also the eigenvalue of the operator $K'\colon A \to A$ defined by $(K'h)(x) = (1/n) \int_0^n f(x - t)\, h(t)\, dt$. Moreover, the function g' defined by $g'(x) = 1/n \int_0^n f(x - t)\, g(t)\, dt$, $x \in \mathbb{R}$, belongs to A and is the eigenvector corresponding to the eigenvalue of K'. Conversely each eigenvalue of K' is also an eigenvalue of K.*

Proof. For each s, $u \in \mathbb{R}$, we have $|g'(s + u) - g'(u)| = |(1/n) \int_0^n f(s + u - t)\, g(t)\, dt - (1/n) \int_0^n f(u - t)\, g(t)\, dt| \leqslant M \| f_s - f \|$, where $M = \sup\{| g(t)|\colon 0 \leqslant t \leqslant n\}$. Therefore, the almost periodicity of f implies the almost periodicity of g'. We also have $(1/n) \int_0^n f(x - t)\, g(t)\, dt = \mu g(x)$, $0 \leqslant x \leqslant n$. Hence $g'(x) = \mu g(x)$, $0 \leqslant x \leqslant n$. Then

$$\begin{aligned}(K'g')(x) &= \frac{1}{n} \int_0^n f(x - t)\, g'(t)\, dt \\ &= \frac{1}{n} \int_0^n f(x - t)\, \mu g(t)\, dt \\ &= \mu \frac{1}{n} \int_0^n f(x - t)\, g(t)\, dt \\ &= \mu g'(x), \qquad x \in \mathbb{R}.\end{aligned}$$

This shows that μ is the eigenvalue of K' with corresponding eigenvector g'. The converse is obvious.

Proof of Theorem 3. We know by Proposition 3 that the operators $T_n\colon A \to A$ defined by $(T_n h)(x) = (1/n) \int_0^n f(x - t)\, h(t)\, dt$ converge to T_f pointwise and $\{T_n\}$ is collectively compact. Hence we apply Theorem 4.8 [1] to conclude that for every open set $\Omega \supset \sigma(T_f)$, there exists N such that $\Omega \supset \sigma(T_n)$ for $n \geqslant N$. Now by Lemma 3, we see that $\Omega \supset \sigma(S_n)$ for $n \geqslant N$.

Remark 4. It may be observed that the last theorem gives information about the Fourier coefficients B_k without the knowledge of the corresponding Fourier exponents λ_k.

Remark 5. In Proposition 3, we have seen that $T_n \to T_f$ pointwise and $\{T_n\}$ is collectively compact. Under such assumptions, Osborn [8, Theorem 2] obtained the estimates for the approximations of the eigenvalues of T_f in terms of the averages of the eigenvalues of T_n. Hence by using these results, we can obtain estimates for approximations of Fourier coefficients of f in terms of the averages of the eigenvalues of Fredholm type operators S_n defined in Theorem 3.

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