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Extended incomplete gamma functions with applications

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Abstract

An extension of the generalized inverse Gaussian density function is proposed. Analogous to a recent useful generalization of the incomplete gamma functions, extensions of the generalized incomplete gamma functions are presented for which the usual properties and representations are naturally and simply extended. Several classical functions including, Abramowitz's functions, Dowson's integral function, Goodwin and Stalon's function, and astrophysical thermonuclear functions are proved to be special cases of these extensions. In addition, extended Meijer G-functions and Fox's H-functions are defined.

1. Introduction

Good [44,53] proposed the generalized inverse Gaussian distribution

$$g(t) := \frac{1}{I(\alpha; a, b)} t^{\alpha-1} e^{-at-bt^{-1}} \quad (t > 0, a > 0, b > 0, -\infty < \alpha < \infty), \tag{1.1}$$

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that was used by Sichel [68,69] to construct a mixture of Poisson distributions, $I(\alpha; a, b)$ being the normalizing constant. Wise [76] used the model (1.1) in biomedicine, and Marcus [53] used it as a unified stochastic model for the power laws in compartment analysis. Barndorff-Nielson and Halgreen [12] showed that the density (1.1) is infinitely divisible and Halgreen [53] showed that (1.1) is self-decomposable (also [49,50]). Chhikara and Folks [32,33] used (1.1) as a lifetime model and discussed its statistical applications. Jorgensen [53] studied the density (1.1) systematically and discussed its applications in different fields like fractures of air-conditioning equipment, traffic data, fracture toughness of MIG welds, and repair time data. For further applications and properties [5–11, 13,17,20,22,26,28–30,35,39,40,44,47,49,50,52,54,55,57,58,66,67,69,70,73–76].

The density (1.1) can further be generalized in infinitely many ways. However, a generalization of (1.1) will be useful if the corresponding cumulative density function and the reliability functions are found to be useful and less cumbersome. A natural extension of (1.1) is the density

$$f(t) := \begin{cases} C(\alpha, a, \beta, b) t^{\alpha-1} e^{-at-bt^{-\beta}} \\ \qquad (t > 0, a > 0, b > 0, \beta > 0, -\infty < \alpha < \infty), \\ 0 \quad \text{elsewhere}, \end{cases} \tag{1.2}$$

where

$$C := C(\alpha, a, \beta, b) = \left(\int_0^\infty t^{\alpha-1} e^{-at-bt^{-\beta}} \right)^{-1}, \tag{1.3}$$

is the normalizing constant.

A number of distributions such as inverse Gaussian distribution, generalized inverse Gaussian distribution, generalized gamma distribution, Weibull, Raleigh, folded-normal, and negative exponential are either special cases of (1.2) or can be obtained by a simple transformation of the variable t. These distributions are applicable in many problems arising in engineering, physical, social and biological sciences. The cumulative density function of the density (1.2) is

$$F(x) = C \int_0^x t^{\alpha-1} e^{-at-bt^{-\beta}} \, dt \quad (x > 0), \tag{1.4}$$

and its reliability function is given by

$$R(x) = 1 - F(x). \tag{1.5}$$

It is to be noted that while the study of the functions $F(x)$ and $R(x)$ is important in statistics and reliability theory, the functions (1.4) and (1.5) cannot be simplified in terms of a *finite combination* of tabulated special functions. The most general class of special functions like Fox's H-functions fail to accommodate the useful functions (1.4) and (1.5).

Chaudhry and Zubair [23] introduced the generalized incomplete gamma functions

$$\gamma(\alpha, x; b) = \int_0^x t^{\alpha-1} e^{-t-bt^{-1}}\, dt \tag{1.6}$$

and

$$\Gamma(\alpha, x; b) = \int_x^\infty t^{\alpha-1} e^{-t-bt^{-1}}\, dt, \tag{1.7}$$

found useful in a variety of heat conduction problems [24,78–81]. The decomposition and extension of these functions were also found to be useful [18,19,27, 80].

Miller and Moskowitz [64] found a representation of the generalized incomplete function (1.7) in terms of the Kampé de Fériet (KdF) functions and discussed its closed form representations. Miller [65] found several reduction formulae of the KdF functions in terms of the function (1.7) and discussed its relations with incomplete Weber integrals.

In this paper we introduce a pair of functions

$$\gamma(\alpha, x; b, \beta) = \int_0^x t^{\alpha-1} e^{-t-bt^{-\beta}}\, dt \quad (x \geqslant 0, b > 0; b = 0, \mathrm{Re}(\alpha) > 0) \tag{1.8}$$

and

$$\Gamma(\alpha, x; b, \beta) = \int_x^\infty t^{\alpha-1} e^{-t-bt^{-\beta}}\, dt \quad (x \geqslant 0, b \geqslant 0), \tag{1.9}$$

and call them extended incomplete gamma functions. We note that the generalized incomplete gamma functions (1.6)–(1.7) are special cases of (1.8)–(1.9) when $\beta = 1$,

$$\gamma(\alpha, x; b; 1) = \gamma(\alpha, x; b), \tag{1.10}$$

$$\Gamma(\alpha, x; b; 1) = \Gamma(\alpha, x; b). \tag{1.11}$$

The cumulative density function (1.4) and the reliability function (1.5) can be simplified in terms of the extended incomplete gamma functions to give

$$F(x) = Ca^{-\alpha}\gamma(\alpha, ax; ba^\beta; \beta), \tag{1.12}$$

$$R(x) = Ca^{-\alpha}\Gamma(\alpha, ax; ba^\beta; \beta). \tag{1.13}$$

Anderson et al. [31] have called the following integral functions

$$I_1(z,\nu) := \int_0^\infty y^\nu \exp(-y - zy^{-1/2})\,dy, \tag{1.14}$$

$$I_2(z,d,\nu) := \int_0^d y^\nu \exp(-y - zy^{-1/2})\,dy, \tag{1.15}$$

$$I_3(z,t,\nu) := \int_0^\infty y^\nu \exp\bigl(-y - (y+t)^{-1/2}\bigr)\,dy, \tag{1.16}$$

and

$$I_4(z,\delta ta,b;\nu) := \int_0^\infty y^\nu \exp(-y - by^\delta - zy^{-1/2})\,dy, \tag{1.17}$$

as the astrophysical thermonuclear functions.

These integral functions arise in the study of the thermonuclear reaction rates [31,61,63] of the stars. The astrophysical thermonuclear functions can be simplified in terms of the extended incomplete gamma functions to give [16,21]

$$I_1(z,\nu) = \Gamma\left(\nu+1, 0; z; \frac{1}{2}\right), \tag{1.18}$$

$$I_2(z,d,\nu) = \gamma\left(\nu+1, d; z; \frac{1}{2}\right), \tag{1.19}$$

$$I_3(z,t,\nu) = e^t \sum_{r=0}^{\nu} \binom{\nu}{r} (-t)^{\nu-r}\,\Gamma\left(r+1, t; z, \frac{1}{2}\right), \tag{1.20}$$

$$I_4(z,\delta,b;\nu) = \sum_{r=0}^{\infty} \frac{(-b)^r}{r!}\,\Gamma\left(\nu + r\delta + 1, 0; z; \frac{1}{2}\right). \tag{1.21}$$

Moreover, in view of (1.18)–(1.21), it is evident that the extended gamma functions will provide a solid foundation for the analytic study of the thermonuclear reaction rates of the stars that are significantly important in astrophysics and space sciences.

The Abramowitz's function [2, p. 1003]

$$f_m(z) := \int_0^\infty t^m e^{-t^2 - zt^{-1}}\,dt, \tag{1.22}$$

has been used in many fields of physics [59,60]. It has several applications in the field of particle and radiation transform. The function (1.22) can be expressed as a special case of the extended incomplete gamma function to give

$$f_m(z) = \frac{1}{2}\Gamma\left(\frac{m+1}{2}, 0; z; \frac{1}{2}\right). \tag{1.23}$$

Thus the usefulness of the extended incomplete gamma functions in physics, particle and radiation transform is evident in view of the relation (1.23).

Another integral function considered by Goodwin and Stalon [2,46] is given by

$$f(x) := \int_0^\infty \frac{e^{-t^2}}{t+x}\,dt. \tag{1.24}$$

The function in (1.24) is related to the exponential integral function via [2, p. 1003, (27.6.3)]

$$f(x) = \frac{1}{2}e^{-x^2}\mathrm{Ei}(x^2) + \sqrt{\pi}e^{-x^2}\int_0^x e^{t^2}\,dt, \tag{1.25}$$

where

$$F(x) := e^{-x^2}\int_0^x e^{t^2}\,dt \tag{1.26}$$

is the Dowson integral function. The function (1.24) can be simplified in terms of the extended incomplete gamma function to give

$$f(x) = \frac{1}{2}e^{-x^2}\Gamma\left(0, x^2; 2x; -\frac{1}{2}\right). \tag{1.27}$$

For further study of the related q-extension of the gamma family, the incomplete cylindrical functions, their properties and applications, we refer to [1,3,4,14,15, 25,34,36,38,41–43,51,56,71,77].

In this paper we have studied some properties of the extended incomplete gamma functions. It is anticipated that the present analysis will stimulate mathematicians and scientists to explore wide applications of these functions. Lastly we have introduced decomposition of Fox's H-functions that has unified all the previous extensions of the incomplete gamma functions.

2. Recurrence formula

The recurrence formula for the extended incomplete gamma function (1.9) naturally and simply extends the recurrence relation of the classical and generalized incomplete gamma functions.

Theorem 2.1.

$$\Gamma(\alpha+1,x;b;\beta)=\alpha\Gamma(\alpha,x;b;\beta)+b\beta\Gamma(\alpha-\beta,x;b;\beta)\\+x^{\alpha}e^{-x-bx^{-\beta}}. \tag{2.1}$$

Proof. Let us define

$$f(t):=e^{-t-bt^{-\beta}}H(t-x), \tag{2.2}$$

where

$$H(t-x):=\begin{cases}1 & \text{if } t>x,\\ 0 & \text{if } t<x,\end{cases}$$

is the Heaviside unit step function. The extended gamma function (1.9) is simply the Mellin transform of the function $f(t)$ in α [37, p. 307]

$$\Gamma(\alpha,x;b;\beta)=M\{f(t);t\to\alpha\}. \tag{2.3}$$

The differentiation of (2.2) in the sense of distribution yields

$$\frac{d}{dt}\{f(t)\}=(-1+b\beta t^{-\beta-1})f(t)+e^{-t-bt^{-\beta}}\delta(t-x), \tag{2.4}$$

where

$$\delta(t-x):=\frac{d}{dt}\big(H(t-x)\big) \tag{2.5}$$

is the Dirac delta function.

The Mellin transform of a function and its derivative are related via [37, p. 307]

$$-(\alpha-1)M\{f(t);t\to\alpha-1\}=M\left\{\frac{d}{dt}\big(f(t)\big);t\to\alpha\right\}. \tag{2.6}$$

From (2.5) and (2.6) we get

$$-(\alpha-1)\Gamma(\alpha-1,x;b;\beta)=-\Gamma(\alpha,x;b;\beta)+b\beta\Gamma(\alpha-\beta-1;x;b;\beta)\\+x^{\alpha-1}e^{-x-bx^{-\beta}} \tag{2.7}$$

which simplifies to give

$$\Gamma(\alpha,x;b;\beta)=(\alpha-1)\Gamma(\alpha-1,x;b;\beta)\\+b\beta\Gamma(\alpha-\beta-1,x;b;\beta)+x^{\alpha-1}e^{-x-bx^{-\beta}}. \tag{2.8}$$

Replacing α by $\alpha+1$ in (2.8) yields (2.1). □

Corollary [23, (15)].

$$\Gamma(\alpha+1,x;b)=\alpha\Gamma(\alpha,x;b)+b\Gamma(\alpha-1,x;b)+x^{\alpha}e^{-x-bx^{-1}}. \tag{2.9}$$

Proof. This follows from (2.1) when we take $\beta = 1$. $\square$

It is to be noted that the substitution $b = 0$ in (2.9) yields the recurrence relation

$$\Gamma(\alpha+1,x) = \alpha\Gamma(\alpha,x) + x^{\alpha}e^{-x}, \tag{2.10}$$

for the classical incomplete gamma function [45, p. 951, (8.356)(2)].

3. Decomposition formula

The decomposition formula

$$\gamma(\alpha,x) + \Gamma(\alpha,x) = \Gamma(\alpha), \tag{3.1}$$

for the classical incomplete gamma functions was proved to be a special case of the decomposition formula [23, (12)]

$$\gamma(\alpha,x;b) + \Gamma(\alpha,x;b) = 2b^{\alpha/2}K_{\alpha}\left(2\sqrt{b}\right), \tag{3.2}$$

for the generalized incomplete gamma functions. One could expect to have a similar formula for the extended incomplete gamma functions. Firstly, we prove the relation between $\Gamma(\alpha,0;b;\beta)$ and Fox's H-functions [61,62].

Theorem 3.1.

$$\Gamma(\alpha,0;b;\beta) = H_{0,2}^{2,0}\left(b \,\middle|\, \begin{matrix} - & - \\ (0,1), & (\alpha,\beta) \end{matrix}\right). \tag{3.3}$$

Proof. Putting $x = 0$ in (1.9) we get the complete integral

$$\Gamma(\alpha,0;b;\beta) = \int_0^{\infty} t^{\alpha-1}e^{-t-bt^{-\beta}}\,dt. \tag{3.4}$$

Multiplying (3.4) by b^{s-1} and integrating afterward with respect to b from $b = 0$ to $b = \infty$, we get

$$\int_0^{\infty} \Gamma(\alpha,0;b;\beta)b^{s-1}\,db = \int_0^{\infty} t^{\alpha-1}e^{-t}\left(\int_0^{\infty} b^{s-1}e^{-bt^{-\beta}}\right)dt. \tag{3.5}$$

However, the integral $\int_0^{\infty} b^{s-1}e^{-bt^{-\beta}}\,db$ in (3.5) can be simplified in terms of the gamma function to give [45, p. 942, (8.312)(2)]

$$\int_0^{\infty} b^{s-1}e^{-bt^{-\beta}}\,db = t^{\beta s}\Gamma(s) \quad (\mathrm{Re}(s) > 0). \tag{3.6}$$

From (3.5) and (3.6) we get

$$\int_0^\infty \Gamma(\alpha,0;b;\beta)b^{s-1}\,db = \Gamma(s)\int_0^\infty t^{\alpha+\beta s-1}e^{-t}\,dt$$
$$= \Gamma(s)\Gamma(\alpha+\beta s) \quad (\mathrm{Re}(\alpha+\beta s)>0). \tag{3.7}$$

Taking the inverse Mellin transform of both sides in (3.7) we get [37,61]

$$\Gamma(\alpha,0;b;\beta) = (2\pi i)^{-1}\int_{C-i\infty}^{C+i\infty} \Gamma(s)\Gamma(\alpha+\beta s)b^{-s}\,ds. \tag{3.8}$$

However, the integral representation (3.8) is the special case of Fox's H-function [62]

$$H^{2,0}_{0,2}\left(b\,\middle|\,\begin{matrix} - & - \\ (0,1), & (\alpha,\beta)\end{matrix}\right) = (2\pi i)^{-1}\int_{C-i\infty}^{C+i\infty} \Gamma(s)\Gamma(\alpha+\beta s)b^{-s}\,ds$$
$$(0<c<1). \tag{3.9}$$

From (3.8) and (3.9) we get the proof of (3.3). □

Corollary.

$$\gamma(\alpha,x;b;\beta) + \Gamma(\alpha,x;b;\beta) = H^{2,0}_{0,2}\left(b\,\middle|\,\begin{matrix} - & - \\ (0,1), & (\alpha,\beta)\end{matrix}\right). \tag{3.10}$$

Proof. This follows from (3.3) and from the fact that

$$\gamma(\alpha,x;b;\beta) + \Gamma(\alpha,x;b;\beta) = \Gamma(\alpha,0;b;\beta). \qquad \square \tag{3.11}$$

Corollary [23, p. 101, (12)].

$$\Gamma(\alpha,x;b) + \gamma(\alpha,x;b) = 2b^{\alpha/2}K_\alpha\bigl(2\sqrt{b}\bigr). \tag{3.12}$$

Proof. The proof of this decomposition formula for the generalized incomplete gamma function follows from (3.10) when we take $\beta = 1$ and use the relation [62, p. 145, (A6)]

$$H^{2,0}_{0,2}\left(b\,\middle|\,\begin{matrix} - & - \\ (0,1), & (\alpha,1)\end{matrix}\right) = G^{2,0}_{0,2}\left(b\,\middle|\,\begin{matrix} - & - \\ 0, & \alpha\end{matrix}\right) = 2b^{\alpha/2}K_\alpha\bigl(2\sqrt{b}\bigr). \tag{3.13}$$

It is to be noted that letting $b \to 0^+$ in (3.12) and using

$$\Gamma(\alpha) = \lim_{b\to 0} 2b^{\alpha/2}K_\alpha\bigl(2\sqrt{b}\bigr), \tag{3.14}$$

we get the decomposition formula

$$\Gamma(\alpha,x) + \gamma(\alpha,x) = \Gamma(\alpha) \tag{3.15}$$

for the classical incomplete gamma functions. □

Theorem 3.2.

$$\begin{aligned}\Gamma\left(\alpha,0;b;\frac{1}{n}\right)&=H_{0,2}^{2,0}\left[b\,\middle|\,\begin{matrix}-, & -\\(0,1), & (\alpha,n)\end{matrix}\right]\\&=(2\pi)^{(1-n)/2}\sqrt{n}\,G_{0,n+1}^{n+1,0}\left[\left(\frac{b}{n}\right)^n\,\middle|\,\begin{matrix}-, & -,\dots, & -\\0, & \frac{1}{n},\frac{2}{n},\dots,\frac{n-1}{n}, & \alpha\end{matrix}\right].\end{aligned}\tag{3.16}$$

Proof. Substituting $\beta=1/n$ in (3.3) and using the transformation relation [62, p. 142, (3.11.6)], [63, p. 4, (1.2.3)] we arrive at

$$\begin{aligned}\Gamma\left(\alpha,0;b;\frac{1}{n}\right)&=nH_{0,2}^{2,0}\left[b^n\,\middle|\,\begin{matrix}-, & -\\(\alpha,1), & (0,n)\end{matrix}\right]\\&=n\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty}\Gamma(\alpha+s)\Gamma(ns)b^{-ns}\,ds.\end{aligned}\tag{3.17}$$

However, the use of the multiplication formula [45, p. 946, (8.335)]

$$\Gamma(mz)=(2\pi)^{(1-m)/2}m^{mz-1/2}\prod_{k=0}^{m-1}\Gamma\left(z+\frac{k}{m}\right)\tag{3.18}$$

for the gamma function yields

$$\begin{aligned}\Gamma\left(\alpha,0;b;\frac{1}{n}\right)&=(2\pi)^{(1-n)/2}\sqrt{n}\frac{1}{2\pi i}\\&\quad\times\int_{C-i\infty}^{C+i\infty}\Gamma(\alpha+s)\prod_{k=0}^{n-1}\Gamma\left(s+\frac{k}{n}\right)(b/n)^{-ns}\,ds,\end{aligned}\tag{3.19}$$

which is exactly (3.16). □

Corollary.

$$\Gamma\left(\alpha,0;b;\frac{1}{2}\right)=\pi^{-1/2}G_{0,3}^{3,0}\left[\frac{b^2}{4}\,\middle|_{0,\frac{1}{2},\alpha}\right].\tag{3.20}$$

Proof. This is a special cases of (3.16) when we take $n=2$. □

Remark. The closed form representation (3.20) is important in view of the relation (1.18) that yields to the representation [16]

$$I_1(z,\nu)=\Gamma\left(\nu+1,0;z;\frac{1}{2}\right)=\pi^{-1/2}G_{0,3}^{3,0}\left[\frac{z^2}{4}\,\middle|_{0,\frac{1}{2},1+\nu}\right].\tag{3.21}$$

In view of (3.3) and the decomposition formula (3.10) it seems natural to regard

$$\Gamma(\alpha,0;b;\beta)=H_{0,2}^{2,0}\left(b\,\middle|\,\begin{matrix}\overline{}, & \overline{}\\(0,1), & (\alpha,\beta)\end{matrix}\right)$$

to be the extended gamma function of the variable b and the parameters α and β. We denote this function by $\Gamma_b(\alpha;\beta)$. The function $\Gamma_b(\alpha;\beta)$ has representation, in terms of the H-functions, similar to that of the generalized Bessel functions $J_\nu^\mu(z)$ and the generalized Mittag Leffler function $E_{\alpha,\beta}(z)$ and thus merits independent investigation.

4. Special cases

The relationship between the extended incomplete gamma functions and some of the known functions including generalized incomplete gamma functions, astrophysical thermonuclear functions, exponential integral function, Dowson's integral function, Abramowitz's function and Goodwin and Stalon's function are exhibited in Section 1. We state some additional special cases that follow from the results due to Buschman [62, p. 12].

Theorem 4.1.

$$\Gamma(\alpha,0;b;-1)=\Gamma(\alpha)(1+b)^{-\alpha}\quad(\operatorname{Re}(1+b)>0), \tag{4.1}$$

$$\Gamma\left(\alpha,0;b;-\frac{1}{2}\right)=2^{1-\alpha}\Gamma(2\alpha)\exp\left(\frac{b^2}{8}\right)D_{-2\alpha}\left(\frac{b}{\sqrt{2}}\right), \tag{4.2}$$

$$\Gamma(\alpha,0;b;-2)=\Gamma(\alpha)(2b)^{-\alpha/2}\exp\left(-\frac{1}{\alpha b}\right)D_{-\alpha}\left(\frac{1}{\sqrt{2b}}\right), \tag{4.3}$$

where $D_{-\alpha}$ are the parabolic cylindrical functions [62, p. 139].

Theorem 4.2 (Series representation).

$$\Gamma(\alpha,x;b;\beta)=\sum_{n=0}^{\infty}\Gamma(\alpha-n\beta,x)\frac{(-b)^n}{n!}\quad(x>0). \tag{4.4}$$

Proof. Replacing $e^{-bt^{-\beta}}$ in (1.9) by its series representation yields the series

$$\Gamma(\alpha,x;b;\beta)=\sum_{n=0}^{\infty}\frac{(-b)^n}{n!}\int_x^\infty t^{\alpha-n\beta-1}e^{-t}\,dt, \tag{4.5}$$

which is exactly (4.4). □

Remark. It seems interesting to explore a uniform asymptotic expansion of the extended incomplete gamma function (1.9). For $x=0$, the function coincides with the Fox H-function. Thus for $x=0$, the known results for the asymptotic expansion of the Fox's H-functions could be utilized. In fact we have (see [62, pp. 9–10] and [75, pp. 147–148])

$$\Gamma(\alpha,0;b;\beta)\sim\begin{cases}\Gamma(\alpha)+\frac{1}{\beta}\Gamma\left(-\frac{\alpha}{\beta}\right)b^{\alpha/\beta}+o(b) \\ \qquad\qquad \text{(for small } b,\ \alpha\neq 0\text{)}, \\ -\frac{1}{\beta}\ln b \quad \text{(for small } b,\ \alpha=0\text{)},\end{cases} \tag{4.6}$$

and

$$\Gamma(\alpha,0;b;\beta)\sim\frac{1}{\beta}\left(\frac{2\pi\beta}{1+\beta}\right)^{1/2}\beta^{(2\alpha+\beta)/2(1+\beta)}b^{(2\alpha-1)/2(\beta+1)} \\ \times\exp\left[-(1+\beta)^{\beta/(1+\beta)}b^{1/(1+\beta)}\right] \quad \text{(for large } b\text{)}. \tag{4.7}$$

5. Laplace transform representations

The utility of a function is increased when it provides a closed form representation of the Laplace transform of a class of functions. This fact is demonstrated in the following theorem.

Theorem 5.1. *Let L be the Laplace transform operator. Then*

$$\Gamma(\nu\beta,x^{-\beta};b;\beta)=\frac{1}{\beta}L\left\{t^{-\nu-1}\exp\left(-\frac{1}{t^{1/\beta}}\right)H(x-t);t\to b\right\} \\ (b>0,\beta>0). \tag{5.1}$$

Proof. According to the definition (1.9)

$$\Gamma(\nu\beta,x^{-\beta};b;\beta)=\int_{x^{-\beta}}^{\infty}t^{\nu\beta-1}\exp(-t-bt^{-\beta})\,dt. \tag{5.2}$$

The substitutions

$$t=\tau^{-\frac{1}{\beta}},\qquad dt=-\frac{1}{\beta}\tau^{-\frac{1}{\beta}-1}\,d\tau \tag{5.3}$$

in (5.2) lead to

$$\Gamma(\nu\beta,x^{-\beta};b;\beta)=\frac{1}{\beta}\int_{0}^{x}\tau^{-\nu-1}\exp\left(-\frac{1}{\tau^{1/\beta}}-b\tau\right)d\tau \tag{5.4}$$

which is exactly (5.1). $\square$

Corollary.

$$\Gamma(\nu\beta, 0; b; \beta) = \frac{1}{\beta} L\left\{t^{-\nu-1} \exp\left(-\frac{1}{t^{1/\beta}}\right); t \to b\right\}. \tag{5.5}$$

Proof. Letting $x \to \infty$ in (5.1) and using $H(\infty - t) \equiv 1$, we reach (5.5). □

Corollary [37, p. 146, (29)].

$$\Gamma(\nu, 0; b; 1) = L\left\{t^{-\nu-1} \exp\left(-\frac{1}{t}\right); t \to b\right\} = 2b^{\nu/2} K_\nu\left(2\sqrt{b}\right). \tag{5.6}$$

Remark. From (3.3) and (5.4) we have the representation

$$L\left\{t^{-\nu-1} \exp\left(-\frac{1}{t^{1/\beta}}\right); t \to b\right\} = \beta H_{0,2}^{2,0}\left(b \,\middle|\, \begin{matrix} - & - \\ (0,1), & (\nu\beta, \beta) \end{matrix}\right). \tag{5.7}$$

Several special cases of (5.1) and (5.6) can be written for various values of the parameters ν and β. For $\beta = 1$, we have

$$\Gamma(\nu, x^{-1}; b; 1) = \Gamma(\nu, x^{-1}; b).$$

Thus we get the identities [18, (23) and (29)]

$$\begin{aligned} &L\left\{t^{-\frac{3}{2}} \exp\left(-\frac{1}{t}\right) H(t) H(x-t); t \to b\right\} \\ &\quad = \frac{\pi}{2}\left[e^{-2\sqrt{b}} \operatorname{Erfc}\left(\frac{1}{\sqrt{x}} - \sqrt{bx}\right) + e^{2\sqrt{b}} \operatorname{Erfc}\left(\frac{1}{\sqrt{x}} + \sqrt{bx}\right)\right] \end{aligned} \tag{5.8}$$

and

$$\begin{aligned} &L\left\{t^{-\frac{1}{2}} \exp\left(-\frac{1}{t}\right) H(t) H(x-t); t \to b\right\} \\ &\quad = \frac{\pi}{2\sqrt{b}}\left[e^{-2\sqrt{b}} \operatorname{Erfc}\left(\frac{1}{\sqrt{x}} - \sqrt{bx}\right) - e^{2\sqrt{b}} \operatorname{Erfc}\left(\frac{1}{\sqrt{x}} + \sqrt{bx}\right)\right]. \end{aligned} \tag{5.9}$$

6. Log-convex property

A real valued function f on (a, b) is convex if

$$f\left(\lambda x + (1-\lambda) y\right) \leqslant \lambda f(x) + (1-\lambda) f(y) \quad (0 < \lambda < 1, x, y \in (a, b)). \tag{6.1}$$

A positive function f on (a, b) is log-convex if $\ln f$ is convex. It is to be noted that the gamma function $\Gamma : (0, \infty) \to (0, \infty)$ is log-convex [72, p. 42]. One would like to see if the extended complete gamma function

$$\Gamma_b(\alpha;\beta) := \Gamma(\alpha,0;b,\beta) = H_{0,2}^{2,0}\left(b \,\middle|\, \overline{(0,1),}\ \overline{(\alpha,\beta)}\right) \tag{6.2}$$

is log-convex or not. This is proved in the following theorem.

Theorem 6.1. *Let* $1 < p < \infty$ *and* $(1/p) + (1/q) = 1$. *Then*

$$\Gamma_b\left(\frac{x}{p}+\frac{y}{q};\beta\right) \leqslant \big(\Gamma_b(x;\beta)\big)^{1/p}\big(\Gamma_b(y;\beta)\big)^{1/q} \quad (b \geqslant 0, x > 0, y > 0). \tag{6.3}$$

Proof. Taking $\alpha = x/p + y/q$ in (3.4), we find

$$\Gamma_b\left(\frac{x}{p}+\frac{y}{q};\beta\right) = \int_0^\infty \big(t^{x-1}e^{-t-b/t^\beta}\big)^{1/p}\big(t^{y-1}e^{-t-b/t^\beta}\big)^{1/q}\,dt. \tag{6.4}$$

Using the Hölder inequality [45, p. 1131, (12.312)], we find

$$\Gamma_b\left(\frac{x}{p}+\frac{y}{q};\beta\right) \leqslant \left(\int_0^\infty t^{x-1}e^{-t-b/t^\beta}\,dt\right)^{1/p}\left(\int_0^\infty t^{y-1}e^{-t-b/t^\beta}\,dt\right)^{1/q}, \tag{6.5}$$

which is exactly (6.3). □

Corollary 6.1.

$$\Gamma_b\left(\frac{x+y}{2};\beta\right) \leqslant \sqrt{\Gamma_b(x;\beta)\Gamma_b(y;\beta)} \quad (x > 0, y > 0, b \geqslant 0, \beta > 0). \tag{6.6}$$

Proof. This is a special case of (6.3) when $p = q = 2$. □

Corollary 6.2.

$$\Gamma_b\left(\frac{1}{2};\beta\right) \leqslant \sqrt{\Gamma_b(x;\beta)\Gamma_b(y;\beta)} \quad (0 < x < 1, b \geqslant 0, \beta > 0). \tag{6.7}$$

Proof. This follows from (6.6) when we take $y = 1 - x$ $(0 < x < 1)$. □

Remark. As the arithmetic mean of two positive numbers is greater or equal to their geometric mean, it follows from (6.6) that

$$\Gamma_b\left(\frac{x+y}{2};\beta\right) \leqslant \sqrt{\Gamma_b(x;\beta)\Gamma_b(y;\beta)} \leqslant \frac{1}{2}\big(\Gamma_b(x;\beta)+\Gamma_b(y;\beta)\big). \tag{6.8}$$

For $\beta = 1$ in (6.8), we find the inequalities

$$\begin{aligned} K_{\frac{x}{2}+\frac{y}{2}}\left(2\sqrt{b}\right) &\leqslant \sqrt{K_x(2\sqrt{b})K_y(2\sqrt{b})} \\ &\leqslant \frac{1}{2}\left(b^{\frac{x-y}{2}} K_x\left(2\sqrt{b}\right) + b^{\frac{y-x}{4}} K_y\left(2\sqrt{b}\right)\right) \\ &(x > 0, y > 0, b > 0), \end{aligned} \tag{6.9}$$

satisfied by the Macdonald function.

7. Incomplete Fox's H-functions

Chaudhry and Zubair introduced the generalized incomplete gamma functions (1.6) and (1.7) that were found to be useful in a variety of heat conduction problems [24,78–81]. These functions have the inverse Mellin transform representations given by

$$\gamma(\alpha, x; b) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \Gamma(s)\gamma(\alpha + s, x)b^{-s}\, ds \tag{7.1}$$

and

$$\Gamma(\alpha, x; b) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \Gamma(s)\Gamma(\alpha + s, x)b^{-s}\, ds, \tag{7.2}$$

where $\gamma(\alpha, x)$ and $\Gamma(\alpha, x)$ are the classical incomplete gamma functions. Chaudhry and Zubair introduced the extensions of the generalized incomplete gamma functions [18,19]

$$\gamma_\nu(\alpha, x; b) := \left(\frac{2b}{\pi}\right)^{1/2} \int_0^x t^{\alpha-\frac{3}{2}} \exp(-t) K_{\nu+\frac{1}{2}}(b/t)\, dt, \tag{7.3}$$

$$\Gamma_\nu(\alpha, x; b) := \left(\frac{2b}{\pi}\right)^{1/2} \int_x^\infty t^{\alpha-\frac{3}{2}} \exp(-t) K_{\nu+\frac{1}{2}}(b/t)\, dt, \tag{7.4}$$

in connection with the extension of the generalized inverse Gaussian distribution [53]. These functions can also be represented as the inverse Mellin transform to give

$$\gamma_\nu(\alpha, x; b) = \frac{1}{\sqrt{4\pi}}\left(\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \Gamma\left(\frac{s-\nu}{2}\right)\Gamma\left(\frac{s+\nu+1}{2}\right) \times \gamma(\alpha+s, x)(b/2)^{-s}\,ds\right) \tag{7.5}$$

and

$$\Gamma_\nu(\alpha, x; b) = \frac{1}{\sqrt{4\pi}}\left(\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \Gamma\left(\frac{s-\nu}{2}\right)\Gamma\left(\frac{s+\nu+1}{2}\right) \times \Gamma(\alpha+s, x)(b/2)^{-s}\,ds\right). \tag{7.6}$$

The extended incomplete gamma functions (1.8) and (1.9) introduced in this paper have the inverse Mellin transform representations given by

$$\gamma(\alpha, x; b; \beta) = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \Gamma(s)\gamma(\alpha+\beta s, x)b^{-s}\,ds \tag{7.7}$$

and

$$\Gamma(\alpha, x; b; \beta) = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \Gamma(s)\Gamma(\alpha+\beta s, x)b^{-s}\,ds \quad (0 < c < 1). \tag{7.8}$$

It follows from (7.1) to (7.8) that none of these incomplete extended functions can be expressed in terms of a finite combination of the Meijer G-functions or the Fox H-functions. However, these functions are found to be useful in a variety of fields of engineering, statistics, astrophysics and applied mathematics. In order to have a uniform approach to these functions we propose to introduce the incomplete Fox H-functions $\gamma_{p,q}^{m,n}((z,x))$ and $\Gamma_{p,q}^{m,n}((z,x))$ as follows:

$$\begin{aligned}
\gamma_{p,q}^{m,n}\big((z,x)\big) &:= \gamma_{p,q}^{m,n}\left((z,x)\,\middle|\, \begin{matrix}\big((a_p,\alpha_p)\big)\\ \big((bq,\beta q)\big)\end{matrix}\right)\\
&:= \gamma_{p,q}^{m,n}\left((z,x)\,\middle|\, \begin{matrix}\big((a_1,\alpha_1),(a_2,\alpha_2),\ldots,(a_p,\alpha_p)\big)\\ \big((b_1,\beta_1),(b_2,\beta_2),\ldots,(b_q,\beta_q)\big)\end{matrix}\right)\\
&:= \frac{1}{2\pi i}\int_L g(s,x)z^{-s}\,ds,
\end{aligned} \tag{7.9}$$

where

$$g(s,x) := \frac{\prod_{j=1}^{m-1}\Gamma(b_j+\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j s)} \times \gamma(b_m+\beta_m s, x), \tag{7.10}$$

and L is the same contour as described in [61, pp. 140–141], [62, pp. 2–3].

Similarly, we define

$$\begin{aligned} \Gamma_{p,q}^{m,n}\big((z,x)\big) &:= \Gamma_{p,q}^{m,n}\left((z,x)\,\middle|\,\begin{matrix}\big((a_p,\alpha_p)\big)\\ \big((bq,\beta q)\big)\end{matrix}\right)\\ &:= \Gamma_{p,q}^{m,n}\left((z,x)\,\middle|\,\begin{matrix}\big((a_1,\alpha_1),(a_2,\alpha_2),\dots,(a_p,\alpha_p)\big)\\ \big((b_1,\beta_1),(b_2,\beta_2),\dots,(b_q,\beta_q)\big)\end{matrix}\right)\\ &:= \frac{1}{2\pi i}\int_L G(s,x)z^{-s}\,ds, \end{aligned} \tag{7.11}$$

where

$$G(s,x) := \frac{\prod_{j=1}^{m-1}\Gamma(b_j+\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j s)} \times \Gamma(b_m+\beta_m s, x), \tag{7.12}$$

and L is the same contour as described in [61, pp. 140–141], [62, pp. 2–3]. The functions (7.9) and (7.11) exist for all $x \geqslant 0$ under the same conditions as stated in [61, pp. 141–142]. It is to be noted that unlike Fox's H-functions the order of $(a_1,\alpha_1),(a_2,\alpha_2),\dots,(a_p,\alpha_p)$ and $(b_1,\beta_1),(b_2,\beta_2),\dots,(b_q,\beta_q)$ in (7.9) and (7.11) is important for $x > 0$.

The functions

$$\gamma_{p,q}^{m,n}\left((z,x)\,\middle|\,\begin{matrix}\big((a_p)\big)\\ \big((b_q)\big)\end{matrix}\right) := \gamma_{p,q}^{m,n}\left((z,x)\,\middle|\,\begin{matrix}\big((a_p,1)\big)\\ \big((b_q,1)\big)\end{matrix}\right) \tag{7.13}$$

and

$$\Gamma_{p,q}^{m,n}\left((z,x)\,\middle|\,\begin{matrix}\big((a_p)\big)\\ \big((b_q)\big)\end{matrix}\right) := \Gamma_{p,q}^{m,n}\left((z,x)\,\middle|\,\begin{matrix}\big((a_p,1)\big)\\ \big((b_q,1)\big)\end{matrix}\right) \tag{7.14}$$

can be regarded as incomplete G-functions [45,61,62].

Theorem 7.1.

$$\gamma_{p,q}^{m,n}\big((z,x)\big) + \Gamma_{p,q}^{m,n}\big((z,x)\big) = H_{p,q}^{m,n}(z). \tag{7.15}$$

Proof. This is a direct consequence of the definitions (7.9)–(7.12). □

Remark. In view of the decomposition formula (7.15), we note that the Fox's H-functions are recovered simply by substituting $x=0$:

$$\Gamma_{p,q}^{m,n}\left((z,0)\,\middle|\,\begin{matrix}\big((a_p,\alpha_p)\big)\\ \big((b_q,\beta_q)\big)\end{matrix}\right)\Bigg| = H_{p,q}^{m,n}\left(z\,\middle|\,\begin{matrix}(a_p,\alpha_p)\\ (b_q,\beta_q)\end{matrix}\right). \tag{7.16}$$

Thus the study of the incomplete Fox H-functions will provide a uniform approach to the study of a variety of incomplete functions useful in various branches of science and engineering. In particular, the functions (7.1)–(7.8) can be simplified in terms of the incomplete Fox H-functions to give

$$\gamma(\alpha,x;b)=\gamma_{0,2}^{2,0}\left((b,x)\,\middle|\,\begin{matrix}- & -\\ \big((0,1), & (\alpha,1)\big)\end{matrix}\right), \tag{7.17}$$

$$\Gamma(\alpha,x;b)=\Gamma_{0,2}^{2,0}\left((b,x)\,\middle|\,\begin{matrix}- & -\\ \big((0,1), & (\alpha,1)\big)\end{matrix}\right), \tag{7.18}$$

$$\gamma_\nu(\alpha,x;b)=\frac{1}{\sqrt{4\pi}}\gamma_{0,3}^{3,0}\left((b,x)\middle|\left(\left(-\frac{\nu}{2},\frac{1}{2}\right),\left(\frac{\nu+1}{2},\frac{1}{2}\right),(\alpha,1)\right)\right), \tag{7.19}$$

$$\Gamma_\nu(\alpha,x;b)=\frac{1}{\sqrt{4\pi}}\Gamma_{0,3}^{3,0}\left((b,x)\middle|\left(\left(-\frac{\nu}{2},\frac{1}{2}\right),\left(\frac{\nu+1}{2},\frac{1}{2}\right),(\alpha,1)\right)\right), \tag{7.20}$$

$$\gamma(\alpha,x;b;\beta)=\gamma_{0,2}^{2,0}\left((b,x)\,\middle|\,\begin{matrix}- & -\\ \big((0,1), & (\alpha,\beta)\big)\end{matrix}\right), \tag{7.21}$$

$$\Gamma(\alpha,x;b;\beta)=\Gamma_{0,2}^{2,0}\left((b,x)\,\middle|\,\begin{matrix}- & -\\ \big((0,1), & (\alpha,\beta)\big)\end{matrix}\right). \tag{7.22}$$

We have just given the basic definition of these functions besides demonstrating the fact that the generalized and extended incomplete gamma functions have been found useful in various branches of science and engineering. However, the basic properties of the H-functions are carried over naturally and simply to the incomplete H-functions. For example, we have

$$\begin{aligned}&\Gamma_{p,q}^{m,n}\left[(x,z)\,\middle|\,\begin{matrix}\big((a_1,\alpha_1),\dots,(a_p,\alpha_p)\big)\\ \big((b_1,\beta_1),\dots,(b_p,\beta_p)\big)\end{matrix}\right]\\ &\quad= k\Gamma_{p,q}^{m,n}\left[(z^k;x)\,\middle|\,\begin{matrix}\big((a_1,k\alpha_1),\dots,(a_p,k\alpha_p)\big)\\ \big((b_1,k\beta_1),\dots,(b_q,kb_q)\big)\end{matrix}\right],\end{aligned} \tag{7.23}$$

$$z_{p,q}^{m,n}\left[(z,x)\left|\begin{matrix}\big((a_1,\alpha_1),\dots,(a_p,\alpha_p)\big)\\ \big((b_1,\beta_1),\dots,(b_p,\beta_p)\big)\end{matrix}\right.\right]$$
$$=\frac{1}{z^{\sigma}}\Gamma_{p,q}^{m,n}\left[(z;x)\left|\begin{matrix}\big((a_1+\sigma\alpha_1,\alpha_1),\dots,(a_p+\sigma\alpha_p,\alpha_p)\big)\\ \big((b_1+\sigma\beta_1,\beta_1),\dots,(b_p+\sigma\beta_q,\beta_q)\big)\end{matrix}\right.\right]. \tag{7.24}$$

It is important to note that several properties of the incomplete Fox H-functions including asymptotic representations, transform representations, differential equations, special cases, and recurrence relations are yet to be explored.

8. Graphical and tabular representation of $\Gamma(\alpha, x; b; \beta)$

For numerical and scientific computations, the extended incomplete gamma function can easily be tabulated by using IMSL FORTRAN subroutines for mathematical applications [48]. In this regard, the values of the function are calculated by using the numerical integration subroutine QDAGI [83]. For $\beta = 1$, the extended incomplete gamma function coincides with the generalized incomplete gamma function (Eq. (1.11)), whose tabular and graphical representations are given in Tables 1–9 and Figs. 1–9 in [23]. For other values of β, the function can be evaluated similarly.

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