Exact solutions for MHD flow of couple stress fluid with heat transfer

Najeeb Alam Khan *, Hassam Khan, Syed Anwar Ali

Department of Mathematical Sciences, University of Karachi, Karachi 75270, Pakistan

Received 23 May 2013; revised 25 August 2014; accepted 21 October 2014
Available online 11 February 2015

Abstract This paper aims at presenting exact solutions for MHD flow of couple stress fluid with heat transfer. The governing partial differential equations (PDEs) for an incompressible MHD flow of couple stress fluid are reduced to ordinary differential equations by employing wave parameter. The methodology is implemented for linearizing the flow equations without extra transformation and restrictive assumptions. Comparison is made with the result obtained previously.

2010 MATHEMATICS SUBJECT CLASSIFICATION: 35C07; 76W99; 35A25

© 2015 Production and hosting by Elsevier B.V. on behalf of Egyptian Mathematical Society.

1. Introduction

Many non-linear science problems can appropriately and exactly be described by mathematical model of non-linear equations. Obtaining an exact solution of a non-linear PDEs [1–4] plays a dynamic role in the non-linear problems. If exact solution is obtainable, facilitate the confirmation of numerical solvers and stability theory. Numerous effective methods such as inverse scattering method, the tanh method, Exp-function method, the group analysis method have been extensively used to obtain exact solutions [5].

Since classical continuum theory enables to explain the rheological flow behavior of a Newtonian lubricant blended with various additives. A few of micro-continuum theories have been proposed [5–8]. The couple stress fluid model is one of the several models that anticipated to portray response characteristics of non-Newtonian fluids. They are a particular non-Newtonian fluid possessing “couple stress” effects. Couple stress fluid theory originated by Vijay Kumar Stokes in his treatise “Theories of Fluids with Microstructure” [9], is one among the polar fluid theories that takes into account couple stresses in addition to the classical Cauchy stress. In fact, the rotation vector is equal to the one-half the curl of the velocity vector as in the case in Newtonian fluids. Second order gradient of the velocity vector, rather than the kinematically independent rotation vector of asymmetric hydrodynamics is introduced into stress constitutive equations and consequently the theory yields only one vector equation to describe the velocity field. Moreover, microstructure of couple stress fluid is mechanically momentous. If the order of the magnitude of microstructure is equal as the characteristic geometric dimension of the problem considered, then the effect of microstructure on a liquid can be felt [10]. The spin field due to micro-rotation of these freely suspended particles in a vis-
cous medium result in an anti-symmetric stress, which is known as couple-stress, and thus forming couple-stress fluid. The study of couple-stress fluid is very useful in understanding various physical problems because in the biomechanical area, this couple stress fluid model has been applied to study the mechanism of peristalsis [11,12]. One of the applications of couple-stress fluid is its use to study the mechanism of lubrication of synovial joints [13], which has grown to be the object of scientific research. The shoulder, knee, hip and ankle joints are the loaded-bearing synovial joints of human body. These joints Normal synovial fluid is clear or yellowish and is a viscous, non-Newtonian fluid. The couple stress fluids are proficient of describing different types of lubricants, suspension fluids, blood, etc. Colloidal fluids, liquid containing long chain molecules as polymeric suspension animal and human blood, polymer-thickened oils, lubricants containing small amount of polymer additive, electro-therochemical fluids and synthetic fluids are examples of these fluids. Moreover, some of the non-Newtonian flow characteristics of blood can be explained by supposing the blood to be a fluid with couple stress. It is well recognized that at low shear stress rates during its flow through narrow vessels, being the suspension of cells, blood behave like a non-Newtonian fluid [14].


The equation of motion of non-Newtonian [25,26] fluids is undoubtedly non-linear and become higher order so for couple stress fluids. Their exact solutions are extraordinary or nonexistent. In special cases solutions have been obtained by Islam et al. [27,28].

The basic aim of this paper was to find the exact solution of two dimensional MHD couple stress fluid further, heat transfer analysis is also taken into account. Traveling wave phenomenon was implemented for obtaining the exact solution of MHD aligned flow of second grade fluid by Khan et al. [29]. They recovered the polynomial solution for the Ref. [30]. Moreover, Khan et al. [31] presented the traveling wave solutions of micropolar fluid. They showed that the result obtained by Shahzad et al. [25] can be found from their investigation as a special case. We offered the solution methodology for obtaining the exact solution of couple stress fluid.

2. Formulation of the problem

The equations of motion of an incompressible MHD couple stress fluids with heat transfer are governed by the system

\[ \nabla \cdot V = 0 \] (1)

\[ \frac{\partial V}{\partial t} + (V \cdot \nabla) V = -\frac{\nabla p}{\rho} - \frac{\mu}{\rho} (\nabla \times \nabla \times V) - \frac{\eta}{\rho} (\nabla \times \nabla \nabla \times V) - \frac{\Omega}{\rho} (\nabla \times H) \times H \] (2)

\[ \nabla \cdot H = 0 \] (3)

\[ \frac{\partial H}{\partial t} + \nabla \times (V \times H) + \frac{1}{\mu_0 \sigma} \nabla \times (\nabla \times H) = 0 \] (4)

\[ \rho C_p \left( \frac{\partial T}{\partial t} + (V \cdot \nabla) T \right) = k \nabla^2 T + \phi \] (5)

where \( V \) is the velocity, \( p \) is the pressure function, \( H \) is the magnetic field, \( \rho \) is the density, \( \mu \) is the constant viscosity, \( \mu' \) is the magnetic permeability, \( \sigma \) is electrical conductivity, \( T \) is the temperature, \( k \) is the thermal conductivity, \( C_p \) is the specific heat, \( \eta \) is material constant for couple stress fluid, \( \phi \) is the dissipation function, \( v = \frac{\eta}{\rho} \) is the kinematic viscosity and \( \eta^* = \frac{\mu}{\rho} \) is the couple stress parameter. Here, we shall consider a MHD couple stress fluid with heat transfer in a plane, taking

\[ V = (u(x, y, t), v(x, y, t), 0), H = (H_1(x, y, t), H_2(x, y, t), 0), p = p(x, y, t), \text{ and } T = T(x, y, t) \] so that our flow Eqs. (1)–(5) take the form

\[ u_x + v_y = 0 \] (6)

\[ u_t + u u_x + v u_y = -\frac{p_x}{\rho} + v(u_{xx} + u_{yy}) - \frac{\eta}{\rho} (u_{xxxx} + 2u_{xyy} + u_{yyyy}) - \frac{\Omega}{\rho} H_x (H_{2x} - H_{1y}) \] (7)

\[ v_t + v u_x + v v_y = -\frac{p_y}{\rho} + v(v_{xx} + v_{yy}) - \frac{\eta}{\rho} (v_{xxxx} + 2v_{xyy} + v_{yyyy}) - \frac{\Omega}{\rho} H_y (H_{2x} - H_{1y}) \] (8)

\[ H_{2x} - H_{1y} = \frac{1}{\mu' \sigma} [H_{2xxx} + H_{2xyy} - H_{1xxy} - H_{1yyy}] + v H_{1xx} + v H_{1yy} + v H_{1} - u H_{2xx} - u H_{2y} - u H_{2yy} - u H_{2x} \] (9)

\[ H_{2x} + H_{1y} = 0 \] (10)

\[ \rho C_p (T_t + u T_x + v T_y) = k (T_{xx} + T_{yy}) + \mu \left[ u_x^2 + 2u_x v_x + v_x^2 + 4v_y^2 \right] + \eta \left[ (u_{xx} + u_{yy})^2 + (v_{xx} + v_{yy})^2 \right] \] (11)

3. Methodology implementation

The method under consideration can be summarized as follows: for the present system of coupled PDEs

\[ R_0 (u_x, u_y) = 0 \] (12)

\[ R_1 (u, v, p_x, u, u_y, u_{xx}, \ldots, v_x, v_y, v_{xx}, \ldots, H_1, H_{1x}, H_{2x}) = 0 \] (13)
where $a_0$, $a_1$, are arbitrary constants of integration. Assuming that $a_1 = 0$. On eliminating pressure $p$, from Eqs. (24) and (25) invoking. Eqs. (29) and (30), we get

$$\eta^* u'' - \frac{v}{(m^2 + n^2)} u'' + \frac{(C + a_0)}{(m^2 + n^2)} u'' = 0$$  \hspace{1cm} (31)$$

If we take $\eta^* \to 0$ reduce to classical Newtonian flow problem. Here, two cases shall be discussed independently.

I. $C + a_0 = 0$.
II. $C + a_0 \neq 0$.

**Case I**

In this case Eq. (31) takes the form

$$u'' - A v'' = 0$$  \hspace{1cm} (32)$$

where $A = \frac{v}{\eta (m^2 + n^2)}$.

Which is on solving gives

$$u = a_2 + a_3 \xi + a_4 e^{\sqrt{2}\xi} + a_5 e^{-\sqrt{2}\xi}$$  \hspace{1cm} (33)$$

Using Eq. (33) in Eq. (29), we find

$$v = -m \left( a_2 + a_3 \xi + a_4 e^{\sqrt{2}\xi} + a_5 e^{-\sqrt{2}\xi} \right) + a_0$$  \hspace{1cm} (34)$$

Substituting Eqs. (30)–(34) in Eq. (26) with $a_1$, we have

$$H''_1 = 0$$  \hspace{1cm} (35)$$

Which on integration gives

$$H_1 = a_6 \xi^2 + a_7 \xi + a_8$$  \hspace{1cm} (36)$$

Utilizing Eq. (35) in Eq. (30), we get

$$H_2 = -\frac{m}{n} (a_6 \xi^2 + a_7 \xi + a_8)$$  \hspace{1cm} (37)$$

The heat equation for this case is

$$T'' + \nu H''_1 + \delta u'' = 0$$  \hspace{1cm} (38)$$

where $\chi = \frac{\mu (m^2 + n^2)}{k \nu}$, $\delta = \frac{\eta (m^2 + n^2)}{k}$.

Which provides the solution

$$T = a_9 e^{\sqrt{2}\xi} + E_1 e^{-\sqrt{2}\xi} + E_2 a_4 e^{\sqrt{2}\xi} + E_3 a_5 e^{-\sqrt{2}\xi} + (AE_3 + E_3) \xi^2 + a_{10} \xi + a_{11}$$  \hspace{1cm} (39)$$

where

$$E_0 = -\frac{1}{4} a_3^2 (\delta A + \chi), \quad E_1 = -\frac{1}{4} a_3^2 (\delta A + \chi),$$

$$E_2 = -\frac{2a_4 \chi}{\sqrt{A}}, \quad E_3 = -Aa_5 a_6 (A - \chi), \quad E_4 = -\frac{1}{2} a_5^2$$

And $a_0$, $i = 1, 2, \ldots, 11$ are constants of integration. Returning back to original variables, we get

$$u(x, y, t) = a_2 + a_3 (mx + ny + Ct) + a_4 e^{\sqrt{3}(mx + ny + Ct)} + a_5 e^{-\sqrt{3}(mx + ny + Ct)}$$  \hspace{1cm} (40)$$

$$v(x, y, t) = -m \left( a_2 + a_3 (mx + ny + Ct) + a_4 e^{\sqrt{3}(mx + ny + Ct)} + a_5 e^{-\sqrt{3}(mx + ny + Ct)} \right) + a_0$$  \hspace{1cm} (41)$$

\[ \text{Exact solutions for MHD flow of couple stress fluid with heat transfer} \]
\[ H_1(x, y, t) = a_0 (mx + ny + Ct)^2 + a_1(mx + ny + Ct) + a_6 \]  
(42)

\[ H_2(x, y, t) = -\frac{m}{n} (a_0 (mx + ny + Ct)^2 + a_1(mx + ny + Ct) + a_6) \]  
(43)

\[ T = E_0 e^{\sqrt{A} (mx + ny + Ct)} + E_1 e^{-\sqrt{A} (mx + ny + Ct)} + E_2 a e^{\sqrt{A} (mx + ny + Ct)} + (AE_1 + E_4)(mx + ny + Ct)^2 \]
\[ + a_{10}(mx + ny + Ct) + a_{11} \]  
(44)

**Case II**

For this case Eq. (31) takes the form

\[ \nu^2 - \nu t' + Bb' = 0 \]  
(45)

where \( A = \frac{(C+D)}{\nu} \), \( B = \frac{(C+D)}{\nu} \).

The auxiliary equation

\[ \mu^3 - \mu m^2 + B = 0 \]  
(46)

Eq. (45) admits the solution

\[ u = a_{12} + a_{13} e^{\xi i} + a_{14} e^{\zeta i} + a_{15} e^{\xi i} \]  
(47)

where \( \lambda_i, i = 1, 2, 3 \) are the roots of equation. Utilizing Eq. (47) in Eq. (30), we find

\[ \psi = -\frac{m(a_{12} + a_{13} e^{\xi i} + a_{14} e^{\zeta i} + a_{15} e^{\xi i})}{n} + a_0 \]  
(48)

Substituting Eqs. (47) and (48) and (30) in Eq. (26), we have

\[ H''^2 + \Gamma H'' = 0 \]  
(49)

where \( \Gamma = \frac{\nu e^{(C+D)}}{\nu^2 m^2} \).

Which gives the solution

\[ H_1 = a_{10} e^{\xi i} + a_{13} \zeta + a_{18} \]  
(50)

Utilizing Eq. (50) in Eq. (30), we get

\[ H_2 = -\frac{m}{n} (a_{10} e^{\xi i} + a_{13} \zeta + a_{18}) \]  
(51)

Invoking Eqs. (47) and (29) in Eq. (28)

\[ T'' - \varepsilon T' + g \mu^2 + \delta \mu^2 = 0 \]  
(52)

where \( \varepsilon = \frac{\mu e^{(C+D)}}{\mu^2 m^2} \), \( \chi = \frac{\mu e^{(C+D)}}{\nu^2 m^2} \), \( \delta = \frac{\mu e^{(C+D)}}{\nu^2 m^2} \).

Which admits the solution

\[ T = b_0 e^{\lambda_1 \zeta} + b_1 e^{\lambda_2 \zeta} + b_2 e^{\lambda_3 \zeta} + b_3 e^{\lambda_4 \zeta} + b_4 e^{\lambda_5 \zeta} + b_5 e^{\lambda_6 \zeta} + b_{20} \]  
(53)

where \( \lambda_0, j = 1, 2, \ldots, 20 \), are the constants of the integration.

\[ b_0 = a_{10} a_{12} (\delta \lambda_1^2 + \chi) \]  
\[ b_1 = -\frac{a_{10} a_{12} (\delta \lambda_2^2 + \chi)}{2(1 + 2\lambda_1)} \]  
\[ b_2 = -\frac{a_{10} a_{12} (\delta \lambda_3^2 + \chi)}{2(1 + 2\lambda_1)} \]  
\[ b_3 = -\frac{a_{10} a_{12} (\delta \lambda_4^2 + \chi)}{(1 + 2\lambda_1)(1 + 2\lambda_1)} \]  
\[ b_4 = -\frac{a_{10} a_{12} (\delta \lambda_5^2 + \chi)}{(1 + 2\lambda_1)(1 + 2\lambda_1)} \]  
\[ b_5 = -\frac{a_{10} a_{12} (\delta \lambda_6^2 + \chi)}{(1 + 2\lambda_1)(1 + 2\lambda_1)} \]  
\[ b_{20} = \frac{a_{10}}{\varepsilon} \]

The velocity components, magnetic field and temperature in the original variables are

\[ u(x, y, t) = a_{12} + a_{13} e^{(mx + ny + Ct)} + a_{14} e^{(sx + ny + Ct)} + a_{15} e^{(tx + ny + Ct)} \]  
(54)

\[ v(x, y, t) = -\frac{m(a_{12} + a_{13} e^{(mx + ny + Ct)} + a_{14} e^{(mx + ny + Ct)} + a_{15} e^{(mx + ny + Ct)})}{n} + a_0 \]  
(55)

\[ H_1(x, y, t) = a_{10} e^{(mx + ny + Ct)} + a_{17} (mx + ny + Ct) + a_{18} \]  
(56)

\[ H_2(x, y, t) = -\frac{m}{n} (a_{10} e^{(mx + ny + Ct)} + a_{17} (mx + ny + Ct) + a_{18}) \]  
(57)

\[ T = b_0 e^{\lambda_1 (mx + ny + Ct)} + b_1 e^{\lambda_2 (mx + ny + Ct)} + b_2 e^{\lambda_3 (mx + ny + Ct)} + b_3 e^{\lambda_4 (mx + ny + Ct)} + b_4 e^{\lambda_5 (mx + ny + Ct)} + b_5 e^{\lambda_6 (mx + ny + Ct)} + b_{20} e^{(mx + ny + Ct)} + a_{20} \]  
(58)

\[ 5. \text{Conclusions} \]

An attempt to obtain exact solutions of MHD flow of couple stress fluid with heat transfer has been made. The methodology in the present work is easy to linearize the equation of MHD flow for couple stress fluid. It should be noted that when \( \eta \to 0 \) and \( C + a_0 < 0 \), there is no couple stresses, exponential type of solutions has been recovered for viscous and non-Newtonian fluids [32–34]. The method has been used in a direct way without any restrictive assumptions and labourious calculations. For future research, we will solve the three dimensional Newtonian and couple stress flow equations by using proposed method. Moreover, this methodology can be employed on other non-Newtonian fluids to obtain an exact solutions, for instance pseudoplastic and rate type fluids.

\[ \text{Acknowledgments} \]

The author Najeed Alam Khan is highly thankful and grateful to the reviewers for careful assessment and suggestion regarding the earlier version of this manuscript. He is also thankful to the Dean of the faculty of Science, University of Karachi for supporting this work.

\[ \text{References} \]


G. Ramananah, Squeeze films between finite plates lubricated by fluids with couple stresses, Wear 17 (1979) 315–320.


