The normalizer of a metabelian group in its integral group ring

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Abstract

In this paper, we discuss two cases where all \( C \)-automorphisms are inner; therefore, the normalizer property holds for those cases. Our results generalize a result of Marciniak and Roggenkamp. As an application of our theorems, we prove that the normalizer property holds for the integral group ring of a split finite metabelian group with a dihedral Sylow 2-subgroup. Our proofs rely heavily on the method of Marciniak and Roggenkamp.

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1. Introduction

Let \( G \) be a group and let \( U(ZG) \) be the group of units of the integral group ring \( ZG \). It is a classical problem in the theory of group rings to investigate the normalizer \( N_{U}(G) \) of \( G \) in \( U(ZG) \) (see [6,13] for details). It is clear that \( N_{U}(G) \) contains \( G \) and \( Z = Z(U(ZG)) \), the subgroup of central units of \( U \).

Problem 43 in [13] asks whether \( N_{U}(G) = GZ \) when \( G \) is finite. For the first time this equality was proved for finite nilpotent groups by Coleman [1], and later extended by Jackowski and Marciniak [5] to all finite groups having a normal Sylow 2-subgroup. In particular, this property holds for all finite groups of odd order. Mazur noted that this question and the isomorphism problem are
closely related (see Mazur [9–11]). Hertweck constructed counterexamples to the normalizer problem, and then, using them and a clever generalization of Mazur’s results, he managed to construct a counterexample to the isomorphism problem [2–4]. Since the problem is connected to the isomorphism problem, it is still of interest to know which groups enjoy the normalizer property.

Recently, Parmenter, Sehgal and the author [7] proved that the normalizer property holds for any finite group $G$ such that $R(G)$ is not trivial, where $R(G)$ denotes the intersection of all non-normal subgroups of $G$. Meanwhile, Marciniak and Roggenkamp [8] proved that this property holds for finite metabelian groups with an abelian Sylow 2-subgroup. In this paper, we discuss two important situations when all $C$-automorphisms are inner; therefore, the normalizer property holds for those cases (Theorems 2.8 and 2.17). Using these results, we extend the above mentioned result of Marciniak and Roggenkamp to some metabelian groups with not necessarily abelian Sylow 2-subgroups. For example, we prove that the normalizer property holds for the integral group ring of a split finite metabelian group with a dihedral Sylow 2-subgroup. Our proofs rely heavily on the method of Marciniak and Roggenkamp in [8].

2. The normalizer $N_{\mathcal{U}}(G)$ for metabelian groups

Any unit $u \in N_{\mathcal{U}}(G)$ determines an automorphism $\rho = \rho_u$ of $G$ such that $\rho(g) = ugu^{-1}$ for all $g \in G$. We now consider the subgroup $\text{Aut}_{\mathcal{U}}(G)$ formed by all such automorphisms and it is not hard to see that the normalizer problem described in [13] is equivalent to Question 3.7 in [5]:

“Is $\text{Aut}_{\mathcal{U}}(G) = \text{Inn}(G)$ for all finite groups?”

It is convenient to use this equivalent form to discuss the normalizer problem here, and we will describe some finite metabelian groups for which the normalizer property holds.

Every automorphism in $\text{Aut}_{\mathcal{U}}(G)$ automatically satisfies several properties described by Coleman and the following definition was introduced by Marciniak and Roggenkamp in [8].

**Definition 2.1.** An automorphism $\rho$ of a finite group $G$ is called a Coleman automorphism, or a $C$-automorphism for short, if $\rho^2 = \rho \circ \rho$ is inner, $\rho$ preserves the conjugacy classes in $G$ and for every Sylow $p$-subgroup $P$ of $G$, we have $\rho|_P = \text{conj}(g)|_P$ for some $g \in G$.

**Remark 2.2.** Our interest in those automorphisms comes from the fact that all automorphisms in $\text{Aut}_{\mathcal{U}}(G)$ are $C$-automorphisms. Therefore, if all $C$-automorphisms of $G$ are inner, then the normalizer property holds for $G$. 
We first describe a necessary and sufficient condition for a $C$-automorphism of a metabelian group to be inner.

**Proposition 2.3.** Let $G$ be a metabelian group and let $B$ be an abelian normal subgroup of $G$ for which the quotient group $A = G/B$ is also abelian. Then a $C$-automorphism $\rho$ of $G$ is inner if and only if $\rho|_{B \cup P} = \text{conj}(g)|_{B \cup P}$ for some $g \in G$, where $P$ is a Sylow 2-subgroup of $G$.

The proof is standard, but we include it for completeness.

**Proof.** The necessity of the proposition is evident. Let $\rho$ be a $C$-automorphism of $G$ such that $\rho|_{B \cup P} = \text{conj}(g)|_{B \cup P}$. Substituting for $\rho$ an new $C$-automorphism $\rho \circ \text{conj}(g^{-1})$, we have $\rho|_B = \text{id}|_B$ and $\rho|_P = \text{id}|_P$. Note that the subgroup generated by $B$ and $P$ is exactly the pre-image $H$ of $A_2$ in $G$; i.e., $H = \langle B, P \rangle$. Thus $\rho|_H = \text{id}|_H$. Since $\rho$ preserves conjugacy classes, obviously it induces the identity on $A$. Now we define a map $\delta : A \to B$ by $\delta(a) = \rho(y)y^{-1}$ for all $a \in A$, where $y$ is any pre-image of $a$ in $G$. It is routine to check that $\delta$ is well defined, $\delta(a) \in B$, and $\delta$ is a 1-cocycle. Thus $[\delta] \in H^1(A, B)$. Note that $H^1(A, B) = H^1(A_2 \times A'_2, B) = H^1(A_2, B) \times H^1(A'_2, B)$. We first note that the restriction of $[\delta]$ to $A_2$ is trivial. This is because for all $a_2 \in A_2$, $\delta(a_2) = \rho(h)h^{-1} = 1$, where $h$ is any pre-image of $a_2$, so $\rho(h) = h$ by assumption ($\ast$). Further, let $k = |A'_2|$. It is well known that the restriction of $[\delta^k]$ to $A'_2$ is trivial. Thus $[\delta^k]$ is trivial and then $\delta^k$ is a coboundary. Therefore, $\rho^k$ is inner. Since $\rho^2$ is inner and $k$ is odd, it follows that $\rho$ is inner. $\square$

**Remark 2.4.** Proposition 2.3 states that a $C$-automorphism $\rho$ is inner if and only if the restriction of $\rho$ to the pre-image $H$ of $A_2$ in $G$ is inner (i.e., $\rho$ acts on $H$ as conjugation by a group element).

Proposition 2.3 can be easily rephrased to give the following proposition.

**Proposition 2.5.** Let $G$ be a metabelian group and let $B$ be an abelian normal subgroup of $G$ for which the quotient group $A = G/B$ is also abelian. Then the normalizer property holds for $G$ if and only if for every $\rho \in \text{Aut}_U(G)$, $\rho|_{B \cup P} = \text{conj}(g)|_{B \cup P}$ for some $g \in G$, where $P$ is a Sylow 2-subgroup of $G$.

**Corollary 2.6.** Let $G$ be a metabelian group and let $B$ be an abelian normal subgroup of $G$ for which the quotient group $G/B = A$ is abelian. If $P$ is a Sylow 2-subgroup of $G$, and $P = B_2 \times D$, where $B_2$ is the Sylow 2-subgroup of $B$ and $D$ is a 2-subgroup of $G$, then a $C$-automorphism $\rho$ is inner if and only if $\rho|_B = \text{conj}(g)|_B$ for some $g \in G$.

**Proof.** The necessity is obvious.
Conversely, multiplying by a conjugation of a group element, we may assume that $\rho|_B = \text{id}|_B$. Since $\rho$ is a $C$-automorphism, $\rho|_P = \text{conj}(h)|_P$. Since $\rho^2$ is inner, then $\rho$ is inner, if some odd power of $\rho$ is inner. By taking some suitable odd power of $\rho$, we may assume that the order of $\rho$ is a power of 2 and $h$ is a 2-element. Note that $\rho$ permutes the set of all Sylow 2-subgroups of $G$. As the order of $\rho$ is a power of 2, and the number of permuted groups is odd, this permutation action has a fixed point. That is to say there exists a Sylow 2-subgroup $P'$ of $G$ such that $\rho(P') = P'$. Without loss of generality, we may assume that $P' = P$. We note that $D$ is abelian since every commutator $[d, d_1]$ in $D$ has image the identity of $A$ and thus $[d, d_1] \in D \cap B_2 = 1$. $\rho(P) = \text{conj}(h)(P) = P$ and thus $h$ is in the normalizer of $P$ in $G$. Since $h$ is a 2-element, it follows that $h$ belongs to $P$. Now we write $h = b_2d$ where $b_2 \in B_2$ and $d \in D$. Since $\text{id}|_{B_2} = \rho|_{B_2} = \text{conj}(b_2d)|_{B_2} = \text{conj}(d)|_{B_2}$, we have that $d \in C(B_2)$. Moreover, as $D$ is abelian, it follows that $d \in C(P)$. Let $\rho_1 = \text{conj}(b_2^{-1}) \circ \rho$. Then $\rho_1|_B = \text{conj}(b_2^{-1}) \circ \rho|_B = \text{conj}(b_2^{-1})|_B = \text{id}|_B$ and $\rho_1|_P = \text{conj}(d)|_P = \text{id}|_P$. It follows from Proposition 2.3 that $\rho_1$ is inner. Therefore $\rho$ is inner and the proof is completed. \hfill $\Box$

**Remark 2.7.** $G$ is called a split metabelian group if $G$ is a semidirect product of an abelian normal subgroup $B$ of $G$ by an abelian subgroup $A$ of $G$, i.e., $G = B \rtimes A$. We note that if $G$ is a split metabelian group, then the assumption on a Sylow 2-subgroup of $G$ in Corollary 2.6 is automatically satisfied.

We now describe a family of finite metabelian groups for which the normalizer property holds.

**Theorem 2.8.** Let $G$ be a finite metabelian group and let $B$ be an abelian normal subgroup of $G$ for which the quotient group $G/B = A$ is abelian. If $P = B_2 \rtimes D$ and $C_{A_2}(B_2) = C_{A_2}(b_2)$ for some $b_2 \in B_2$, where $P$ is a Sylow 2-subgroup of $G$, and $B_2$ and $A_2$ are Sylow 2-subgroups of $B$ and $A$, respectively, then every $C$-automorphism of $G$ is inner; therefore, the normalizer property holds for $G$.

We prove Theorem 2.8 by means of the following two lemmas.

**Lemma 2.9.** Let $P$ be a finite abelian 2-group acting on a finite abelian group $M$ of odd order. Then there is an element $m \in M$ such that $C_P(M) = C_P(m)$.

**Proof.** $M$ can be regarded as a $P(\mathbb{Z}P)$-module and thus $M$ is a direct sum of indecomposable $P$-submodules $M_i$, i.e., $M = \sum M_i$. If for all submodules $M_i$, $C_P(M_i) = C_P(m_i)$ for some $m_i \in M_i$, then $C_P(M) = C_P(m)$ for $m = \sum m_i$. So it is enough to prove the lemma for $M$ indecomposable. Let $Q = C_P(M)$. Then $M$ is a faithful $R = P/Q$-module. Suppose that the result is not true. Then for any nonzero element $m$ of $M$, $C_P(M) \neq C_P(m)$. Thus there exists an
element \( p \in C_P(m) \), \( p \notin C_P(M) = Q \), and then the image \( g \) of \( p \) in \( R \) is not the identity. Therefore, we have that \( gm = m \) with \( m \neq 0 \) and \( 1 \neq g \in R \). Since \( g \) is a 2-element, by replacing \( g \) by a suitable power of \( g \), we can assume that \( g^2 = 1 \). Note that multiplication by \( 1 - g \) is an endomorphism of the \( R \)-module \( M \), but it is not an isomorphism. By the Fitting lemma [12, 3.3.5, p. 82] \( 1 - g \) is a nilpotent endomorphism of \( M \). Note that \((1 - g)^{2^k} = 2^{1+2+\ldots+2^{k-1}}(1 - g)\).

Since the order of \( M \) is odd, \( 1 - g \) annihilates \( M \), i.e., \( g \in C_R(M) = 1 \). This contradiction completes the proof. \( \square \)

**Remark 2.10.** Lemma 2.9 remains true for any finite abelian \( p \)-group \( P \) acting on a finite abelian group \( M \) of order prime to \( p \).

**Lemma 2.11.** Let \( G \) be a finite metabelian group and let \( B \) be an abelian normal subgroup of \( G \) for which the quotient group \( G/B = A \) is also abelian. Suppose that Sylow 2-subgroup \( A_2 \) of \( A \) has the property that \( CA_2(B_2) = CA_2(b_2) \) for some \( b_2 \in B_2 \) where \( B_2 \) is Sylow 2-subgroup of \( B \). Then any \( C \)-automorphism \( \rho \) of \( G \) acts as an inner automorphism of \( B \), i.e., \( \rho|_B = \text{conj}(g)|_B \) for some \( g \in G \).

**Proof.** \( B \) is a direct product of its Sylow subgroups \( B_p \), which are normal in \( G \).

Let \( \rho \) be a \( C \)-automorphism of \( G \). It acts on each \( B_p \) as an inner automorphism. As we mentioned earlier, by taking a sufficiently high odd power, we can assume that \( \rho \) is of 2-power order. We can also assume that \( \rho \) acts on \( B_p \) as conjugation by a 2-element \( h_p \), acts on a Sylow 2-subgroup \( P \) of \( G \) as conjugation by a 2-element \( h_2 \) and that \( \rho^2 \) is a conjugation by a 2-element. It is routine to check that \( \rho \) is a \( C \)-automorphism of the pre-image \( H \) of \( A_2 \) in \( G \) (for example, it follows from Lemma 1 in [9] that \( \rho \) preserves conjugacy classes in \( H \)), so we can assume that \( G = H \), i.e., \( A = A_2 \). Now there exist \( b_p \in B_p \) such that \( CA_2(B_p) = CA_2(b_p) \) by Lemma 2.9 and our assumptions. Let \( b = \prod b_p \) be the product of the \( b_p \)'s. Then there is \( g \in G \) such that \( gb^{-1}g^{-1} = \rho(b) \). Hence \( \prod gb_pg^{-1} = gbg^{-1} = \prod \rho(b_p) = \prod h_p b_p h_p^{-1} \) and thus \( gb_p g^{-1} = h_p b_p h_p^{-1} \) for all \( p \). It follows that \( g^{-1} h_p b_p h_p^{-1} \) centralizes \( b_p \). Since this element acts on \( B_p \) as its image in \( A \), we conclude that \( g^{-1} h_p \) also centralizes \( B_p \). Now for any element \( b' = \prod b'_p \in B \), we have \( \rho(b') = \prod \rho(b'_p) = \prod h_p b'_p h_p^{-1} = \prod h_p (h_p^{-1} gb'_p g^{-1} h_p) h_p^{-1} = \prod gb'_p g^{-1} = gb'g^{-1} \). This shows that \( \rho \) acts on \( B \) as conjugation by \( g \). \( \square \)

Now Theorem 2.8 follows directly from Lemma 2.11 and Corollary 2.6.

Next we discuss several corollaries of Theorem 2.8 and Proposition 2.3. The first one is the result proved by Marciniak and Roggenkamp [8, Theorem 12.3].

**Corollary 2.12.** Let \( G \) be a finite metabelian group and let \( B \) be an abelian normal subgroup of \( G \) for which the quotient group \( G/B = A \) is also abelian. If a Sylow 2-subgroup \( P \) of \( G \) is abelian, then every \( C \)-automorphism of \( G \) is inner.
Proof. First notice that $C_{A_2}(B_2) = A_2 = C_{A_2}(b_2)$ for any $b_2 \in B_2$. From Lemma 2.11 it follows that for every $C$-automorphism $\rho$, $\rho|_B = \text{conj}(g)|_B$ for some $g \in G$. Conjugating by a group element, we may assume that $\rho|_B = \text{id}|_B$. As we remarked earlier, we may also assume that the order of $\rho$ is a power of 2. Thus $\rho$ fixes a Sylow 2-subgroup $P$ of $G$. Now $\rho|_P = \text{conj}(h)|_P$. After taking a suitable odd power of $\rho$, we may assume that $h$ is a 2-element; therefore, $h$ belongs to $P$. Since $P$ is an abelian subgroup, we have $\rho|_P = \text{id}|_P$. The result follows from Proposition 2.3.

Corollary 2.13. Let $G$ be a finite metabelian group and let $B$ be an abelian normal subgroup of $G$ for which the quotient group $G/B = A$ is also abelian. If $B_2$, the Sylow 2-subgroup of $B$, is cyclic, then the restriction to $B$ of every $C$-automorphism $\rho$ of $G$ is inner, i.e., $\rho|_B = \text{conj}(g)|_B$ for some $g \in G$. In addition, if $P = B_2 \rtimes D$ where $P$ is a Sylow 2-subgroup of $G$, then every $C$-automorphism of $G$ is inner.

Corollary 2.14. Let $G$ be a finite metabelian group and let $B$ be an abelian normal subgroup of $G$ for which the quotient group $G/B = A$ is also abelian. If $P = C_{2^n} \rtimes C_2$, where $P$ is a Sylow 2-subgroup of $G$ and $C_m$ is the cyclic group of order $m$, then the restriction to $B$ of every $C$-automorphism $\rho$ of $G$ is inner, i.e., $\rho|_B = \text{conj}(g)|_B$ for some $g \in G$.

Proof. We need only show that $C_{A_2}(B_2) = C_{A_2}(b_2)$ for some $b_2 \in B_2$. Let $P = C_{2^n} \rtimes C_2 = \langle x \rangle \rtimes \langle y \rangle$ with $x^{2^n} = y^2 = 1, x\bar{y} = x^i$. We note that since $B_2$ is a subgroup of $P$, $B_2$ can be generated by at most two elements of $P$.

Case 1. If $B_2$ is cyclic, then it is obvious that $C_{A_2}(B_2) = C_{A_2}(b_2)$ for some $b_2 \in B_2$ being a generator of $B_2$.

Case 2. If $B_2 = \langle x^i, x^j y \rangle$, then in the quotient group $A_2 = \bar{x}^j \bar{y} = 1$, therefore $\bar{y} = \bar{x}^{-j}$. It follows that $A_2 = \langle \bar{x}, \bar{y} \rangle = \langle \bar{x} \rangle$ is cyclic. Therefore, $A_2$ commutes with $x^i$, and thus $C_{A_2}(B_2) = C_{A_2}(x^i y)$.

In both cases we have proved that $C_{A_2}(B_2) = C_{A_2}(b_2)$ for some $b_2 \in B_2$. The corollary follows from Lemma 2.11. $\square$

The next result follows directly from Corollaries 2.6 and 2.14.

Corollary 2.15. Let $G = B \rtimes A$ be a split finite metabelian group, where $B$ is an abelian normal subgroup and $A$ is abelian. If a Sylow 2-subgroup $P$ of $G$ satisfies the condition $P = C_{2^n} \rtimes C_2$, then every $C$-automorphism of $G$ is inner.

Remark 2.16. In particular, under assumption of Corollary 2.15 it follows that if a Sylow 2-subgroup of $G$ is a dihedral group, then every $C$-automorphism is inner. Therefore, the normalizer property holds for $G$. 
Next we discuss another situation for which the normalizer property holds.

**Theorem 2.17.** Let $G$ be a finite metabelian group and let $B$ be an abelian normal subgroup of $G$ for which the quotient group $G/B = A$ is also abelian. If $A_2$, the Sylow 2-subgroup of $A$, is cyclic, then every $C$-automorphism of $G$ is inner.

**Proof.** We first show that $C_{A_2}(B_2) = C_{A_2}(b_2)$ for some $b_2 \in B_2$. Since $A_2$ is a cyclic 2-group, its subgroups are linearly ordered by inclusion. We can take $b_2$ to be any element $b \in B_2$ such that $C_{A_2}(b)$ is minimal. Now

$$C_{A_2}(b_2) = \bigcap_{b' \in B} C_{A_2}(b') = C_{A_2}(B_2).$$

It follows from Lemma 2.11 that for every $C$-automorphism $\rho$ of $G$, $\rho|_B = \text{conj}(g)|_B$ for some $g \in G$.

Next we show that there exists a Sylow 2-subgroup $P$ of $G$ such that $\rho|_P = \text{conj}(h)|_P$. For the same reason mentioned before, we may assume that $h$ is a 2-element, therefore $h \in P$. It is clear that there is an element $g \in P$ such that $P = \langle B_2, g \rangle$ and $g$ maps to a generator $a$ of $A_2$. Write $h = b_2 g^i$ where $b_2 \in B_2$. As $\rho|_{B_2} = \text{id}|_{B_2}$, for all $b \in B_2$ we have $b = \rho(b) = \text{conj}(b_2 g^i)(b) = \text{conj}(g^i)(b)$. Thus $g^i \in C(B_2)$. Since $g^i$ commutes with $g$, it follows that $g^i \in C(P)$. Let $\rho_1 = \text{conj}(b_2^{-1}) \circ \rho$. Then

$$\rho_1|_B = \text{conj}(b_2^{-1}) \circ \rho|_B = \text{conj}(b_2^{-1})|_B = \text{id}|_B$$

and

$$\rho_1|_P = \text{conj}(g^i)|_P = \text{id}|_P.$$

Applying Proposition 2.3 to our situation, we conclude that $\rho$ is inner and this completes the proof. $\square$

The following corollary is the finite version of Theorem 2 in [7] which holds for several families of groups such as dihedral groups and $Q^*$ groups.

**Corollary 2.18.** Let $G = \langle H, g \rangle$ be a finite group, where $H$ is an abelian subgroup of index 2. Then the normalizer property holds for $G$.

Finally we note that Mazur [9] showed that there exist finite metabelian groups with the normalizer property, but for which not all $C$-automorphisms are inner. Another example is as follows.

**Remark 2.19.** In [8], Marciniak and Roggenkamp gave an example of a finite metabelian group of order 384 ($G \cong (C_2^4 \times C_3) \rtimes C_2^3$) for which there exists a $C$-automorphism which is not inner.
Proposition 2.20. Let $G$ be a finite metabelian group and let $B$ be an abelian normal subgroup of $G$ for which the quotient group $G/B = A$ is also abelian. If $U(ZA)$ has only trivial units, then the normalizer property holds for $G$. In particular, it holds for the group $(C_2^4 \times C_3) \rtimes C_2$.

Proof. Let $u \in N_U(G)$. Then $u = \alpha_0a_0 + \sum \alpha_ia_i$, where $\alpha_i$ are in $\mathbb{Z}B$, $a_0 = 1$, and all $a_i$ form a right transversal to $B$ in $G$.

In $\mathbb{Z}(G/B) \cong \mathbb{Z}A$, we have $\bar{u} = \epsilon(\alpha_0) + \sum \epsilon(\alpha_i)a_i$, where $\epsilon$ denotes the augmentation map. Since $\mathbb{Z}(G/B) \cong \mathbb{Z}A$ has only trivial units $\epsilon(\alpha_l) = \pm 1$ and $\epsilon(\alpha_i) = 0$ for all $i \neq l$. Multiplying by $\pm a^{-1}_l$, if necessary, we may assume that $\epsilon(\alpha_0) = 1$ and $\epsilon(\alpha_i) = 0$ for all $i \neq 0$. Let $\rho$ be the automorphism of $G$ such that $\rho(g) = ugu^{-1}$ for all $g \in G$. We will show that $\rho|_{B \cup P} = \text{conj}(b_1)|_{B \cup P}$ for some $b_1 \in B$, where $P$ is a Sylow 2-subgroup of $G$. Thus, the result will follow from Proposition 2.5. Since $\rho$ preserves the conjugacy classes in $G$, it follows that $\rho(b) \in B$ for every $b \in B$. So there exists $b_0 \in B$ such that $b_0bu = ub$. This implies that $b_0ba_0 = a_0b$. Therefore, $(b_0 - 1)a_0 = 0$. This means that $\epsilon(\alpha_0)$ is divisible by the order of $b_0$, forcing $b_0 = 1$. Hence $[u, b] = 1$ for every $b \in B$ and then $\rho|_B = \text{id}|_B$. Let us rewrite $u = \sum u(x)x$ where $u(x) \in \mathbb{Z}$ and $x \in G$. Since $u = \rho(g)ug^{-1}$, we have $\sum u(x)x = \sum u(x)\rho(g)xg^{-1}$ (*). Note that $\rho(g)xg^{-1} \in B$ provided that $x \in B$. We can define a group action $\sigma_h$ of $P$ on the set $B$ as follows: $\sigma_h(x) = \rho(h)xh^{-1}$ for all $x \in B$. It follows from (*) that $u(x)$ is a constant on each orbit of $x$. Since $P$ is a 2-subgroup, every orbit of this action must have a 2-power length. Therefore, we have that

$$1 = \epsilon(\alpha_0) = \sum_{x \in B} u(x) = \sum c_i2^{l_i},$$

where $2^{l_i}$ is the length of the orbit of $x_i$ and $u(x_i) = c_j$. This forces $p^{l_j} = 1$ for some $j$; hence there is a fixed point of this action, say $b_1 = x_j \in B$. Thus $\rho(h)b_1h^{-1} = b_1$ for all $h \in P$. Consequently, $\rho|_P = \text{conj}(b_1)|_P$ and then $\rho|_{B \cup P} = \text{conj}(b_1)|_{B \cup P}$. Therefore, $\rho$ is inner and the normalizer property holds for $G$. \( \square \)

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