Europ. J. Combinatorics (1984) 5, 207–222

Dominance Orders, Capelli Operators, and Straightening of Bideterminants

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The study of the action of Capelli operators on bideterminants leads to a necessary condition—formulated in terms of the column dominance order of standard tableaux—for the non-vanishing of the straightening coefficients.

1. INTRODUCTION

In connection with invariant theoretical studies Rota *et al.* [5, 6, 12] proved that the bideterminants $(S|T)$ of standard bitableaux (S, T) form an R-basis of the polynomial ring

$$R_m^n := R[X_{ij}/ i=1,\ldots,m; j=1,\ldots,n],$$

where R is a commutative ring with $1_R \neq 0$. Thus in particular every bideterminant $(U|V)$ is a linear combination of bideterminants of standard bitableaux

$$(U|V) = \sum_{(S,T)\text{ standard}} a_{UV,ST}(S|T).$$

Since bideterminants play a fundamental role in the invariant and representation theory of classical groups, details about the *straightening coefficients* $a_{UV,ST}$ would help to investigate various problems in this field.

Désarménien has mentioned in the appendix to ref. [12] that in general the number of standard bideterminants which might lie in the support of $(U|V)$, i.e. $a_{UV,ST} \neq 0$, is very large. Thus one of the crucial problems is to reduce *a priori* the number of candidates (S, T). We are concerned here with this problem and especially with the question:

what is implied by $a_{UV,ST} \neq 0$?

As far as we know, the best of the previous answers to this question is due to de Concini, Eisenbud and Procesi [3]:

$$a_{UV,ST} \neq 0 \textit{ implies } U \underset{r}{\trianglelefteq} S \textit{ and } V \underset{r}{\trianglelefteq} T. \tag{1.1}$$

Here $\underset{r}{\trianglelefteq}$ denotes a special quasi-order on tableaux, the so-called *row dominance.*

They derived this result by transferring Hodge's theorem on Plücker coordinates into R_m^n. Hence their methods are quite different from the use of Capelli operators proposed by Rota *et al.* As the Capelli operators $C(S, T)$ of standard bitableaux (S, T) are a main tool for the explicit evaluation of the straightening coefficients, we shall use these operators to attack our problem.

In Section 4 we prove two fundamental properties of Capelli operators from which we can derive that

$$a_{UV,ST} \neq 0 \textit{ implies } {}^{s}U \underset{c}{\trianglerighteq} S \textit{ and } {}^{s}V \underset{c}{\trianglerighteq} T. \tag{1.2}$$

(Here ${}^{s}U$ denotes the *standardization* of U, see Section 2, and $\underset{c}{\trianglelefteq}$ denotes the *column dominance* quasi-order on tableaux, see Section 3.)

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The proofs of these results turned out to be very elementary and it was natural to ask how these dominance orders are related. In fact we show in Section 3 that for standard tableaux A and B with the same content

$$A \underset{r}{\trianglelefteq} B \text{ iff } B \underset{c}{\trianglelefteq} A, \tag{1.3}$$

hence neither rows nor columns are favoured under dominance!

This yields the following slight improvement of (1.1) which is by (1.3) equivalent to (1.2):

$$a_{UV,ST} \neq 0 \text{ implies } {}^sU \underset{r}{\trianglelefteq} S \text{ and } {}^sV \underset{r}{\trianglelefteq} T. \tag{1.4}$$

As another consequence of the two fundamental properties of Capelli operators we get the following result of Désarménien [4, p. 59; 12, p. 78]:

$$(U|V) \neq 0 \text{ implies } a_{UV,{}^sU{}^sV} = \pm 1. \tag{1.5}$$

Now (1.5) shows that the bounds of the estimates in (1.2) and (1.4) cannot be improved.

In Section 5 we prove a third fundamental property of Capelli operators, which generalizes a result of Désarménien [4, theorem 4] to arbitrary shapes.

Combining these results we get an improved algorithm for calculating the straightening coefficients.

2. Standardization of Young Tableaux

A *partition* λ of $n \in \mathbb{N} := \{1, 2, \ldots\}$ is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h)$ of positive integers with

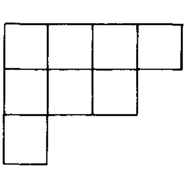
$$\lambda_1 + \cdots + \lambda_h = n \text{ and } \lambda_1 \geqslant \cdots \geqslant \lambda_h > 0.$$

For purely technical reasons λ will be identified with all sequences $(\lambda_1, \ldots, \lambda_h, 0, \ldots, 0)$, and we put $\lambda_{h+i} := 0$ for all $i \in \mathbb{N}$. The (*Young*) *diagram* of a partition λ, also denoted by λ, is formally defined by

$$\lambda := \bigcup_{i=1}^{h} \{(i, 1), \ldots, (i, \lambda_i)\},$$

but we often think of this set as an array of squares arranged as follows:

EXAMPLE.

$(4, 3, 1) =$ 

The *conjugate* of a partition λ is the partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$, $\lambda'_i := |\{j/\lambda_j \geqslant i\}|$, whose diagram is the transpose of the diagram λ. In the above example $(4, 3, 1)' = (3, 2, 2, 1)$.

Every function T mapping the diagram λ into a set M is called a (*Young*) *tableau* of *shape* $|T| := \lambda$ or, for short, a λ-tableau; such a T will be viewed as a λ-shaped matrix $T = (t_{ij})$, where $t_{ij} := T((i, j))$ is the *entry* belonging to the ith *row* and jth *column* of T.

In this paper the range M of a tableau $T : \lambda \to M$ will always be a totally ordered set, often $M = \{1, \ldots, k\} =: \underline{k}$, for some k.

The *content* $\alpha = (\alpha_1, \ldots, \alpha_k)$ of a tableau $T:\lambda \to \underline{k}$, defined by

$$\alpha_r := |\{(i,j) \in |T| / t_{ij} = r\}|,$$

specifies the number of occurrences of any $r \in \underline{k}$ in T.

A tableau T is *row-injective* if no entry is repeated in any of its rows, it is ***normal*** if the entries in each row are increasing from left to right; T is *standard* if it is normal and if the entries in each column are nondecreasing from top to bottom.

If T is a tableau, ${}^cT$ (resp. ${}^rT$) denotes the tableau obtained from T by writing the entries in each column (resp. row) of T in nondecreasing order.

EXAMPLE. Let

$$T = \begin{matrix} 6 & 4 & 2 & 3 \\ 3 & 1 & 4 & \\ 4 & & & \end{matrix} .$$

Then T is a row-injective $(4, 3, 1)$-tableau of content $(1, 1, 2, 3, 0, 1, 0, \ldots, 0)$.

$$ {}^rT = \begin{matrix} 2 & 3 & 4 & 6 \\ 1 & 3 & 4 & \\ 4 & & & \end{matrix} $$

is normal;

$$ {}^cT = \begin{matrix} 3 & 1 & 2 & 3 \\ 4 & 4 & 4 & \\ 6 & & & \end{matrix} $$

is not row-injective; but

$$ {}^c({}^rT) = \begin{matrix} 1 & 3 & 4 & 6 \\ 2 & 3 & 4 & \\ 4 & & & \end{matrix} $$

is standard (see below), while

$$ {}^r({}^cT) = \begin{matrix} 1 & 2 & 3 & 3 \\ 4 & 4 & 4 & \\ 6 & & & \end{matrix} $$

is not standard!

The concepts and operations just defined are combined in the following sorting lemma [4, pp. 52; 6, pp. 15; 12, pp. 70] for which we shall give a short proof below.

SORTING LEMMA 2.1. *Let T be a row-injective tableau. Then the tableau ${}^sT := {}^c({}^rT)$ is standard.*

Here ${}^sT$ is called the *standardization* of T.

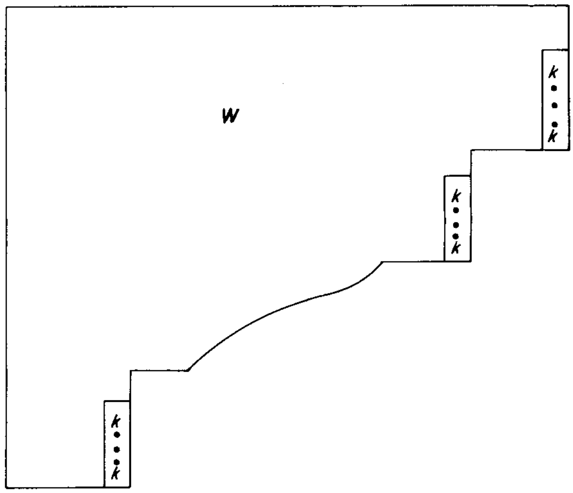
PROOF OF LEMMA 2.1. The proof is by induction on k, $k := \max\{t_{ij}/(i,j) \in |T|\}$. The case $k = 1$ is trivial. So let $k > 1$.

By normality of $U := {}^rT$, every k in U is placed at the end of a suitable row. Now permuting rows of equal length of U (as a whole), one gets another tableau V, where

$$V \text{ is normal, and } {}^cU = {}^cV. \tag{2.1}$$

In particular the rows of U can be permuted in such a way that all 'k-free' rows of length l precede all rows of same length terminating with k:

$V =$



Let W denote the 'k-free' part of V.

Of course, W is again a normal tableau and, by maximality of k, going from V to ${}^{c}V$ the ks will stay at their positions, i.e. $V \mapsto {}^{c}V$ is essentially described by $W \mapsto {}^{c}W$. By induction, ${}^{c}W$ is standard, hence, together with (2.1), ${}^{c}V = {}^{c}U = {}^{c}({}^{r}T)$ is standard.

3. Dominance and Total Orders of Partitions and Tableaux

A partition [diagram] λ is *dominated* by another partition [diagram] μ, abbreviated $\lambda \trianglelefteq \mu$, if $\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$, for all i.

A short proof of the following well-known result can be found in [10, p. 6].

$$\textit{Let } \lambda \textit{ and } \mu \textit{ be partitions of } n \in \mathbb{N}. \textit{ Then } \lambda \trianglelefteq \mu \textit{ iff } \lambda' \trianglerighteq \mu'. \tag{3.1}$$

Write $\lambda < \mu$, where λ and μ are two different partitions, if the first non-vanishing difference $\mu_i - \lambda_i$ is positive. This *lexicographic order* is known to refine the dominance (partial) order:

$$\lambda \triangleleft \mu \textit{ implies } \lambda < \mu. \tag{3.2}$$

For the remaining part of this section the entries in every tableau are assumed to be positive integers.

To a tableau T and $p, q \in \mathbb{N}$ let

$$a^{\mathrm{r}}_{pq}(T) := |\{(i,j) \in |T| / i \leq p \text{ and } t_{ij} \leq q\}|,$$

resp.

$$a^{\mathrm{c}}_{qp}(T) := |\{(i,j) \in |T| / j \leq p \text{ and } t_{ij} \leq q\}|,$$

denote the number of elements $\leq q$ in the first p rows (resp. columns) of T; similarly we define

$$z^{\mathrm{r}}_{pq}(T) := \{|(i,j) \in |T| / i \geq p \text{ and } t_{ij} \geq q\}|$$

and

$$z^{\mathrm{c}}_{qp}(T) := \{|(i,j) \in |T| / j \geq p \text{ and } t_{ij} \geq q\}|.$$

The tableau S is *row-dominated* (resp. *column-dominated, reverse row-dominated, reverse column-dominated*) by the tableau T, for short: $S \underset{r}{\trianglelefteq} T$ (resp. $S \underset{c}{\trianglelefteq} T$, $S \underset{r}{\sqsubseteq} T$, $S \underset{c}{\sqsubseteq} T$) if for all $p, q \in \mathbb{N}$: $a^{r}_{pq}(S) \leq a^{r}_{pq}(T)$ (resp. $a^{c}_{pq}(S) \leq a^{c}_{pq}(T)$, $z^{r}_{pq}(S) \leq z^{r}_{pq}(T)$, $z^{c}_{pq}(S) \leq z^{c}_{pq}(T)$).

All these dominance relations are quasi-orders on the set of all tableaux (i.e. $\underset{r}{\trianglelefteq}, \ldots, \underset{c}{\sqsubseteq}$ are reflexive and transitive but not necessarily antisymmetric). However, when restricted to the set of standard tableaux these dominance relations become (partial) orders.

To get a connection between dominance of partitions and some of the dominance relations of tableaux, let q be the maximum of all entries in the λ-tableau T. Then by $t_{ij} \leq q$, for all $(i, j) \in |T|$, $a^{r}_{pq}(T)$ (resp. $a^{c}_{qp}(T)$) is the number of *all* entries in the first p rows (resp. columns) of T, i.e. for this q:

$$a^{r}_{pq}(T) = \lambda_1 + \cdots + \lambda_p \text{ and } a^{c}_{qp}(T) = \lambda'_1 + \cdots + \lambda'_p.$$

Hence for tableaux U and V the following is valid, cf. [3, p. 137]:

$$U \underset{r}{\trianglelefteq} V \textit{ implies } |U| \trianglelefteq |V|. \tag{3.3}$$

$$U \underset{c}{\trianglelefteq} V \textit{ implies } |U|' \trianglelefteq |V|'. \tag{3.4}$$

The proof of the following theorem indicates that the connection between certain dominance orders is even more comprehensive.

THEOREM 3.1. *Let A, B be two standard tableaux with the same content. Then*

$$A \underset{r}{\trianglelefteq} B \textit{ iff } B \underset{c}{\trianglelefteq} A.$$

PROOF. If T is a standard tableau and $q \in \mathbb{N}$ let T_q denote the restriction of the map T to the domain

$$|T_q| := \{(i, j) \in |T| / t_{ij} \leq q\}.$$

EXAMPLE. If

$$T = \begin{matrix} 1 & 2 & 4 \\ 1 & 3 & 4, \\ 2 & & \end{matrix}$$

then

$$T_1 = \begin{matrix} 1 \\ 1, \end{matrix} \qquad T_2 = \begin{matrix} 1 & 2 \\ 1 & \\ 2 & \end{matrix}, \qquad T_3 = \begin{matrix} 1 & 2 \\ 1 & 3, \\ 2 & \end{matrix}$$

and $T_k = T$ for all $k \geq 4$.

Notice that all T_q are standard. This holds in general, as one can easily show.

Let T be a standard tableau. Then all T_q, $q \in \mathbb{N}$, are standard. (3.5)

Moreover, a standard tableau T can be uniquely recovered from the sequence $(|T_1|, |T_2|, \ldots)$ of shapes.

If $|T_q|_i$ (resp. $|T_q|'_j$) denotes the length of the ith row (resp. jth column) of T_q, then

$$a^{r}_{pq}(T) = |T_q|_1 + \cdots + |T_q|_p, \tag{3.6}$$

$$a^{c}_{qp}(T) = |T_q|'_1 + \cdots + |T_q|'_p. \tag{3.7}$$

After these preliminaries we can now finish the proof of (3.1).

$$
\begin{aligned}
A \underset{r}{\trianglelefteq} B &\Leftrightarrow \forall_q (\forall_p a^{r}_{pq}(A) \leqslant a^{r}_{pq}(B)) \\
&\Leftrightarrow \forall_q |A_q| \trianglelefteq |B_q| && \text{[by (3.6)]} \\
&\Leftrightarrow \forall_q |A_q|' \trianglerighteq |B_q|' && \text{[by (3.1)]} \\
&\Leftrightarrow \forall_q (\forall_p a^{c}_{qp}(A) \geqslant a^{c}_{qp}(B)) && \text{[by (3.7)]} \\
&\Leftrightarrow A \underset{c}{\trianglerighteq} B.
\end{aligned}
$$

[As A and B are standard tableaux with the same content, the diagrams $|A_q|$ and $|B_q|$ have the same number of squares, so (3.1) is indeed applicable.]

Now let us briefly discuss the reverse dominance orders.

Example. The standard tableaux

$$A := \begin{matrix} 1 & 2 & 3 & 5 \\ 4 & & & \end{matrix}, \qquad B := \begin{matrix} 1 & 2 & 4 \\ 3 & 5 & \end{matrix}$$

are both of content (1^5); $A \underset{r}{\sqsubseteq} B$, but A and B are incomparable with respect to $\underset{c}{\sqsubseteq}$.

Nevertheless we have

Theorem 3.2. *Let A and B are standard tableaux of the same content and of the same shape. Then*

$$A \underset{r}{\trianglelefteq} B \Leftrightarrow A \underset{r}{\sqsubseteq} B \Leftrightarrow B \underset{c}{\sqsubseteq} A \Leftrightarrow B \underset{c}{\trianglelefteq} A.$$

Proof. Use (3.1), (3.7), (3.8) and the characterization of dominance of partitions in [3, Proposition 1.1].

In the representation theory of classical groups certain total orders of standard tableaux have been widely used: lexicographic orders, and, in connection with induction and subduction of representations, last letter sequences. In the remaining part of this section we show that all these total orderings are refinements of dominance orders.

First let us recall some definitions.

We write $U \underset{r}{\leqslant} V$ iff $|U| < |V|$ or $|U| = |V|$ and the vector $(u_{11}, u_{12}, \ldots, u_{21}, u_{22}, \ldots)$ of the associated row sequence of U is lexicographically not smaller than the corresponding vector of V. Similarly $U \underset{c}{\leqslant} V$ iff $|U|' < |V|'$ or $|U| = |V|$ and $(u_{11}, u_{21}, \ldots, u_{12}, u_{22}, \ldots) \geqslant (v_{11}, v_{21}, \ldots, v_{12}, v_{22}, \ldots)$. Here $\underset{r}{\leqslant}$ (resp. $\underset{c}{\leqslant}$) is the *row-* (resp. *column-*) *lexicographic order* of tableaux.

Obviously the following holds.

Let U, V be standard tableaux of the same content. Then

(a) $$U \underset{r}{\trianglelefteq} V \text{ implies } U \underset{r}{\leqslant} V; \tag{3.8}$$

(b) $$U \underset{c}{\trianglelefteq} V \text{ implies } U \underset{c}{\leqslant} V.$$

If the number x occurs in the standard tableau U exactly s times and if these xs are placed in rows (resp. columns) with index $i_1 > \cdots > i_s$ (resp. $j_1 \geqslant \cdots \geqslant j_s$), then let $\rho_x(U) := (i_1, \ldots, i_s)$ and $\gamma_x(U) := (j_1, \ldots, j_s)$.

Now the *row-* (resp. *column-*) *last letter sequence* of standard tableaux of content $\alpha=(\alpha_1,\dots,\alpha_m)$ can be defined as follows: $U <_{\mathrm{r},lls} V$ (resp. $U <_{\mathrm{c},lls} V$) iff there exists an $x \in \underline{m}$ such that $\rho_x(U) > \rho_x(V)$ [resp. $\gamma_x(U) > \gamma_x(V)$] and for all $y > x$, $\rho_y(U) = \rho_y(V)$ [resp. $\gamma_y(U) = \gamma_y(V)$].

It is straightforward to prove the following assertion.

For standard tableaux U and V of the same content

(a) $$U \underset{\mathrm{r}}{\sqsubseteq} V \textit{ implies } U \leqslant_{\mathrm{r},lls} V, \textit{ and} \tag{3.9}$$

(b) $$U \underset{\mathrm{c}}{\sqsubseteq} V \textit{ implies } U \leqslant_{\mathrm{c},lls} V.$$

If further U and V are of the same shape then, by Theorem 3.2, $\underset{\mathrm{r}}{\sqsubseteq}$ (resp. $\underset{\mathrm{c}}{\sqsubseteq}$) *can be replaced by* $\underset{\mathrm{r}}{\trianglelefteq}$ (resp. $\underset{\mathrm{c}}{\trianglelefteq}$).

4. Bideterminants and Capelli Operators

Bideterminants are elements of the polynomial ring

$$R_m^n := R[X_{ij} / i = 1, \dots, m; j = 1, \dots, n]$$

over the commutative ring R in the $m \cdot n$ indeterminates $(i|j) := X_{ij}$, where $i \in \underline{m}$ is the *letter-index* and $j \in \underline{n}$ the *place-index*. This polynomial ring, the so-called *letterplace algebra* in m letters and n places, turned out to be a very useful tool e.g. in invariant theory and representation theory, see the references.

A *bitableau* is a pair (S, T) of tableaux with the same shape λ, where all s_{ij} (resp. t_{ij}) are, for the present, elements of $\underline{m}$ (resp. $\underline{n}$). λ is the *shape* of (S, T). A bitableau (S, T) is called *standard* (*normal*, ...) if both S and T have this property. The *content* of the bitableau (S, T) is the pair of vectors (α, β) where α_i (resp. β_j) is the number of occurrences of i in S (resp. j in T).

To every bitableau (S, T) corresponds the *bideterminant*

$$(S|T) := \prod_i \det((s_{ij}|t_{ik})) \in R_m^n.$$

Example.

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 5 & 2 & 4 & 6 \\ 2 & 3 & & 4 & 5 & \\ 4 & & & 5 & & \end{array}\right) = \det\begin{pmatrix} (1|2) & (1|4) & (1|6) \\ (3|2) & (3|4) & (3|6) \\ (5|2) & (5|4) & (5|6) \end{pmatrix} \cdot \det\begin{pmatrix} (2|4) & (2|5) \\ (3|4) & (3|5) \end{pmatrix} \cdot (4|5)$$

is the bideterminant of the standard bitableau

$$\left(\begin{array}{ccc} 1 & 3 & 5 \\ 2 & 3 & \\ 4 & & \end{array}, \begin{array}{ccc} 2 & 4 & 6 \\ 4 & 5 & \\ 5 & & \end{array}\right)$$

From the well-known properties of the determinant one easily deduces that for a bitableau (U, V)

$$(U|V) \neq 0 \textit{ iff } (U, V) \textit{ is row-injective.} \tag{4.1}$$

If (U, V) is row-injective then

$$({}^{\mathrm{r}}U|{}^{\mathrm{r}}V) = \pm (U|V) \neq 0. \tag{4.2}$$

In three highly interesting studies, Rota *et al.* [5, 6, 12] showed:

THEOREM 4.1. *The bideterminants of the standard bitableaux form an R-basis of the polynomial ring* R_m^n.

Moreover they proved the following

STRAIGHTENING FORMULA 4.2. *Every bideterminant* $(U|V)$ *is a linear combination, with integer coefficients, of bideterminants of standard bitableaux* (S, T): $(U|V) = \sum a_{UV,ST}(S|T)$; *furthermore if* $a_{UV,ST} \neq 0$ *then* $U \underset{\mathrm{r}}{\leqslant} S$, $V \underset{\mathrm{r}}{\leqslant} T$, *and the contents of* (U, V) *and* (S, T) *coincide.*

We are now going to improve this straightening formula. This will be done with the help of various substitution processes. In order to distinguish between letters (resp. places) changed by substitution from those yet untouched, we extend the previous letter (resp. place) alphabet by adjoining m *coloured* letters $\underline{1}, \ldots, \underline{m}$ (resp. n *coloured* places $\underline{1}, \ldots, \underline{n}$). Hence the following investigations will be done in the *coloured letter place algebra*

$$\underline{R}_m^n := R[(A|B)/A = 1, \ldots, m, \underline{1}, \ldots, \underline{m};\ B = 1, \ldots, n, \underline{1}, \ldots, \underline{n}].$$

The extensions of the notions of bitableau and bideterminant to these larger alphabets are obvious.

The *letter* (resp. *place*) set polarization operator $D_L^r(\underline{j}, i)$ (resp. $D_P^r(\underline{j}, i)$) is the R-linear operator $\underline{R}_m^n \to \underline{R}_m^n$ mapping every monomial

$$M := \prod_t (A_t|B_t) \in \underline{R}_m^n$$

in which the uncoloured letter (resp. place) i occurs exactly s times onto the sum of the $\binom{s}{r}$ monomials obtained from M by replacing each subset of r letters (resp. places) i by r coloured letters (resp. places) $\underline{j}$.

Consequently,

$$\textit{if } s < r \textit{ then } D_L^r(\underline{j}, i)M = 0 \tag{4.3}$$

and if $r = 0$ then $D_L^0(\underline{j}, i) = D_P^0(\underline{j}, i)$ is the identity operator.

EXAMPLE.

$$D_L^2(\underline{3}, 1): \left(\begin{array}{c|c} 1 & \underline{2} \\ 1 & 2 \\ 1 & 3 \\ 3 & 1 \\ \underline{4} & \underline{5} \end{array}\right) \mapsto \left(\begin{array}{c|c} \underline{3} & \underline{2} \\ \underline{3} & 2 \\ 1 & 3 \\ 3 & 1 \\ \underline{4} & \underline{5} \end{array}\right) + \left(\begin{array}{c|c} \underline{3} & \underline{2} \\ 1 & 2 \\ \underline{3} & 3 \\ 3 & 1 \\ \underline{4} & \underline{5} \end{array}\right) + \left(\begin{array}{c|c} 1 & \underline{2} \\ \underline{3} & 2 \\ \underline{3} & 3 \\ 3 & 1 \\ \underline{4} & \underline{5} \end{array}\right),$$

while $D_P^2(\underline{3}, 1)$ maps the same monomial onto 0.

It is easy to see that

$$\textit{the polarization operators commute mutually.} \tag{4.4}$$

For bideterminants, the set polarization operators act in the following simple way [5, p. 73].

Let (U, V) be a bitableau, $i, j \in \underline{m}$ and $r \in \mathbb{N}$. If the letter i occurs in U exactly s times, let $U^1, \ldots, U^z$ denote the distinct $z = \binom{s}{r}$ tableaux obtained from U by replacing each subset of r letters i in U by r coloured letters $\underline{j}$. Then

$$D_L^r(\underline{j}, i)(U|V) = \sum_t (U^t|V). \tag{4.5}$$

An analogous result does hold for the places.

EXAMPLE

$$D_P^2(\underline{3}, 1)\left(U\,\middle|\,\begin{array}{ccc} 1 & \underline{3} & 4 \\ \underline{2} & 1 & 5 \\ 1 & \underline{1} & \end{array}\right) = \left(U\,\middle|\,\begin{array}{ccc} \underline{3} & \underline{3} & 4 \\ \underline{2} & \underline{3} & 5 \\ 1 & \underline{1} & \end{array}\right) + \left(U\,\middle|\,\begin{array}{ccc} \underline{3} & \underline{3} & 4 \\ \underline{2} & 1 & 5 \\ \underline{3} & \underline{1} & \end{array}\right) + \left(U\,\middle|\,\begin{array}{ccc} 1 & \underline{3} & 4 \\ \underline{2} & \underline{3} & 5 \\ \underline{3} & \underline{1} & \end{array}\right)$$

$$= \left(U\,\middle|\,\begin{array}{ccc} 1 & \underline{3} & 4 \\ \underline{2} & \underline{3} & 5 \\ \underline{3} & 1 & \end{array}\right)$$

[by (4.1)].

The Capelli operators are composed of those polarization operators.

To a tableau S let $c_i(S, j)$ denote the number of occurrences of i in the jth column of S:

$$c_i(S, j) := |\{k / s_{kj} = i\}|.$$

If (S, T) is an uncoloured bitableau of shape λ then [1, 5, 12]

$$C_L(S) := \prod_{1 \le j \le \lambda_1} \prod_{1 \le i \le m} D_L^{c_i(S,j)}(\underline{j}, i)$$

is the *Capelli letter operator* of S,

$$C_P(T) := \prod_{1 \le j \le \lambda_1} \prod_{1 \le i \le n} D_P^{c_i(T,j)}(\underline{j}, i)$$

is the *Capelli place operator* of T, and

$$C(S, T) := C_L(S) \circ C_P(T)$$

is the *Capelli operator* of the bitableau (S, T).

EXAMPLE. Suppose

$$(S, T) = \begin{pmatrix} 1 & 3 & 5 & 1 & 2 & 4 \\ 1 & 3 & & 3 & 4 & \\ 1 & & & 3 & & \\ 2 & & , & 4 & & \end{pmatrix}.$$

Then

$$C(S, T) = D_L^3(\underline{1}, 1) D_L^1(\underline{1}, 2) D_L^2(\underline{2}, 3) D_L^1(\underline{3}, 5) \cdot$$
$$D_P^1(\underline{1}, 1) D_P^2(\underline{1}, 3) D_P^1(\underline{1}, 4) D_P^1(\underline{2}, 2) D_P^1(\underline{2}, 4) D_P^1(\underline{3}, 4).$$

All non-trivial results about Capelli operators, which we could find in the literature, are consequences of three fundamental Theorems 4.3, 4.4 and 5.1. [In the sequel $\underline{T}_\lambda$ denotes the standard λ-tableau whose jth column equals $(\underline{j}, \dots, \underline{j})$.]

THEOREM 4.3. *If* $(S, T)=({}^sU, {}^sX)$ *is the standardization of the normal* λ*-bitableau* (U, X) *then*

(a) $C_L(S)(U|X)=(\underline{T}_\lambda|X)\neq 0$,

(b) $C_P(T)(U|X)=(U|\underline{T}_\lambda)\neq 0$, *and*

(c) $C(S, T)(U|X)=(\underline{T}_\lambda|\underline{T}_\lambda)\neq 0$.

This theorem generalizes results in [4, 5, 6, 12].

PROOF. The proof of (a), similar to our proof of the Sorting Lemma 2.1, is by induction on k, $k:=\max\{u_{ij}/(i,j)\in|U|\}$.

The case $k=1$ is trivial. So let $k>1$.

Permuting simultaneously in U and X rows of equal length, every resulting bitableau (V, Y) is normal as well. Moreover

$$(U|X)=(V|Y) \tag{4.6}$$

and

$$({}^sU, {}^sX)=(S, T)=({}^sV, {}^sY). \tag{4.7}$$

In particular we can assume that V has the same special form as in the proof of Lemma 2.1; once more, W denotes the 'k-free' part of V.

By (4.7), (4.4), and the definition of W we have

$$C_L(S)=C_L(U)=C_L(V)=C_L(W)\cdot C_L(V\setminus W), \tag{4.8}$$

where $C_L(V\setminus W):=\prod_q D_L^{c_k(V,q)}(\underline{q}, k)$ is the Capelli letter operator of the skew tableau $V\setminus W$.

At first let us investigate the action of $C_L(V\setminus W)$ on the bideterminant $(V|Y)$. By (4.5), $C_L(V\setminus W)$ replaces each k in V by a suitable coloured letter; more precisely just $c_k(S, q)$ of the ks in V have to be substituted by $\underline{q}$. Generally this can be done in several ways.

Since $c_k(S, q)=c_k(V, q)$ there is one distinguished way of colouring: replace each k in the qth column of V by $\underline{q}$ and denote the resulting tableau by $\underline{V}^k$. Let $\bar{\underline{V}}$ be the set of all tableaux $\underline{V}$, $\underline{V}\neq\underline{V}^k$, which result from V by substituting $c_k(S, q)$-times the letter k in V by $\underline{q}$, $1\leqslant q\leqslant\lambda_1$.

For example, let

$$U=\begin{matrix}1&4&6\\2&3&6\\1&4&5\\1&6&\\3&5&\end{matrix}.$$

Then $k=6$, and one possible V is

$$V=\begin{matrix}1&4&5\\1&4&6\\2&3&6\\3&5&\\1&6&\end{matrix};$$

here

$$\begin{array}{ccc}1 & 4 & 5\\ 1 & 4 & \underline{3}\\ 2 & 3 & \underline{3}\\ 3 & 5 & \\ 1 & \underline{2} & \end{array} \quad\text{and}\quad \underline{\underline{V}}=\left\{\begin{array}{ccc}1 & 4 & 5\\ 1 & 4 & \underline{3}\\ 2 & 3 & \underline{2}\\ 3 & 5 & \\ 1 & \underline{3} & \end{array},\ \begin{array}{ccc}1 & 4 & 5\\ 1 & 4 & \underline{2}\\ 2 & 3 & \underline{3}\\ 3 & 5 & \\ 1 & \underline{3} & \end{array}\right\}$$

where the first tableau is $\underline{V}^6=$.

In general $\underline{V}^k$ and all $\underline{V}\in\underline{\underline{V}}$ share the following property:

Restricting the maps $\underline{V}^k$ and all $\underline{V}\in\underline{\underline{V}}$ to the domain $|W|$ yields always W. (4.9)

But there is one crucial difference between $\underline{V}^k$ and the elements of $\underline{\underline{V}}$, based on the pigeonhole principle:

To every $\underline{V}\in\underline{\underline{V}}$ there exists an index q such that the qth column of $\underline{V}$ contains a coloured letter $\underline{p}$, where $p<q$. (4.10)

Keeping the notation we can write by (4.5)

$$C_L(V\backslash W)(V|Y)=(\underline{V}^k|Y)+\textstyle\sum_{\underline{V}\in\underline{\underline{V}}}(\underline{V}|Y). \tag{4.11}$$

Now we separate with suitable Laplace's expansions [1, part II, pp. 158] all coloured letters from all uncoloured ones.

In the above example such an expansion looks as follows (we use the Scottish convention, cf. [6, pp. 188]):

$$\left(\underbrace{\begin{array}{ccc}1 & 4 & 5\\ 1 & 4 & \underline{3}\\ 2 & 3 & \underline{2}\\ 3 & 5 & \\ 1 & \underline{3} & \end{array}}_{=\underline{V}}\left|\ Y\right.\right)=\left(\underbrace{\begin{array}{ccc}1 & 4 & 5\\ 1 & 4 & \\ 2 & 3 & \\ 3 & 5 & \\ 1 & & \end{array}}_{=W}\left|\ \underbrace{\begin{array}{ccc}y_{11} & y_{12} & y_{13}\\ \overset{\sigma_1}{y_{21}} & \overset{\sigma_1}{y_{22}} & \\ \overset{\sigma_2}{y_{31}} & \overset{\sigma_2}{y_{32}} & \\ y_{41} & y_{41} & \\ \overset{\sigma_3}{y_{51}} & & \end{array}}_{=:\overset{\sigma}{Y}^{(1)}}\right.\right)\left(\underbrace{\begin{array}{c}\underline{3}\\ \underline{2}\\ \underline{3}\end{array}}_{=:\underline{V}\backslash W}\left|\ \underbrace{\begin{array}{c}\overset{\sigma_1}{y_{23}}\\ \overset{\sigma_2}{y_{33}}\\ \overset{\sigma_3}{y_{52}}\end{array}}_{=:\overset{\sigma}{Y}^{(2)}}\right.\right)$$

If we choose a corresponding notation in the general case, we can reformulate, by (4.9), the right hand side of (4.11):

$$C_L(V\backslash W)(V|Y)=(W|Y^{\sigma_{(1)}})\cdot[(\underline{V}^k\backslash W|Y^{\sigma_{(2)}})+\textstyle\sum_{\underline{V}\in\underline{\underline{V}}}(\underline{V}\backslash W|Y^{\sigma_{(2)}})] \tag{4.12}$$

Now we apply $C_L(W)=C_L({}^sW)$ to (4.12); this will leave every term within the square brackets unchanged. By induction, we can assume that

$$C_L({}^sW)(W|Y^{\sigma_{(1)}})=(\underline{T}_{|W|}|Y^{\sigma_{(1)}}). \tag{4.13}$$

Together with (4.7, 4.8, 4.12) equation (4.13) yields:

$$C_L(S)(U|X)=(\underline{T}_{|W|}|Y^{\sigma_{(1)}})[(\underline{V}^k\backslash W|Y^{\sigma_{(2)}})+\textstyle\sum_{\underline{V}\in\underline{\underline{V}}}(\underline{V}\backslash W|Y^{\sigma_{(2)}}] \tag{4.14}$$

Next reversing all Laplace's expansions we see by (4.10) that all terms in (4.18), which correspond to an element of $\underline{\underline{V}}$, vanish while the term $(\underline{T}_\lambda|Y)=(\underline{T}_\lambda|X)$ is left over. By (4.2) we have $(\underline{T}_\lambda|X)\neq 0$. This proves (a).

The other statements can be proved similarly.

The second fundamental Theorem reads as follows.

THEOREM 4.4. *Suppose* (S, T) *and* (U, V) *are uncoloured bitableaux. If* (S, T) *is standard then*

(a) $$C_L(S)(U|V) \neq 0 \text{ implies } S \underset{c}{\trianglelefteq} {}^sU;$$

(b) $$C_P(T)(U|V) \neq 0 \text{ implies } T \underset{c}{\trianglelefteq} {}^sV; \text{ and}$$

(c) $$C(S, T)(U|V) \neq 0 \text{ implies } S \underset{c}{\trianglelefteq} {}^sU \text{ and } T \underset{c}{\trianglelefteq} {}^sU.$$

PROOF. (a) can be proved as follows. By (4.8) and the assumption one can assume without loss of generality that (U, V) is normal. To every $(q, p) \in \mathbb{N}^2$ let

$$C_{qp}(S) := \prod_{j \leq p} \prod_{i \leq q} D_L^{c_i(S,j)}(\underline{j}, i)$$

denote the '(q, p)-part' of $C_L(S)$. From (4.4) and the assumption follows

$$C_{qp}(S)(U|V) \neq 0, \text{ for all } (q, p). \tag{4.15}$$

According to the action of the letter set polarization operators on bideterminants, see (4.5), and by the fact that only bideterminants of row-injective bitableaux will contribute non-zero terms to (4.15), the tableau U has the following property:

It is possible to find simultaneously for all pairs (i, j), $i \leq q$ and $j \leq p$, $c_i(S, j)$-times the letter i in U such that, after having replaced these is by $\underline{j}$s, a row-injective tableau $\underline{U}_{qp}$ results. If $\underline{U}_{qp}$ has in its ith row exactly s coloured letters, these s elements are substitutes of entries $u_{ij} \leq q$. Hence in the ith row of U there are at least s elements $\leq q$. By normality of U

$$u_{i1} \leq \cdots \leq u_{is} \leq q. \tag{4.16}$$

Furthermore at most $\underline{1}, \ldots, \underline{p}$ can occur as coloured letters in $\underline{U}_{qp}$. Since $\underline{U}_{qp}$ is row-injective the last remark together with (4.16) yields for the ith row of U

$$s \leq p. \tag{4.17}$$

Now the total number of coloured elements in $\underline{U}_{qp}$ is

$$\sum_{i \leq q} \sum_{j \leq p} c_i(S, j) = a^c_{qp}(S). \tag{4.18}$$

By (4.16)–(4.18) there are at least $a^c_{qp}(S)$ elements $\leq q$ in the first p columns of U, i.e.

$$a^c_{qp}(S) \leq a^c_{qp}(U). \tag{4.19}$$

Finally

$$a^c_{qp}(U) = a^c_{qp}({}^cU) \tag{4.20}$$

and

$${}^cU = {}^sU, \tag{4.21}$$

hence by (4.19)–(4.21) S is column-dominated by ${}^sU$. This proves (a). The second statement follows by symmetry, and (c) is a consequence of (a) and (b).

We can now prove two fundamental properties of straightening coefficients.

THEOREM 4.5. *If the bideterminant to the bitableau (U, V) is expressed as a linear combination of standard ones:*

$$(U|V) = \sum_{(S,T)\text{ standard}} a_{UV,ST}(S|T),$$

then

(a) $$a_{UV,ST} \neq 0 \text{ implies } {}^sU \underset{c}{\trianglerighteq} S \text{ and } {}^sV \underset{c}{\trianglerighteq} T; \text{ and}$$

(b) $$(U|V) \neq 0 \text{ implies } a_{UV,{}^sU{}^sV} = \pm 1.$$

PROOF. (a) is proved as follows. According to the Straightening Formula (4.2) we can write

$$(U|V) = A + B,$$

where $A = \sum \alpha_{ST}(S|T)$ (resp. $B = \sum \beta_{ST}(S|T)$) is a linear combination of bideterminants of standard bitableaux (S, T) of the same content and satisfying (resp. *not* satisfying) $(S, T) \underset{c}{\trianglelefteq} ({}^sU, {}^sV)$.

We have to show that $B = 0$.

Assume $B \neq 0$. Let (W, Z) be any $\underset{c}{\trianglelefteq}$-maximal element of the non-empty set $\{(S, T)/\beta_{ST} \neq 0\}$. We apply the Capelli operator $C(W, Z)$ to the identity

$$(U|V) - A = B. \tag{4.22}$$

By Theorems 4.3 and 4.4 and the $\underset{c}{\trianglelefteq}$-maximality of (W, Z)

$$C(W, Z)B = \beta_{WZ}(\underline{T}_\lambda|\underline{T}_\lambda).$$

where λ is the shape of (W, Z). Moreover,

$$\text{the coefficient of the monomial} \qquad \{\underline{T}_\lambda|\underline{T}_\lambda\} := \prod_{(i,j)\in\lambda} (\underline{j}|\underline{j}) \text{ in } (\underline{T}_\lambda|\underline{T}_\lambda) \text{ is } 1_R. \tag{4.23}$$

Hence

$$\beta_{WZ}(\underline{T}_\lambda|\underline{T}_\lambda) = \beta_{WZ}\{\underline{T}_\lambda|\underline{T}_\lambda\} + \cdots$$

is not the zero polynomial.

On the other hand, $C(W, Z)$ annihilates $(U|V)$ by Theorem 4.4, and maps A onto 0; the last statement is true for $C(W, Z)(S|T) \neq 0$ and $\alpha_{ST} \neq 0$ would imply $(W, Z) \underset{c}{\trianglelefteq} (S, T)$, by Theorem 4.4, and $(S, T) \underset{c}{\trianglelefteq} ({}^sU, {}^sV)$, be definition of A, and this would give $(W, Z) \underset{c}{\trianglelefteq} ({}^sU, {}^sV)$, contrary to the constructions of B and (W, Z).

Hence $C(W, Z)$ annihilates the left hand side of (4.22) and maps the right hand side onto a non-zero polynomial. This contradiction yields $B = 0$ and the proof of (a) is done.

(b) is proved as follows. By (4.1) and (4.2), we can assume without loss of generality that (U, V) is a normal bitableau of shape μ, say. According to (a), Theorems 4.3 and 4.4

$$(\underline{T}_\mu|\underline{T}_\mu) = C({}^sU, {}^sV)(U|V) = a_{UV,{}^sU{}^sV}(\underline{T}_\mu|\underline{T}_\mu),$$

hence, by (4.23), $a_{UV,{}^sU{}^sV} = 1_R$.

5. How to Calculate the Straightening Coefficients?

Let (U, V) be a row-injective bitableau of content (α, β). By Theorem 4.5,

$$(U|V)=\sum a_{UV,ST}(S|T)=\sum_{j=1}^{h} x_j(S^j|T^j), \tag{5.1}$$

where the summation is over all h standard bitableau (S^j, T^j) of content (α, β) which satisfy ${}^{s}U \underset{c}{\trianglerighteq} S^j$ and ${}^{s}V \underset{c}{\trianglerighteq} T^j$. We can assume that

$$(S^1, T^1) \underset{c}{\leqslant} \cdots \underset{c}{\leqslant} (S^h, T^h)=({}^{s}U, {}^{s}V). \tag{5.2}$$

Let λ^i denote the shape of (S^i, T^i). With these notations and by (4.23), parts of Theorem 4.4 and Theorem 4.3 can be reformulated as follows:

$$\textit{If } i>j \textit{ then } C(S^i, T^i)(S^j|T^j)=0; \tag{5.3}$$

$$C(S^i, T^i)(S^i|T^i)=(\underline{T}_{\lambda^i}|\underline{T}_{\lambda^i})=1_R \cdot \{\underline{T}_{\lambda^i}|\underline{T}_{\lambda^i}\}+\cdots. \tag{5.4}$$

Applying the h Capelli operators $C(S^i, T^i)$ to (5.1) we get the following matrix equation over $\underline{R}_m^n$:

$$(C(S^i, T^i)(S^j|T^j))_{h\times h} \cdot (x_j)_{h\times 1}=(C(S^i, T^i)(U|V))_{h\times 1}. \tag{5.5}$$

By (5.3), the $h \times h$ matrix on the left hand side of (5.5) has upper triangular form. Considering in the ith row of equation (5.5) the coefficient of the monomial $\{\underline{T}_{\lambda^i}|\underline{T}_{\lambda^i}\}$, we get by (5.3)–(5.4) the following matrix equation over R, from which we can calculate the straightening coefficients $x_i=a_{UV,S^iT^i}$:

$$\begin{pmatrix} 1 & & \xi_{ij} \\ & \ddots & \\ 0 & & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ \vdots \\ x_h \end{pmatrix}=\begin{pmatrix} b_1(U, V) \\ \vdots \\ b_h(U, V) \end{pmatrix}, \tag{5.6}$$

here ξ_{ij} [resp. $b_i(U, V)$] is the coefficient of $\{\underline{T}_{\lambda^i}|\underline{T}_{\lambda^i}\}$ in $C(S^i, T^i)(S^j|T^j)$ [resp. $C(S^i, T^i)(U|V)$].

To formulate the third fundamental Theorem about Capelli operators, which describes the coefficients ξ_{ij} and $b_i(U, V)$, we have to introduce further notations.

If D is a finite subset of $\mathbb{N}\times\mathbb{N}$, every function $T: D\to\mathbb{N}$ will be called a (*generalized*) *tableau* of *shape* D. The extensions of the notions of bitableau and bideterminant, as well as the extensions of the operations $T\mapsto {}^{r}T$ and $T\mapsto {}^{c}T$ to these more general shapes are obvious. If (U, V) is a row-injective bitableau of shape D, then we call the quotient

$$\operatorname{sgn}(U, V):=({}^{r}U|{}^{r}V):(U|V),$$

which is by (4.2) equal to ± 1, the *sign* of (U, V).

To every tableau T of shape D we can associate two further generalized tableaux: the *left-shifted* tableau $\leftarrow T$, and the *top-shifted* tableau $\uparrow T$; their definitions will become clear by the following

Example. Suppose

$$T=\begin{array}{|c|c|c|c|c|c|c|}\hline & 2 & & & & & \\ \hline & 1 & & & 5 & & \\ \hline & & & 6 & 7 & & 2 \\ \hline 3 & & & & 1 & 3 & 4 \\ \hline\end{array}$$

Then

$$\leftarrow T=\begin{array}{|c|c|c|c|}\hline 2 & & & \\ \hline 1 & 5 & & \\ \hline 6 & 7 & 2 & \\ \hline 3 & 1 & 3 & 4 \\ \hline\end{array}\qquad \uparrow T=\begin{array}{|c|c|c|c|c|c|c|}\hline 3 & 2 & & 6 & 5 & 3 & 2 \\ \hline & 1 & & & 7 & & 4 \\ \hline & & & & 1 & & \\ \hline\end{array}, \quad \textit{and} \quad {}^{r}(\uparrow T)=\begin{array}{|c|c|c|c|c|c|c|}\hline 2 & 2 & & 3 & 3 & 5 & 6 \\ \hline & 1 & & & 4 & & 7 \\ \hline & & & & 1 & & \\ \hline\end{array}$$

With these notations, the coefficients ξ_{ij} and $b_i(U, V)$ can be characterized as follows.

THEOREM 5.1. *Let (U, V) be a normal bitableau, and let (S, T) be a standard λ-bitableau of the same content. Then the coefficient of the monomial $\{T_\lambda | T_\lambda\}$ in $C(S, T)(U|V)$ equals*

$$\sum_{(X,Y)} \operatorname{sgn}(X, Y),$$

where the summation is over all generalized bitableaux (X, Y) such that

$$(U, V) = ({}^{r}(\leftarrow X), {}^{r}(\leftarrow Y)), \text{ and} \tag{1}$$

$$(S, T) = ({}^{c}(\uparrow X), {}^{c}(\uparrow Y)). \tag{2}$$

[As usual, the empty sum is defined to be 0.]

PROOF. By (4.5) we have $C(S, T)(U|V) = \sum 4(\underline{U}|\underline{V})$, where the summation is over all bitableaux $(\underline{U}, \underline{V})$ of shape $|U|$ such that
(a) $(\underline{U}, \underline{V})$ is row-injective, and
(b) $\underline{U} = (\underline{u}_{ij})$ [(resp. $\underline{V} = (\underline{v}_{ij})$) can be obtained from U (resp. V) by substituting $c_i(S, j)$— (resp. $c_i(T, j)$] times the number i by $\underline{j}$.
The monomial $\{T_\lambda | T_\lambda\}$ occurs in the bideterminant of such a bitableau $(\underline{U}, \underline{V})$ iff for all i
(c) the ith row of $\underline{U}$ equals up to a permutation of its elements the ith row of $\underline{V}$.
Now suppose the bitableau $(\underline{U}, \underline{V})$ satisfies (a)–(c). Then the elements of $(\underline{U}, \underline{V})$ tell us how to rearrange and to shift the elements of (U, V) to get a generalized bitableau (X, Y) satisfying (1) and (2): If $\underline{u}_{ij} = \underline{a}$ (resp. $\underline{v}_{ij} = \underline{b}$) then one has to shift u_{ij} (resp. v_{ij}) to the position (i, a) [resp. (i, b)], i.e.: $x_{ia} := u_{ij}$ resp. $y_{ib} := v_{ij}$.

EXAMPLE. If

$$(U, V) = \begin{pmatrix} 1 & 4 & 6 & & 1 & 2 & 5 \\ 2 & 4 & & , & 1 & 4 & \\ 2 & 4 & & & 3 & 6 & \end{pmatrix} \quad \text{and} \quad (S, T) = \begin{pmatrix} 1 & 2 & 4 & 6 & & 1 & 2 & 3 & 4 \\ 2 & 4 & & & , & 1 & 4 & & \\ 4 & & & & & 6 & & & \end{pmatrix}$$

then one possible $(\underline{U}, \underline{V})$ is

$$(\underline{U}, \underline{V}) = \begin{pmatrix} \underline{1} & \underline{2} & \underline{4} & & \underline{1} & \underline{2} & \underline{4} \\ \underline{2} & \underline{1} & & , & \underline{1} & \underline{2} & \\ \underline{1} & \underline{3} & & & \underline{3} & \underline{1} & \end{pmatrix}.$$

Here

$$X = \begin{array}{|c|c|c|c|} \hline 1 & 4 & & 6 \\ \hline 4 & 2 & & \\ \hline 2 & & 4 & \\ \hline \end{array} \quad \text{and } Y = \begin{array}{|c|c|c|c|} \hline 1 & 2 & & 4 \\ \hline 1 & 4 & & \\ \hline 6 & & 3 & \\ \hline \end{array}$$

It is straightforward to show that by $(\underline{U}, \underline{V}) \mapsto (X, Y)$ the set of all bitableaux $(\underline{U}, \underline{V})$ satisfying (a)–(c) is mapped bijectively onto the set of all bitableaux (X, Y) satisfying (1) and (2).
Using this interpretation of $(\underline{U}, \underline{V})$, our assertion follows immediately.

We would like to mention two special cases of Theorem 5.1. If the bitableaux in the theorem are both of content $((1^n), (1^n))$, then one easily shows that in the theorem there is at most one generalized bitableau (X, Y) satisfying (1) and (2). Hence, using the notation of (5.6), we have the following

COROLLARY 5.2. *If* (U, V) *is injective then* ξ_{ij}, $b_i(U, V) \in \{0, 1, -1\}$.

The second corollary is a reformulation of a result of Désarménien [4, theorem 4].

COROLLARY. 5.3. *Suppose the bitableaux in Theorem* 5.1 *are both of shape* λ. *Then all generalized bitableaux* (X, Y) *which occur in Theorem* 5.1 *are of shape* λ. *In particular conditions* (1) *and* (2) *in Theorem* 5.1 *can be replaced by*

(1)′ $$(U, V) = ({}^{r}X, {}^{r}Y) \text{ and}$$

(2)′ $$(S, T) = ({}^{c}X, {}^{c}Y),$$

respectively.

PROOF. If the shape D of a generalized tableau W is not a Young diagram, then one easily shows that $\leftarrow W$ and $\uparrow W$ have different shapes. This remark together with $|S| = |U|$ proves our statement.

REFERENCES

1. M. Clausen, Letter place algebras and a characteristic-free approach to the representation theory of the general linear and symmetric groups, I, *Advances in Math.* **33** (1979), 161–191; II, *Advances in Math.* **38** (1980), 152–177.
2. C. de Concini, Symplectic standard tableaux, *Advances in Math.* **34** (1979), 1–27.
3. C. de Concini, D. Eisenbud and C. Procesi, Young diagrams and determinantal varieties, *Inventiones math.* **56** (1980), 129–165.
4. J. Désarménien, An algorithm for the Rota straightening formula, *Discrete Math.* **30** (1980), 51-68.
5. J. Désarménien, J. P. S. Kung and G.-C. Rota, Invariant theory, Young bitableaux and combinatorics, *Advances in Math.* **27** (1978), 63–92.
6. P. Doubilet, G.-C. Rota and J. Stein, On the foundations of combinatorial theory IX. Combinatorial methods in invariant theory, *Stud. Appl. Math.* **53** (1974), 185–216.
7. M. Gardner, Mathematical games: On the fine art of putting players, pills and points into their proper pigeonholes, *Sci. Amer.*, August (1980), 14–18.
8. J. A. Green, *Polynomial Representations of* GL_n, *Lecture Notes in Math. 830*, Springer-Verlag, Berlin, 1980.
9. G. D. James, *The Representation Theory of the Symmetric Groups, Lecture Notes in Math. 682*, Springer-Verlag, Berlin, 1978.
10. I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Clarendon Press, Oxford, 1979.
11. K. Pommerening, Invarianten unipotenter Gruppen, *Math. Z.* **176** (1981), 359–374.
12. G.-C. Rota, *Théorie combinatoire des invariants classiques* [with an appendix by J. Désarménien], Séries de Mathématiques pures et appliquées, IRMA, Strasbourg, 1977.

Received 1 July 1981

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