Eigenvalues of Nonnegative Symmetric Matrices

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ABSTRACT

Some necessary and some sufficient conditions are found for $n$ real numbers to be eigenvalues of an $n \times n$ nonnegative (or alternatively, positive) symmetric matrix and for $2n$ real numbers to be eigenvalues and diagonal entries of an $n \times n$ nonnegative symmetric matrix.

1. INTRODUCTION

In recent years, many papers about eigenvalues of nonnegative or positive matrices have appeared. The geometric method of Sulejmanova [9] was essentially applied by H. Perfect [6], Ciarlet [1], and B. Kellogg [4] to obtain sufficient conditions for $n$ real numbers $\lambda_1, \ldots, \lambda_n$ to be eigenvalues of a nonnegative (or positive) $n \times n$ matrix. H. Perfect [7] and F. Salzmann [8] used theorems by A. Brauer and E. V. Haynsworth to obtain other sufficient conditions.

None of the mentioned methods was used for the case of symmetric matrices, though the assumption that $\lambda_1, \ldots, \lambda_n$ are real could be considered more natural for this case than for the case of general matrices.

In Section 2, we shall present a simple but powerful lemma and show in Theorem (2,5) that practically all known sufficient conditions for real numbers $\lambda_1, \ldots, \lambda_n$ to be eigenvalues of a nonnegative $n \times n$ matrix are also sufficient for the existence of a nonnegative (or positive) symmetric matrix with these eigenvalues.

Moreover, our proof seems to be simpler and the theorems are valid not only for the field of reals but for any ordered field which is Euclidean, i.e., has the property that it contains square roots of all its positive elements.

In Section 3, results similar to those in Section 2 for positive matrices are proved.

In Section 4, we shall investigate relations between the eigenvalues and the diagonal entries of a nonnegative symmetric matrix.

2. SUFFICIENT CONDITIONS FOR EIGENVALUES OF NONNEGATIVE MATRICES

We shall introduce the following notation. We denote by \( P_n \) the set of all \( n \)-tuples \( (\lambda_1; \lambda_2, \ldots, \lambda_n) \), where \( \lambda_2, \ldots, \lambda_n \) are considered unordered, such that there exists a nonnegative \( n \times n \) matrix with the Perron root \( \lambda_1 \) and the remaining eigenvalues \( \lambda_2, \ldots, \lambda_n \). Similarly, we denote by \( S_n \) the set of those \( n \)-tuples in \( P_n \) for which there exists a symmetric nonnegative \( n \times n \) matrix with these eigenvalues.

Let us recall now that in [9] Sulejmanova announced, and H. Perfect [6] proved, that if \( \lambda_1 > \lambda_2 > \cdots > \lambda_n \) are real numbers such that

\[
\lambda_1 + \sum_{i, \lambda_i < 0} \lambda_i > 0, \tag{1}
\]

then \((\lambda_1; \lambda_2, \ldots, \lambda_n) \in P_n\).

Salzmann [8] improved this by showing that if \( \lambda_1 \geq \cdots \geq \lambda_n \) and

\[
\frac{1}{2} (\lambda_i + \lambda_{n+1-i}) < \frac{1}{n} \sum_{i=1}^{n} \lambda_i, \quad i = 2, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor, \tag{S}
\]

then \((\lambda_1; \lambda_2, \ldots, \lambda_n) \in P_n\).

If we modify the conditions in the paper [4] by R. B. Kellogg for the case of nonnegative matrices, we can state them (in a somewhat different, but equivalent formulation) as follows:

If \( \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_N \), if \( K = \{ i \in \{1, \ldots, \lfloor N/2 \rfloor \} \mid \lambda_i > 0 \} \) and \( \lambda_i + \lambda_{N+1-i} < 0 \), and if \( M \) is the greatest index \( j \) (\( 0 < j < N \)) for which \( \lambda_j > 0 \) then

\[
\lambda_0 + \sum_{i \in K} (\lambda_i + \lambda_{N+1-i}) + \lambda_{N+1-k} > 0 \text{ for all } k \in K,
\]

\[
\lambda_0 + \sum_{i \in K} (\lambda_i + \lambda_{N+1-i}) + \sum_{j=M+1}^{N-M} \lambda_j > 0 \tag{K}
\]

implies \((\lambda_0; \lambda_1, \ldots, \lambda_N) \in P_{N+1}\).
We shall show first:

**Theorem 2.1.** If Salzman's conditions (S) are satisfied for \((\lambda_0; \lambda_1, \ldots, \lambda_N)\) then the following inequality

\[
\lambda_0 + \lambda_N + \sum_{i=0}^{N} \lambda_i \geq \frac{1}{2} \sum_{i=1}^{N-1} |\lambda_i + \lambda_{N-i}|
\]  

(2)

(where \(\lambda_0 > \lambda_1 > \cdots > \lambda_N\)) is satisfied as well.

If (2) is satisfied then the conditions (K) are satisfied as well. The converse is not true in either case.

**Proof.** Assume that \(\lambda_0 > \lambda_1 > \cdots > \lambda_N\). Then (S) has the form

\[
\frac{1}{2}(\lambda_i + \lambda_{N-i}) \leq \frac{1}{N+1} \sum_{i=0}^{N} \lambda_i, \quad i = 1, \ldots, \lceil \frac{N}{2} \rceil.
\]

\[
\sum_{i=0}^{N} \lambda_i > 0.
\]

(S')

Denote

\[M_0 = \{1, \ldots, N-1\},\]

\[M_1 = \{i \in M_0 | \lambda_i + \lambda_{N-i} < 0\},\]

\[M_2 = M_0 \setminus M_1.\]

If

\[S_0 = \lambda_0 + \lambda_N,\]

\[S_1 = \frac{1}{2} \sum_{i \in M_1} (\lambda_i + \lambda_{N-i}),\]

\[S_2 = \frac{1}{2} \sum_{i \in M_2} (\lambda_i + \lambda_{N-i}),\]

then the second inequality in (S') can be written as

\[S_0 + S_1 + S_2 > 0,\]

\[S_0 + (S_0 + S_1 + S_2) > S_2 - S_1,\]

\[3\]
which is equivalent to

$$S_0 + S_1 \geq 0. \quad (4)$$

Let \((S')\) be satisfied. If \(M_2\) is void, \(S_0 = 0\) and (3) is identical with (4). If \(M_2\) has \(m_2 \geq 1\) elements, then summing the first inequalities over \(i \in M_2\), we obtain

$$S_2 \leq \frac{m_2}{N+1} (S_0 + S_1 + S_2),$$

which is equivalent to

$$S_0 + S_1 \geq \left( \frac{N+1}{m_2} - 1 \right) S_2.$$

Since \(m_2 \leq N - 1\) and \(S_2 > 0\), (4) follows and the proof of the first part is complete.

Now let (4) (which is equivalent to (2)) be satisfied. We shall show that (K) is then satisfied as well. We shall denote

$$K_1 = \{ i \in \{1, \ldots, \lfloor \frac{1}{2} (N-1) \rfloor \} \mid \lambda_i + \lambda_{N-i} < 0 \},$$

$$K_1^+ = \{ i \in K_1 \mid \lambda_i > 0 \},$$

and use the abbreviation

$$(\lambda_{\frac{1}{2}N})^- \text{ in the following sense:}$$

$$(\lambda_{\frac{1}{2}N})^- = \lambda_{\frac{1}{2}N}, \text{ if } N \text{ is even and } \lambda_{\frac{1}{2}N} < 0,$$

$$(\lambda_{\frac{1}{2}N})^- = 0, \text{ otherwise.}$$

It is clear that (4) can then be written in the form

$$\lambda_0 + \lambda_N + \sum_{i \in K_1} (\lambda_i + \lambda_{N-i}) + (\lambda_{\frac{1}{2}N})^- > 0. \quad (4')$$

Assume that (K) is not fulfilled. Let first

$$\lambda_0 + \sum_{i \in K} (\lambda_i + \lambda_{N+1-i}) + \lambda_{N+1-k} < 0 \text{ for some } k \in K.$$
Then
\[ \lambda_0 + \sum_{i \in K, i < k} (\lambda_i + \lambda_{N+1-i}) + \lambda_{N+1-k} < \lambda_0 + \lambda_N + \sum_{i \in K_1} (\lambda_i + \lambda_{N-i}) + (\frac{1}{2}N)^-, \]

which is equivalent to
\[ \sum_{i \in K, i < k} (\lambda_i + \lambda_{N-i}) + \sum_{i \in K, i < k} (\lambda_{N+1-i} - \lambda_{N-i}) + \lambda_{N+1-k} < \sum_{i \in K_1, i < k} (\lambda_i + \lambda_{N-i}) + \sum_{i \in K_1, i > k} (\lambda_i + \lambda_{N-i}) + \lambda_N + (\frac{1}{2}N)^-. \]

Since \( K_1^+ \subseteq K \) and
\[ \sum_{i \in K, i < k} (\lambda_{N+1-i} - \lambda_{N-i}) > k \cdot \sum_{i=1}^{k-1} (\lambda_{N+1-i} - \lambda_{N-i}) = \lambda_N - \lambda_{N+1-k}, \]
it follows that
\[ \sum_{i \in K \setminus K_1^+, i < k} (\lambda_i + \lambda_{N-i}) + \lambda_N < \sum_{i \in K_1 \setminus K_1^+, i < k} (\lambda_i + \lambda_{N-i}) + \sum_{i \in K_1^+, i > k} (\lambda_i + \lambda_{N-i}) + \lambda_N + (\frac{1}{2}N)^- \]

However, \( k \leq N/2 \),
\[ \sum_{i \in K \setminus K_1^+, i < k} (\lambda_i + \lambda_{N-i}) > 0, \]
while the similar sums over \( K_1 \setminus K_1^+ \) as well as over \( K_1^+, i > k \) are nonpositive. Consequently,
\[ 0 < (\frac{1}{2}N)^-, \]
which is a contradiction.

Assume now that
\[ \lambda_0 + \sum_{i \in K} (\lambda_i + \lambda_{N+1-i}) + \sum_{j=M+1}^{N-M} \lambda_j < 0. \]
Let us distinguish two cases:

A. $M < N - M$; we have then

$$
\lambda_0 + \sum_{i \in K} (\lambda_i + \lambda_{N-i}) + \sum_{i \in K} (\lambda_{N+1-i} - \lambda_{N-i}) + \sum_{i = M+1}^{N-M} \lambda_i
$$

$$
< \lambda_0 + \lambda_N + \sum_{i \in K} (\lambda_i + \lambda_{N-i}) + (\lambda_{\frac{1}{2}N})^-
$$

which implies, by

$$
\sum_{i \in K} (\lambda_{N+1-i} - \lambda_{N-i}) \geq \sum_{i = 1}^{M} (\lambda_{N+1-i} - \lambda_{N-i}) = \lambda_N - \lambda_{N-M}
$$

that

$$
\sum_{i \in K \setminus K_1} (\lambda_i + \lambda_{N-i}) + \lambda_N - \lambda_N + \sum_{i = M+1}^{N-M} \lambda_i
$$

$$
< \lambda_N + \sum_{i \in K \setminus K_1^+} (\lambda_i + \lambda_{N-i}) + (\lambda_{\frac{1}{2}N})^-.
$$

However,

$$
\sum_{i \in K \setminus K_1^+} (\lambda_i + \lambda_{N-i}) + (\lambda_{\frac{1}{2}N})^- = \sum_{i = M+1}^{N-M-1} \lambda_i,
$$

if $M + 1 < N - M - 1$ and zero otherwise. It follows that

$$
\sum_{i \in K \setminus K_1} (\lambda_i + \lambda_{N-i}) < 0,
$$

which is a contradiction.

B. $M > N - M$. Then $(\lambda_{\frac{1}{2}N})^- = 0$ and $K_1 = K_1^+$. We have

$$
\lambda_0 + \sum_{i \in K} (\lambda_i + \lambda_{N+1-i}) < \lambda_0 + \lambda_N + \sum_{i \in K_1} (\lambda_i + \lambda_{N-i}).
$$

This is easily seen to be equivalent to

$$
\sum_{i \in K} (\lambda_i + \lambda_{N-i}) + \sum_{i \in K} (\lambda_{N+1-i} - \lambda_{N-i}) < \lambda_N + \sum_{i \in K_1} (\lambda_i + \lambda_{N-i}),
$$
so that
\[ \sum_{i \in K \setminus K_1^+} (\lambda_i + \lambda_{N-i}) + \sum_{i \in K} (\lambda_{N+1-i} - \lambda_{N-i}) < \lambda_N. \] (5)

Let \( m = \max_{i \in K} i \). Thus \( m < \left[ \frac{1}{2} N \right] \) and \( \lambda_m > 0, \lambda_m + \lambda_{N+1-m} < 0 \). Assume \( m \in K_1 \), i.e., \( m < \left[ \frac{1}{2} (N-1) \right] \) and \( \lambda_m + \lambda_{N-m} < 0 \). If \( N \) is odd and \( m = \frac{1}{2} (N-1) \), we have \( N - m = \frac{1}{2} (N+1) < M \), \( \lambda_{N-m} > 0 \), \( \lambda_m > 0 \), a contradiction. In all other cases, \( m < \frac{1}{2} (N-1) \) and \( m + 1 < \left[ \frac{1}{2} N \right] < M \) so that \( \lambda_{m+1} > 0 \) and
\[ \lambda_{m+1} + \lambda_{N-m} < \lambda_m + \lambda_{N-m} < 0. \]

Thus \( m + 1 \in K \), a contradiction. Consequently, \( m \in K \setminus K_1 \) and we obtain that the left-hand side of (5) is not less than
\[ \sum_{i \in K \setminus K_1} (\lambda_i + \lambda_{N-i}) + \lambda_m + \lambda_{N-m} + \sum_{i = 1}^{m} (\lambda_{N+1-i} - \lambda_{N-i}) \]
\[ = \sum_{i \in K \setminus K_1, i \neq m} (\lambda_i + \lambda_{N-i}) + \lambda_N + \lambda_m. \]

This is a contradiction to (5) since the first and third summands are nonnegative.

The example (for \( N = 3 \)) \( \{5; 4, -2, -4\} \) shows that (S) can be fulfilled even if (2) is not, the example (for \( N = 4 \)) \( \{3; 2, -1, -1, -3\} \) shows the same about (K) and (S). The proof is complete.

We shall now present the announced lemma. This was already used in [2] to supply a simple and more general proof of the well-known Horn’s theorem [3] on eigenvalues and diagonal entries of symmetric matrices. In this lemma as well as in the sequel, we shall assume that all numbers belong to an ordered field which is Euclidean, i.e., that it contains square roots of all its positive elements.

**Lemma 2.2.** Let \( A \) be a symmetric \( m \times m \) matrix with eigenvalues \( \alpha_1, \ldots, \alpha_m \), let \( u, \|u\| = 1 \), be a unit eigenvector corresponding to \( \alpha_1 \); let \( B \) be a symmetric \( n \times n \) matrix with eigenvalues \( \beta_1, \ldots, \beta_n \), let \( v, \|v\| = 1 \), be a unit eigenvector corresponding to \( \beta_1 \).

Then for any \( \rho \), the matrix
\[ C = \begin{pmatrix} A & \rho uv^T \\ \rho vu^T & B \end{pmatrix} \]
has eigenvalues $\alpha_2, \ldots, \alpha_m, \beta_2, \ldots, \beta_n, \gamma_1, \gamma_2$, where $\gamma_1, \gamma_2$ are eigenvalues of the matrix

$$\hat{C} = \begin{pmatrix} \alpha_1 & \rho \\ \rho & \beta_1 \end{pmatrix}.$$

Proof. Let $u_1, u_2, \ldots, u_m$ form an orthonormal system of eigenvectors of $A$,

$$Au_i = \alpha_i u_i, \quad i = 2, \ldots, m.$$

A direct verification shows that the vectors

$$\begin{pmatrix} u_i \\ 0 \end{pmatrix}, \quad i = 2, \ldots, m$$

with $m + n$ rows, are eigenvectors of $C$ with corresponding eigenvalues $\alpha_i, i = 2, \ldots, m$.

Similarly, let $v_1, v_2, \ldots, v_n$ form an orthonormal system of eigenvectors of $B$,

$$Bv_i = \beta_i v_i, \quad i = 2, \ldots, n.$$

Then

$$\begin{pmatrix} 0 \\ v_i \end{pmatrix}, \quad i = 2, \ldots, n$$

are eigenvectors of $C$ corresponding to eigenvalues $\beta_2, \ldots, \beta_n$ of $C$.

Let $\gamma_1, \gamma_2$ be eigenvalues of the matrix $C$, let $(r_i, s_i)^T, i = 1, 2$, be eigenvectors of $C$ corresponding to $\gamma_1, \gamma_2$ and forming an orthonormal system.

Then

$$\begin{pmatrix} r_i u \\ s_i v \end{pmatrix}$$

is easily seen to be an eigenvector of $C$ corresponding to $\gamma_i, i = 1, 2$. Since all the mentioned eigenvectors of $C$ form an orthonormal set of $m + n$ vectors, the set of eigenvalues described above is the complete set of eigenvalues for $C$. The proof is complete.

Theorem 2.3. If $(\alpha_0; \alpha_1, \ldots, \alpha_m - 1) \in S_m, (\beta_0; \beta_1, \ldots, \beta_n - 1) \in S_n$ and $\alpha_0 > \beta_0$ then for any $\sigma > 0$,

$$(\alpha_0 + \sigma; \beta_0 - \sigma, \alpha_1, \ldots, \alpha_{m-1}, \beta_1, \ldots, \beta_{n-1}) \in S_{m+n}.$$
Proof. Since \((a_0; a_1, \ldots, a_{m-1}) \in S_m\), there exists a nonnegative symmetric
\(m \times m\) matrix \(A\) with eigenvalues \(a_0, \ldots, a_{m-1}\), with the Perron root \(a_0\) and
the corresponding unit nonnegative eigenvector \(u\). Similarly, there exists a
nonnegative symmetric \(n \times n\) matrix \(B\) with eigenvalues \(\beta_0, \beta_1, \ldots, \beta_{n-1}\), with
the Perron root \(\beta_0\) and the corresponding unit nonnegative eigenvector \(v\). If
\(\sigma > 0\), choose \(\rho = (\sigma(a_0 - \beta_0 + \sigma))^{1/2}\). It follows from Lemma (2,2) that the
nonnegative symmetric matrix
\[
\begin{pmatrix}
A & \rho uv^T \\
\rho vu^T & B
\end{pmatrix}
\]
has eigenvalues \(a_0 + \sigma, \beta_0 - \sigma, a_1, \ldots, a_{m-1}, \beta_1, \ldots, \beta_{n-1}\) and the Perron root \(a_0 + \sigma\). The proof is complete.

Let us apply this Theorem to prove essentially Sulejmanova's result for
the case of symmetric matrices.

**Theorem 2.4.** Let \(\lambda_0 > 0 > \lambda_1 > \cdots > \lambda_N, \sum_{i=0}^N \lambda_i > 0\). Then
\[(\lambda_0; \lambda_1, \ldots, \lambda_N) \in S_{N+1}.
\]

Proof. We shall proceed by induction with respect to \(N\). If \(N=0\), the
theorem is clear. If \(N=1\), the matrix
\[
\begin{pmatrix}
0 & (\lambda_0 \lambda_1)^{1/2} \\
(\lambda_0 \lambda_1)^{1/2} & \lambda_0 + \lambda_1
\end{pmatrix}
\]
is easily seen to satisfy the conditions.

Now let \(N > 2\) and assume the result is true for all smaller systems. The
system
\[
\lambda_0 = \lambda_0 + \lambda_1, \quad \lambda_1 = \lambda_2, \ldots, \lambda_{N-1} = \lambda_N
\]
clearly satisfies the assumptions. By the induction hypothesis, \((\lambda_0; \lambda_1, \ldots, \lambda_{N-1}) \in S_N\). Since \((0) \in S_1\), we can apply Theorem (2,3) with \(\sigma = |\lambda_1|\) and
obtain that
\[(\lambda_0; \lambda_1, \lambda_2, \ldots, \lambda_N) \in S_{N+1}.
\]
The proof is complete.

We are now able to prove the main theorem of this Section:

**Theorem 2.5.** Let \(\lambda_0 > \lambda_1 > \cdots > \lambda_N\) satisfy the system \((K)\). Then there
exists a symmetric nonnegative \((N+1) \times (N+1)\) matrix with eigenvalues
\(\lambda_0, \ldots, \lambda_N\).
Proof. We shall proceed by induction with respect to $N$. If $N = 0$, $K$ is void, $M = 0$ and the theorem is true. If $N = 1$, $K$ is void as well and for $M = 0$ the assertion follows from Theorem (2, 4).

Thus, let $N > 2$ and suppose the theorem true for all smaller systems of numbers $\lambda$. Let $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_N$ satisfy $(K)$.

If $M > \lfloor \frac{N}{2} \rfloor$ then $\lambda_M$ does not occur in $(K)$ at all since $M \notin K$ and $\lambda_{N+1-M} > \lambda_M$ implies $N+1 - M \notin K$. Thus, the system of numbers obtained from $\lambda_0, \lambda_1, \ldots, \lambda_N$ by omitting $\lambda_M$ satisfies $(K)$ as well and, by the induction hypothesis, belongs to $S_N$. Since $(\lambda_M) \in S_1$, we have $(\lambda_0; \lambda_1, \ldots, \lambda_N) \in S_{N+1}$ by Theorem (2, 3).

Suppose now that $M \leq \lfloor \frac{N}{2} \rfloor$. If $M = 0$, the theorem is true by Theorem (2, 4). Thus, let $M > 1$. If $K \neq \{1, \ldots, M\}$, there exists an integer $1 < j < M$, such that $j \notin K$. Thus $\lambda_j > 0$, $\lambda_j + \lambda_{N+1-j} > 0$. Since neither $\lambda_j$ nor $\lambda_{N+1-j}$ occur in $(K)$, the system obtained from $\lambda_0, \ldots, \lambda_N$ by omitting both $\lambda_j$ and $\lambda_{N+1-j}$ also satisfies $(K)$ and a similar reasoning yields, since $(\lambda_j; \lambda_{N+1-j}) \in S_2$, that $(\lambda_0; \lambda_1, \ldots, \lambda_N) \in S_{N+1}$.

It remains to prove the theorem for the case that $1 \leq M \leq \lfloor \frac{N}{2} \rfloor$ and $K = \{1, \ldots, M\}$. The system $(K)$ has then the form

$$
\sum_{j=0}^{k} \lambda_j + \sum_{i=0}^{k} \lambda_{N-i} > 0, \quad k = 0, \ldots, M-1, \quad (6)
$$

$$
\sum_{i=0}^{N} \lambda_i > 0. \quad (7)
$$

The fact that $K = \{1, \ldots, M\}$ means that

$$
\lambda_j > 0, \quad (8)
$$

$$
\lambda_j + \lambda_{N+1-j} < 0, \quad i = 1, \ldots, M. \quad (9)
$$

Define

$$
\epsilon_1 = \lambda_0 + \lambda_N,
$$

$$
\epsilon_2 = - (\lambda_1 + \lambda_N).
$$

By (6) and (9), $\epsilon_1 > 0$, $\epsilon_2 > 0$.

Define

$$
\epsilon = \min (\epsilon_1, \epsilon_2),
$$

$$
\lambda_0' = \lambda_0 - \epsilon, \lambda_1' = \lambda_1 + \epsilon, \lambda_i' = \lambda_i, \quad i = 2, \ldots, N.
$$

Since

$$
\lambda_0 - \lambda_1 = \epsilon_1 + \epsilon_2 \geq 2 \epsilon,
$$

$$
\lambda_0 - \lambda_i = \epsilon_i + \epsilon_2 \geq 2 \epsilon,
$$

$$
\lambda_i - \lambda_i' = \epsilon_i \geq \epsilon,
$$

$$
\lambda_i' - \lambda_i'' = \epsilon_i - \epsilon \geq \epsilon.
$$
we have

\[ \lambda_0 > \lambda_i. \]  

(10)

If \( \varepsilon = \varepsilon_1 \), the system \( (\lambda_1'; \lambda_2'; \ldots; \lambda_{N-1}') \in S_{N-1} \) by the induction hypothesis since \( \lambda_1' > \lambda_2' > \cdots > \lambda_{N-1}' \) and

\[ \sum_{i=1}^{s} \lambda_i' + \sum_{i=1}^{s} \lambda_{N-i}' = \sum_{i=0}^{s} \lambda_i + \sum_{i=0}^{s} \lambda_{N-i} > 0 \]

by (6) for \( s = 1, \ldots, M - 1 \),

\[ \sum_{i=1}^{N-1} \lambda_i' - \sum_{i=0}^{N} \lambda_i > 0 \]

by (7).

Since

\( \lambda_0' > \lambda_N' \)

and

\( \lambda_0' + \lambda_N' = 0 \),

\( (\lambda_0'; \lambda_N') \in S_2 \). According to (10) and Theorem (2, 3), applied to \( \sigma = \varepsilon \), \( (\lambda_0'; \lambda_1', \ldots, \lambda_N') \in S_{N+1} \). Now let \( \varepsilon = \varepsilon_2 \). Then \( (\lambda_0'; \lambda_2', \ldots, \lambda_{N-1}') \in S_{N-1} \) since \( \lambda_0' > \lambda_2' \) and

\[ \lambda_0' + \sum_{i=2}^{s} \lambda_i' + \sum_{i=1}^{s} \lambda_{N-i}' = \sum_{i=0}^{s} \lambda_i + \sum_{i=0}^{s} \lambda_{N-i} > 0, \quad s = 1, \ldots, M - 1, \]

as well as

\[ \lambda_0' + \sum_{i=2}^{N-1} \lambda_i' = \sum_{i=0}^{N} \lambda_i > 0. \]

Since also

\( \lambda_1' > \lambda_N' \)

and

\( \lambda_1' + \lambda_N' = 0 \),

we have \( (\lambda_1'; \lambda_N') \in S_2 \) and, as before, Theorem (2, 3) yields

\( (\lambda_0'; \lambda_1', \ldots, \lambda_N') \in S_{N+1} \).

The proof is complete.
3. SUFFICIENT CONDITIONS FOR EIGENVALUES OF
POSITIVE SYMMETRIC MATRICES

In this section, we shall formulate sufficient conditions similar to those in
Section 2 for the case of positive matrices. Analogously to \( S_n \), we define \( \hat{S}_n \).
Thus an \( n \)-tuple of elements of our ordered Euclidean field \((\lambda_0; \lambda_1, \ldots, \lambda_{n-1})\)
belongs to \( \hat{S}_n \), if and only if there exists a positive symmetric \( n \times n \) matrix
with the Perron root \( \lambda_0 \) and the remaining (unordered) eigenvalues \( \lambda_1, \ldots, \lambda_{n-1} \).

We shall prove first a lemma and a useful theorem.

**Lemma 3.1.** Let \( \alpha > \beta > 0, \sigma > 0 \). Then there exists a number \( c > 0 \) such
that the matrix

\[
\begin{pmatrix}
\alpha & c \\
c & \beta
\end{pmatrix}
\]

has eigenvalues \( \alpha + \sigma, \beta - \sigma \).

**Proof.** It suffices to choose \( c = (\sigma(\alpha - \beta + \sigma))^{1/2} \).

**Theorem 3.2.** If \((\lambda_0; \lambda_1, \ldots, \lambda_{n-1}) \in S_n \) and if \( \epsilon > 0 \) then

\((\lambda_0 + \epsilon; \lambda_1, \ldots, \lambda_{n-1}) \in \hat{S}_n \).

**Proof.** Let \( A > 0 \) realize \((\lambda_0; \lambda_1, \ldots, \lambda_{n-1}) \in S_n \), let

\[
A = \begin{pmatrix}
A_1 & 0 & \cdots & 0 \\
0 & A_2 & \cdots & 0 \\
& & \ddots & \vdots \\
0 & 0 & \cdots & A_k
\end{pmatrix},
\]

where \( A_i, i = 1, \ldots, k \), are irreducible and \( \lambda_0 \) is an eigenvalue of \( A_1 \). We shall
proceed by induction with respect to \( k \).

If \( k = 1 \), if

\[ A_1 u_1 = \lambda_0 u_1, \quad u_1 > 0, \quad \|u_1\| = 1, \]

the matrix

\[
\tilde{A} = A + \epsilon u_1 u_1^T
\]
is clearly positive and realizes

$$(\lambda_0 + \epsilon; \lambda_1, \ldots, \lambda_{n-1}) \in \hat{S}_n.$$  

Now let $k \geq 2$ and assume the assertion true if $A$ has less than $k$ irreducible diagonal blocks.

We can choose the notation in such a way that

$$
\begin{pmatrix}
A_1 & 0 & \cdots & 0 \\
0 & A_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_{k-1}
\end{pmatrix}
$$

realizes $(\lambda_0; \lambda_1, \ldots, \lambda_{t-1}) \in S_t$ while $A_k$ realizes $(\lambda_t; \lambda_{t+1}, \ldots, \lambda_{n-1}) \in S_{n-t}$, $1 \leq t < n$.

There exist numbers $\epsilon'$ and $\epsilon''$ such that

$$\epsilon' > \epsilon'' > 0,$$

$$\epsilon' + \epsilon'' = \epsilon.$$

By the induction hypothesis, there exists a matrix $\tilde{A} > 0$ which realizes $(\lambda_0 + \epsilon'; \lambda_1, \ldots, \lambda_{t-1}) \in \tilde{S}_t$ and a matrix $\tilde{B} > 0$ which realizes $(\lambda_t + \epsilon''; \lambda_{t+1}, \ldots, \lambda_{n-1}) \in \tilde{S}_{n-t}$.

Since

$$\lambda_0 + \epsilon' > \lambda_t + \epsilon'',$$

there exists, by Lemma 3.1, a number $c > 0$ such that

$$
\begin{pmatrix}
\lambda_0 + \epsilon' & c \\
c & \lambda_t + \epsilon''
\end{pmatrix}
$$

has eigenvalues $\lambda_0 + \epsilon' + \epsilon'' = \lambda_0 + \epsilon$ and $\lambda_t$. By Lemma 2.2, the matrix

$$
\begin{pmatrix}
\hat{A} & cuv^T \\
cvu^T & \hat{B}
\end{pmatrix}
$$

where $u > 0$ is a unit eigenvector of $\hat{A}$ corresponding to $\lambda_0 + \epsilon'$ and $v > 0$ is a unit eigenvector of $\hat{B}$ corresponding to $\lambda_t + \epsilon''$, realizes

$$(\lambda_0 + \epsilon; \lambda_1, \ldots, \lambda_{n-1}) \in \hat{S}_n.$$  

The proof is complete.
This theorem enables us to prove a theorem analogous to Theorem 2.5.

**THEOREM 3.3.** Let $\lambda_0 > \lambda_1 > \cdots > \lambda_N$ satisfy the following system

$$
\lambda_0 + \sum_{i \in K} \left( \lambda_i + \lambda_{N+1-i} \right) + \lambda_{N+1-k} > 0 \text{ for all } k \in K,
$$

$$
\lambda_0 + \sum_{i \in K} \left( \lambda_i + \lambda_{N+1-i} \right) + \sum_{j = M+1}^{N-M} \lambda_j > 0,
$$

where

$$
K = \{ i \in \{1, \ldots, \lfloor \frac{1}{2} N \rfloor \} | \lambda_i > 0 \text{ and } \lambda_i + \lambda_{N+1-i} < 0 \},
$$

and $M$ is the greatest index $j$ ($0 < j < N$) for which $\lambda_j > 0$. Then there exists a positive symmetric $(N+1) \times (N+1)$ matrix with the eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_N$.

*Proof.* Let $(K)$ be satisfied for $\lambda_0 > \lambda_1 > \cdots > \lambda_N$. Then there is an $\epsilon > 0$ such that $\lambda_0 > \lambda_1 > \cdots > \lambda_N$ where $\lambda'_0 = \lambda_0 - \epsilon$ satisfies $(K)$ in Section 2. By Theorem 2.5,

$$
(\lambda'_0, \lambda_1, \ldots, \lambda_N) \in S_{N+1}.
$$

By Theorem 3.2,

$$
(\lambda'_0, \lambda_1, \ldots, \lambda_N) \in \hat{S}_{N+1}.
$$

The proof is complete.

For sake of completeness, we include two theorems.

**THEOREM 3.4.** Let $(\lambda_0; \lambda_1, \ldots, \lambda_{n-1}) \in \hat{S}_n$. Then there exists an $\epsilon > 0$ such that for all $\sigma < \epsilon$, $(\lambda_0 - \sigma; \lambda_1, \ldots, \lambda_{n-1}) \in \hat{S}_n$.

*Proof.* Let $(\lambda_0; \lambda_1, \ldots, \lambda_{n-1}) \in \hat{S}_n$ be realized by a positive symmetric $n \times n$ matrix $A$. Let $Au = \lambda_0 u$ with $u > 0$, $\|u\| = 1$. It follows that there exists a positive number $\epsilon$ such that

$$
A - \epsilon uu^T > 0.
$$

Consequently, whenever $\sigma < \epsilon$,

$$
A - \sigma uu^T > 0.
$$
and this matrix is easily seen to realize
\[(\lambda_0 - \sigma; \lambda_1, \ldots, \lambda_{n-1}) \in \hat{S}_n.\]
The proof is complete.

**Theorem 3.5.** If \((a_0; a_1, \ldots, a_{m-1}) \in \hat{S}_m, (\beta_0; \beta_1, \ldots, \beta_{n-1}) \in S_n, \alpha_0 > \beta_0,\)
then for any \(\sigma > 0\)
\[(a_0 + \sigma; \beta_0 - \sigma, a_1, \ldots, a_{m-1}, \beta_1, \ldots, \beta_{n-1}) \in \hat{S}_{m+n}.\]
If \((a_0; a_1, \ldots, a_{m-1}) \in \hat{S}_m, (\alpha_0; \beta_1, \ldots, \beta_{n-1}) \in \hat{S}_n, \sigma > 0\) then
\[(a_0 + \sigma; \beta_0 - \sigma, a_1, \ldots, a_{m-1}, \beta_1, \ldots, \beta_{n-1}) \in \hat{S}_{m+n}.\]

**Remark.** The example \(m = n = 1, \alpha_0 = 1, \sigma = 0\) shows that the condition \(\sigma > 0\) cannot be weakened to \(\sigma \geq 0\).

**Proof.** Let the assumptions of the first part be fulfilled. Then there exists, by Theorem 3.4, an \(\epsilon > 0\) such that \((a_0 - \epsilon; a_1, \ldots, a_{m-1}) \in \hat{S}_m\) and \(\alpha_0 - \epsilon > \beta_0 + \epsilon\). By (3.1), there exists a number \(c > 0\) such that
\[
\left( \begin{array}{cc}
\alpha_0 - \epsilon & c \\
c & \beta_0 + \epsilon
\end{array} \right)
\]
has eigenvalues \(\alpha_0 + \sigma, \beta_0 - \sigma\). If \(\tilde{A}\) realizes \((\alpha_0 - \epsilon; a_1, \ldots, a_{m-1}) \in \hat{S}_m, \tilde{B}\) (existing by Theorem 3.2) realizes \((\beta_0 + \epsilon; \beta_1, \ldots, \beta_{n-1}) \in \hat{S}_n\) and if \(A\)
\[= (a_0 - \epsilon)u, u > 0, \|u\| = 1, \ Bv = (\beta_0 + \epsilon)v, v > 0, \|v\| = 1\) then the positive symmetric matrix
\[
\left( \begin{array}{cc}
\tilde{A} & cuv^T \\
cuv^T & \tilde{B}
\end{array} \right)
\]
realizes, by Lemma 2.2,
\[(\alpha_0 + \sigma; \beta_0 - \sigma, a_1, \ldots, a_{m-1}, \beta_1, \ldots, \beta_{n-1}) \in \hat{S}_{m+n}.\]
The last assertion follows in the same manner from Lemma 3.1 and Lemma 2.2.
The proof is complete.
4. EIGENVALUES AND DIAGONAL ENTRIES OF NONNEGATIVE SYMMETRIC MATRICES

We shall introduce another notation. We denote by $S^*_n$ the set of all $2n$-tuples $(\lambda_1; \lambda_2, \ldots, \lambda_n | a_1, a_2, \ldots, a_n)$ of numbers from our ordered Euclidean field such that there exists a nonnegative symmetric $n \times n$ matrix with the Perron root $\lambda_1$, the remaining eigenvalues $\lambda_2, \ldots, \lambda_n$ and the diagonal entries $a_1, \ldots, a_n$. The numbers $\lambda_2, \ldots, \lambda_n$, as well as the numbers $a_1, \ldots, a_n$, are considered as unordered.

Lemma 4.1. Let $\lambda_1, \lambda_2, a_1, a_2$ satisfy

$$\lambda_1 \geq \max(a_1, a_2)$$

and

$$\lambda_1 + \lambda_2 = a_1 + a_2.$$

Then $(\lambda_1; \lambda_2 | a_1, a_2) \in S^*_2$.

Proof. The matrix

$$\begin{pmatrix}
a_1 & t \\
t & a_2
\end{pmatrix}$$

where $t = (\lambda_1 - a_1)^{\frac{1}{2}}(\lambda_2 - a_2)^{\frac{1}{2}}$ is easily seen to realize $(\lambda_1; \lambda_2 | a_1, a_2) \in S^*_2$.

Theorem 4.2. Let $\lambda_1, \lambda_2, \ldots, \lambda_n, a_1, a_2, \ldots, a_n$ satisfy the conditions

$$a_i \geq 0, \quad i = 1, \ldots, n,$$

$$a_1 = \max_i a_i,$$

$$\lambda_j \leq a_j, \quad j = 2, \ldots, n,$$

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=2}^{n} a_i.$$

Then,

$$(\lambda_1; \lambda_2, \ldots, \lambda_n | a_1, a_2, \ldots, a_n) \in S^*_n.$$

Proof. We shall use induction with respect to $n$. The assertion is trivial for $n = 1$ and follows from Lemma 4.1 for $n = 2$. Thus, let $n > 2$ and assume
the assertion true for all $2k$-tuples with $k < n$. Let $\lambda_i, a_i, i = 1, \ldots, n$ satisfy the conditions. Define

$$\begin{align*}
\lambda'_1 &= \lambda_1 + \lambda_n - a_n, \\
\lambda'_{i+1} &= \lambda_i, \quad i = 2, \ldots, n - 1.
\end{align*}$$

The numbers $\lambda'_1, \lambda'_2, \ldots, \lambda'_{n-1}, a_1, a_2, \ldots, a_{n-1}$ satisfy the conditions. By the induction hypothesis, there exists a nonnegative symmetric matrix $A$ realizing $\left(\lambda'_1; \lambda'_2, \ldots, \lambda'_{n-1} | a_1, \ldots, a_{n-1}\right) \in S^*_n$. According to Lemma 4.1, there exists a nonnegative symmetric matrix $X$ realizing $\left(\lambda'_1; \lambda'_2, \ldots, \lambda'_{n-1} | a_1, \ldots, a_{n-1}\right) \in S^*_n$. The proof is complete.

**Remark 4.3.** Theorem 2.4 follows clearly from Theorem 4.2 if we choose $a_1 = 1 + 1 + a_i = 0, i = 2, \ldots, n$. It can also be strengthened if we choose the $a_i$'s in a more suitable way. However, the resulting theorem would be weaker than Theorem 2.5.

**Theorem 4.4.** Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n, a_1 \geq a_2 \geq \ldots \geq a_n$ satisfy

$$\sum_{i=1}^{s} \lambda_i \geq \sum_{i=1}^{s} a_i, \quad s = 1, \ldots, n - 1$$

and

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_i$$

$$\lambda_k \leq a_{k-1}, \quad k = 2, \ldots, n - 1.$$
Then \((\lambda_1; \lambda_2, \ldots, \lambda_n|a_1, \ldots, a_n) \in S_n^*\).

**Proof.** We shall use induction with respect to \(n\). The assertion is trivial for \(n = 1\) and follows from Lemma 4.1 for \(n = 2\). Let \(n > 3\) and suppose the assertion true for \(2k\)-tuples with \(k < n\). Let \(\lambda_i, a_i, i = 1, \ldots, n\) satisfy the above conditions. Define

\[
\lambda'_2 = \lambda_1 + \lambda_2 - a_1.
\]

The system \(\lambda'_2, \lambda_3, \ldots, \lambda_n, a_2, \ldots, a_n\) is easily seen to satisfy the analogous system. By the induction hypothesis, there exists a nonnegative symmetric \((n - 1) \times (n - 1)\) matrix \(\tilde{A}\) with the Perron root \(\lambda'_2\), the remaining eigenvalues \(\lambda_3, \ldots, \lambda_n\) and diagonal entries \(a_2, \ldots, a_n\). Since

\[
\lambda_1 > \lambda'_2 \quad \text{and} \quad \lambda_1 > a_1, \quad \lambda_1 + \lambda_2 = \lambda'_2 + a_1,
\]

there exists by Lemma 4.1 a nonnegative \(2 \times 2\) matrix

\[
\begin{pmatrix}
\lambda'_2 & \sigma \\
\sigma & a_1
\end{pmatrix}
\]

with eigenvalues \(\lambda_1, \lambda_2\). By Lemma 2.2, if \(Au = \lambda'_2 u, u > 0, \|u\| = 1\), the nonnegative matrix

\[
\begin{pmatrix}
\tilde{A} & \sigma u \\
\sigma u^T & a_1
\end{pmatrix}
\]

realizes

\(\lambda_1; \lambda_2, \ldots, \lambda_n|a_1, \ldots, a_n) \in S_n^*\).

The proof is complete.

We shall turn now to necessary conditions for \((\lambda_1; \lambda_2, \ldots, \lambda_n|a_1, \ldots, a_n)\) to belong to \(S_n^*\).

First of all, let us recall the well-known Horn’s conditions [3] for eigenvalues \(\lambda_1 > \ldots > \lambda_n\) and diagonal entries \(a_1 > \ldots > a_n\) of any (not necessarily nonnegative) symmetric \(n \times n\) matrix:

\[
\sum_{i=1}^{s} \lambda_i > \sum_{i=1}^{s} a_i, \quad s = 1, \ldots, n - 1,
\]

\[
\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_i. \quad (11)
\]
These conditions are not only necessary but also sufficient for the existence of a symmetric $n \times n$ matrix with eigenvalues $\lambda_i$ and diagonal entries $a_i$. This is even true [2] for any ordered Euclidean field.

The conditions (11) may be also written in the form

$$\sum_{i=s}^{n} \lambda_i < \sum_{i=s}^{n} a_i, \quad s=2, \ldots, n,$$

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_i. \quad (12)$$

For $A$ nonnegative, the following theorem yields other independent conditions.

**THEOREM 4.5.** If $(\lambda_1; \lambda_2, \ldots, \lambda_n | a_1, \ldots, a_n)$ belongs to $S_n^*$ then $a_i > 0$, $i=1; \ldots, n$,

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_i,$$

and

$$\sum_{i=1}^{n} \lambda_i^k > \sum_{i=1}^{n} a_i^k, \quad k=2,3,\ldots.$$

**Proof.** Let $(\lambda_1; \lambda_2, \ldots, \lambda_n | a_1, \ldots, a_n) \in S_n^*$. Then there exists a nonnegative symmetric matrix $A$ with diagonal entries $a_1, \ldots, a_n$ and eigenvalues $\lambda_1, \ldots, \lambda_n$. Thus, of course $a_i > 0$ and for the trace

$$\sum_{i=1}^{n} \lambda_i = \text{tr}A = \sum_{i=1}^{n} a_i.$$ 

Similarly,

$$\sum_{i=1}^{k} \lambda_i^k = \text{tr}A^k.$$ 

However, since $A \succ 0$, 

$$\text{tr}A^k > \sum_{i=1}^{n} a_i^k$$

and the proof is complete.
To obtain other necessary conditions, let us prove first two lemmas.

**Lemma 4.6.** Let \( P = (p_{ik}) \) be an \( m \times m \) nonnegative matrix, \( u, v \) non-negative column vectors with \( m \) coordinates each. Then

\[
v^T P u \geq \left( \min_{i=1, \ldots, m} p_{ii} \right) v^T u.
\]

**Proof.** Follows immediately from \( P \geq (\min_i p_{ii}) I \), where \( I \) is the \( m \times m \) identity matrix.

**Lemma 4.7.** Let \( \lambda_1 > \lambda_2 > \ldots > \lambda_n, a_1 > a_2 > \ldots > a_n, n > 2 \). If \( (\lambda_1; \lambda_2; \ldots, \lambda_n | a_1, \ldots, a_n) \in S_n^+ \) then

\[
\lambda_1 + \lambda_n \geq a_{n-1} + a_n.
\]

**Remark.** This follows also from a theorem proved by Lidskij [5] and Wielandt [10] on eigenvalues of the sum of two Hermitian matrices \( A \) and \( B \) if we choose as \( A \) the diagonal submatrix of our matrix. We include a straightforward simple proof here since it seems to be of independent interest.

**Proof.** Let \( A = (a_{ik}) \) be a symmetric matrix with eigenvalues \( \lambda_1, \ldots, \lambda_n \) and diagonal entries \( a_i \) (in some order). Assume first that \( A \) is irreducible. Then \( \lambda_1 \) corresponds to a positive eigenvector \( u \) of \( A \), \( \lambda_n \) corresponds to an eigenvector \( v \) which is not nonnegative. Without loss of generality, we can assume that the first \( m \) \( (1 \leq m \leq n-1) \) coordinates of \( v \) are positive, the remaining \( n - m \) nonpositive. The corresponding partitioning of \( A, u, v \) be

\[
A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},
\]

where \( A_{1k} \), \( u_1, v_1 \) have \( m \) rows.

We have

\[
u_1 > 0, \quad u_2 > 0,
\]

\[
v_1 > 0, \quad -v_2 > 0,
\]

\[
Au = \lambda_1 u, \quad (13)
\]

\[
Av = \lambda_n v. \quad (14)
\]
Clearly \( \lambda_1 \neq \lambda_n \), since otherwise \( A \) would be a multiple of \( I \) and thus reducible. It follows that \( u \) and \( v \) are orthogonal:

\[
v^T u = 0, \text{ i.e.} \\
v_1^T u_1 = -v_2^T u_2.
\]

By (13),

\[
A_{11} u_1 + A_{12} u_2 = \lambda_1 u_1,
\]

by (14)

\[
A_{12}^T v_1 + A_{22} v_2 = \lambda_n v_2.
\]

Multiplying the first of these equalities by \( v_1^T \), the second by \(-u_2^T\) and adding, we obtain

\[
v_1^T A_{11} u_1 - u_2^T A_{22} v_2 = \lambda_1 v_1^T u_1 - \lambda_n u_2^T v_2.
\]

According to (15), this can be written in the form

\[
(\lambda_1 + \lambda_n) v_1^T u_1 = v_1^T A_{11} u_1 + u_2^T A_{22}(-v_2).
\]

By Lemma 4.6, we can estimate the right-hand side from below by

\[
\sigma_1(v_1^T u_1) + \sigma_2(u_2^T(-v_2)) = (\sigma_1 + \sigma_2)(v_1^T u_1)
\]

where \( \sigma_1, \sigma_2 \), respectively are the minimal diagonal entries of \( A_{11}, A_{22} \), respectively. Thus,

\[
\lambda_1 + \lambda_n \geq \sigma_1 + \sigma_2 > a_{n-1} + a_n.
\]

Now let \( A \) be reducible, say

\[
A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},
\]

where \( A_1 \) is that irreducible diagonal submatrix which has the eigenvalue \( \lambda_n \).
If \( A_1 \) has dimension greater than one, we obtain the assertion from the already proved assertion for \( A_1 \). If \( A_1 \) has dimension one, we have

\[
\lambda_n = a_i > a_n
\]

for some \( i \). By (12) for \( s = n \),

\[
\lambda_n < a_n
\]

so that \( \lambda_n = a_n \) and the assertion follows from (11) for \( s = 1 \).
The proof is complete.
We are able now to prove the main theorem of this Section:

**THEOREM 4.8.** If \( \lambda_1 \geq \cdots \geq \lambda_n \) are eigenvalues and \( a_1 \geq \cdots \geq a_n \) diagonal entries of a nonnegative symmetric matrix then

\[
\lambda_1 > a_1, \quad \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_i,
\]

and

\[
\sum_{i=1}^{s} \lambda_i + \lambda_k > \sum_{i=1}^{s-1} a_i + a_{k-1} + a_k
\]  

(16) for all \( s \) and \( k, 1 < s < k < n \).

If \( n = 1, 2, \) or \( 3 \), these conditions, together with \( a_n > 0 \), are also sufficient for the existence of a nonnegative symmetric \( n \times n \) matrix with eigenvalues \( \lambda_i \) and diagonal entries \( a_i \). For \( n > 4 \) these conditions are not sufficient.

**Proof.** The first two conditions as well as the conditions (16) for \( k = s + 1 \) being fulfilled by (11) and the condition (16) for \( s = 1 \) and \( k = n \) being fulfilled by Lemma 4.7, we shall prove the remaining conditions in (16) by induction with respect to \( n \). For \( n = 1, 2 \), there is nothing left. Let \( n > 3 \) and assume all conditions (16) are fulfilled for matrices with \( n - 1 \) rows and columns.

We shall need the well known separation theorem:

If \( \lambda_1 > \cdots \geq \lambda_n \) are eigenvalues of a symmetric \( n \times n \) matrix \( A \) and \( \hat{\lambda}_1 \geq \cdots \geq \hat{\lambda}_{n-1} \) (possibly in an ordered extension of the given ordered field) eigenvalues of a principal \( (n-1) \times (n-1) \) submatrix \( \hat{A} \) of \( A \) then

\[
\lambda_i > \hat{\lambda}_i > \lambda_{i+1}, \quad i = 1, \ldots, n-1.
\]

Let first \((s,k)\) be a pair such that \( k < n \). Let \( \hat{A} \) be the principal submatrix of \( A \) obtained by deleting the row and column of \( A \) containing the diagonal entry \( a_n \). The eigenvalues \( \hat{\lambda}_i \) of \( \hat{A} \) satisfy by the induction hypothesis the inequality

\[
\sum_{i=1}^{s} \hat{\lambda}_i + \hat{\lambda}_k > \sum_{i=1}^{s-1} a_i + a_{k-1} + a_k.
\]
By (17), we obtain really

\[ \sum_{i=1}^{s} \lambda_i + \lambda_k > \sum_{i=1}^{s-1} a_i + a_{k-1} + a_k. \]

Now let \( k = n \) but \( s > 1 \). Let \( \hat{A} \) be this time the principal submatrix of \( A \) obtained by deleting the row and column containing \( a_1 \). The eigenvalues \( \hat{\lambda}_i \) of \( \hat{A} \) satisfy then by the induction hypothesis the inequality

\[ \sum_{i=1}^{s-1} \hat{\lambda}_i + \hat{\lambda}_{n-1} > \sum_{i=2}^{s-1} a_i + a_{n-1} + a_n. \]

Since \( \sum_{i=1}^{n-1} \hat{\lambda}_i = \sum_{i=2}^{n} a_i \), this is equivalent to

\[ \sum_{i=s}^{n-2} \hat{\lambda}_i < \sum_{i=s}^{n-2} a_i. \]

By (17), this implies

\[ \sum_{i=s+1}^{n-1} \lambda_i < \sum_{i=s}^{n-2} a_i \]

which is equivalent to

\[ \sum_{i=1}^{s} \lambda_i + \lambda_n > \sum_{i=1}^{s-1} a_i + a_{n-1} + a_n. \]

The proof of the inequalities is complete.

Sufficiency is clear for \( n = 1 \) and follows from Lemma 4.1 for \( n = 2 \). For \( n = 3 \), it is easily seen that the assumptions of Theorem 4.4 are identical with the assertion of this Theorem. For \( n = 4 \), the Salzmann-type example \( \lambda_1 = -4, \lambda_3 = -1, \lambda_4 = -3, a_1 = a_2 = 2, a_3 = a_4 = 0 \) satisfies the conditions, but it is easily seen that no nonnegative matrix has these eigenvalues and diagonal entries. The proof is complete.

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