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SEMISIMPLE STABLE AND SUPERSTABLE GROUPS

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Introduction

Characterizations of semisimple structures prove the equivalence of two kinds of conditions. Very roughly, one (the null radical) characterizes semisimplicity as the absence of a certain sort of substructure (e.g. no normal Abelian subgroup); the other (reducibility) characterizes a semisimple structure as a direct product of simples. Theorems of this sort require some sort of chain condition on ideals (normal subgroups) as a hypothesis. In this paper we consider versions of these results which make some definability hypothesis on the ideas. Not only do these variations arise naturally in model theory but they directly generalize the analysis of semisimple algebraic groups. Section 1 deals with several variations on the 'null radical' formulation and shows that under appropriate stability hypotheses the conditions involving definable subgroups are equivalent to those involving arbitrary subgroups. In Section 2 we prove a weak form of a decomposition into simple groups. In Section 3 we introduce the notion of an α-semiregular group. These refine (for superstable groups) Hrushovski's notion of a semiregular group. In Section 4 we prove a theorem decomposing a superstable semisimple group (with monomial U-rank) as a product of α-semiregular groups. This generalizes Lascar's analysis of semisimple groups of finite Morley rank. Finally we show that any superstable group is an extension of a solvable group by an ω-stable group. We conclude with a list of problems. Some arose here; others seem more pressing than before in the light of the work here.

For general algebraic background see [18] or [11]. Our model-theoretic

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terminology follows [2] and [14]. For the interaction of group theory and model theory see [6] and [16].

1. What is a semisimple group?

The first section of this paper concerns a foundational and pedagogical issue. We consider the specialization to several categories of groups of a natural abstract definition of 'semisimplicity'. We show that they in fact reduce to the same definition which does not depend on the category. At the same time we outline the properties of semisimplicity and two major results about semisimple groups which one would like to prove in each category.

We do not use the word category in the standard technical sense. Rather by a category **K** of groups we mean a class of groups and specific notion of 'subgroup'. We will consider six such categories. Our results are new only for the last two. They specialize to yield new proofs of old results for the third and fourth cases.

We need some technical notation to define the categories. We work in a fixed language L with a distinguished binary operation. A stable group is a stable structure for this language which is a group with respect to the distinguished operation.

1.1. Definition. A subgroup H of a group G is *type-definable* or ($\bigwedge$-definable) if the universe of H is the intersection of a family of at most $|L|$ definable subsets $\phi_i(G)$ and the conjunction of the ϕ_i define a subgroup in every group elementarily equivalent to G which contains the parameters of the ϕ_i.

Poizat proved in [17] the important fact that if G is stable the ϕ_i in this definition can be chosen to define subgroups. In fact, most of our type-definable subgroups will arise in this way. On the other hand, Lascar and Berline in [6] point out that, again when G is stable, the intersection of any family of definable subgroups is equal to the intersection of at most $|L|$ definable subgroups. The following proposition follows easily from compactness. The saturation hypothesis on G is necessary to show the minimal subgroup is nontrivial (and in fact infinite).

1.2. Proposition. *If G is a $|L|^+$-saturated stable group, then the intersection of any family of infinite type-definable subgroups of G which is closed under finite intersection is an infinite type-definable subgroup of G.*

One of our principal goals is to examine the common principles behind the study of semisimple finite groups and the study of semisimple algebraic groups. Two key factors arise in each case. A certain class of subgroups is considered (all in first case; Zariski closed in the second). This class of subgroup satisfies certain chain conditions. In order to treat uniformly these two cases and various model theoretically defined classes we speak of categories of groups.

1.3. Some categories of groups

Groups. The class of all groups with K-subgroup meaning subgroup.

Finite groups. The class of finite groups with K-subgroup meaning subgroup.

Algebraic groups. The class of algebraic groups over an algebraically closed field with K-subgroup meaning closed (in the Zariski topology) connected subgroup.

ω-Stable groups. The class of groups definable in an ω-stable structure with K-subgroup meaning definable connected subgroup.

Superstable groups. The class of groups definable in a superstable structure with K-subgroup meaning $\bigwedge$-definable connected subgroup.

Stable groups. The class of groups definable in a stable structure with K-subgroup meaning $\bigwedge$-definable connected subgroup.

A Zariski-closed subgroup of an algebraic group over an algebraically closed field is definable and every such group is ω-stable with finite rank so the fourth class generalizes the third.

Finite groups of course satisfy the descending chain condition on subgroups. ω-stable groups and a fortiori groups in category 3 satisfy the descending chain condition on K-subgroups. Proposition 1.2 provides a weaker kind of minimal condition for families of K-subgroups in categories 5 and 6. While infinite descending chains of K-subgroups may exist we can still find a K-lower bound. In this last case we must restrict to saturated models to guarantee that the lower bound remains infinite. We will be able to transport most of the important results back to arbitrary models.

It would be plausible in the last four cases to omit the adjective connected in the definition of 'subgroup'. There are two reasons for including it. On the one hand it widens the applicability of the theory, e.g., since SL(2, $\mathscr{C}$) as well as PSL(2, $\mathscr{C}$) is now semisimple. On the other hand it yields the maximal (i.e. ascending chain) condition on normal solvable subgroups which allow us to prove the existence of a solvable radical (Proposition 1.7). By requiring K-subgroups to be connected we obtain the maximal condition for the cases of algebraic groups, ω-stable groups of finite rank, and superstable groups of finite U-rank.

We now outline the general definition and program for discussing semisimple groups in each of the categories. There are two connotations to the word semisimple. One is 'direct product of simples'; the other is 'absence of a radical'. Under certain finiteness conditions precise versions of these notions are proved equivalent in various contexts. Our intention here is to provide a common model-theoretic generalization of several of these contexts. We begin with the 'radical' version; the 'product of simples' version shows up in Definition 1.19.

1.4. Definition. (i) The group G is $\mathbb{K}$-*simple* if it contains no proper normal $\mathbb{K}$-subgroup.

(ii) The group G is $\mathbb{K}$-*semisimple* if it contains no nontrivial normal solvable $\mathbb{K}$-subgroup.

(iii) The group K is the *solvable* $\mathbb{K}$-*radical* of G if K is a maximal normal solvable $\mathbb{K}$-subgroup of G.

1.5. Program. Now the general program is to prove two results:

(i) Every group in the class is an extension of a solvable group by a semisimple group.

(ii) Classify the semisimple groups.

Applying this schema in the model-theoretic context we have two possible definitions.

1.6. Definition. (i) The group G is *definably simple* if it contains no infinite normal definable subgroup.

(ii) The group G is $\bigwedge$-*definably simple* if it contains no infinite normal $\bigwedge$-definable subgroup.

However, Poizat's proof [17] that a proper normal $\bigwedge$-definable subgroup of a saturated group is contained in a proper normal definable subgroup and the fact [6] that a group is definably simple in the sense of (i) just if every G' elementarily equivalent to G is also definably simple shows that the definitions in 1.6 coalesce. We call the resulting notion *definable simplicity*.

In pure group theory a simple group is one with no normal subgroups. A slight generalization, sometimes called almost simple, is to only require that there be no infinite normal subgroups. In this paper we work entirely in the more general situation. A second nuance to this problem arises because under certain definability conditions 'infinite' can be replaced by 'connected'. We discuss this in Theorem 1.9.

For superstable groups [6] proved a group is simple if and only if it is definably simple. It remains open whether this can be extended to stable groups. We show below that the analogous result does hold for stable semisimple groups.

The following proposition, which shows the existence of a 'solvable radical', is obvious, noting that the product of two normal solvable groups is solvable. Note, however, that it would fail if we attempted to define an 'abelian radical' but since a product of normal nilpotent subgroups is nilpotent a similar argument constructs a 'nilpotent radical'.

1.7. Proposition. *If* $\mathbb{K}$ *satisfies the ascending chain condition on* $\mathbb{K}$*-subgroups and the product of two* $\mathbb{K}$*-subgroups is a* $\mathbb{K}$*-subgroup, then each* $G \in \mathbb{K}$ *contains a unique maximal normal solvable* $\mathbb{K}$*-subgroup.*

Of course the ascending chain condition fails miserably in the category of all groups. It succeeds equally trivially for finite groups. It holds for non-trivial reasons for algebraic groups and certain model-theoretically defined classes. In particular we have the following corollary which specializes to the categories of algebraic groups and ω-stable groups of finite rank.

1.8. Corollary. *If G is a superstable group with finite U-rank, then G is solvable by semisimple.*

There are three parameters distinguishing the six conditions in the following theorem. The subgroup may be definable, $\bigwedge$-definable, or satisfy no definability condition. It may be solvable or abelian. It may be infinite or connected. The last two are simply appropriate variants for the category of the same condition. A definable connected group is infinite; a $\bigwedge$-definable connected group is infinite if the ambient structure is $|L|^+$-saturated.

1.9. Theorem. *Let G be a stable group. The first four of the following are equivalent; if G is $|T|^+$-saturated all six conditions are equivalent.*

(i) *G has no normal infinite solvable subgroup.*
(ii) *G has no normal infinite definable solvable subgroup.*
(iii) *G has no normal infinite abelian subgroup.*
(iv) *G has no normal infinite definable abelian subgroup.*
(v) *G has no normal connected $\bigwedge$-definable solvable subgroup.*
(vi) *G has no normal connected $\bigwedge$-definable abelian subgroup.*

Proof. The implications (i)$\rightarrow$(ii) and (iii)$\rightarrow$(iv) are trivial. It is easy to see that (ii) implies (iii) since if H is a normal infinite abelian subgroup of G then $Z(C_G(H))$ is a definable normal infinite abelian subgroup of G. Corollary 1.14 which we will now prove, shows (ii) implies (i) and (iv) implies (iii). Thus there are two equivalent conditions for 'solvable', two equivalent conditions for 'abelian', and the 'solvable' conditions easily imply the 'abelian' conditions. Since [5] shows that a solvable stable group has a definable abelian subgroup of the same cardinality, (iv) implies (ii) and the 'abelian' conditions imply the 'solvable' conditions. If G is $|L|^+$-saturated condition (v) is 'sandwiched' between (i) and (ii) by taking connected components and similarly for (iii), (vi) and (iv). □

Note that in the case of ω-stable groups of finite rank (and algebraic groups) the descending central series is definable and the equivalence of the 'solvable' and 'abelian' case is much easier.

We say a stable group is *semisimple* if it satisfies any of the equivalent conditions of Theorem 1.9. Note that a definably simple group is semisimple.

1.10. Notation. For any subset X of the group G, let $\bar{X}$ denote the intersection of the type-definable subgroups containing X.

We rely on the following result from [17]. $N_G(H)$ denotes the normalizer of H in G, the set of elements g of G such that $H^g = gHg^{-1} = H$.

1.11. Lemma. *If H is a type-definable $|T|^+$-saturated subgroup of a stable group G, then $N_G(H)$ is also type-definable.*

The following facts have long been known when $\bar{A}$ denotes the definable closure of a subset of an ω-stable group. We extend the result to stable groups.

1.12. Theorem. *Suppose G is stable, $|T|^+$-saturated and $A \lhd B \subseteq G$. If B/A is abelian so is $\bar{B}/\bar{A}$.*

Proof. Suppose C and D are type-definable so is each D_c.

We first observe that $A \lhd B$ implies B normalizes $\bar{A}$. For, if not, for some $b \in B\bar{A} \cap \bar{A}^b$ is a type-definable subgroup of G which properly contains A and is properly contained in $\bar{A}$. Thus $B \subseteq N_G(\bar{A})$ which is type-definable by the previous lemma. We further deduce: $\bar{B}$ normalizes $\bar{A}$.

Now applying the first paragraph of the proof with $\bar{B}$ as C and $\bar{A}$ as D, we see that for each $b \in B$, $\bar{A}_b$ is a type-definable group. Consequently, so is $H = \bigcap_{b\in B} \bar{A}_b$. Since B/A is abelian, $B \subseteq H$ and so $\bar{B} = H$. Let $\bar{H} = \bigcap_{b \in \bar{B}} \bar{A}_b$. Since $H \supseteq \bar{B}$, $[\bar{B}, B] \subseteq A$ and so $B \subseteq \bar{H}$, i.e. $\bar{B} = \bar{H}$. By the definition of $\bar{H}$, $[\bar{H}, \bar{B}] \subseteq \bar{A}$ so $[\bar{B}, \bar{B}] \subseteq \bar{A}$ as required. $\square$

A first application of this result is the following corollary (which also appears in [16]).

1.13. Corollary. *Let H be an n-step solvable subgroup of a stable $|T|^+$-saturated group G.*

(i) *$\bar{H}$ is an n-step solvable type-definable subgroup of G.*

(ii) *There is a definable n-step solvable subgroup $\hat{H}$ containing H.*

Proof. (i) Induct on n. If $n = 1$, note that $\bar{H} \subseteq Z(C_G(H))$ (which is actually definable). Suppose the lemma holds for $k < n$ and H is n-step solvable. By Theorem 1.12, $\bar{H}/(\widetilde{H'})$ is abelian and by induction $(\widetilde{H'})$ is $n-1$-step solvable so we finish.

(ii) For any $|T|^+$-saturated G any first order property such as k-step solvability which holds of a type-definable subgroup of G can be seen by a straightforward compactness argument to hold of a definable supergroup. $\square$

The following corollary completes the proof of Theorem 1.9.

1.14. Corollary. *Let H be an n-step solvable subgroup of a stable group G. There is a definable n-step solvable group $\hat{H}$ containing H.*

Proof. Let G^* be a $|T|^+$-saturated elementary extension of G. Applying Corollary 1.13(ii), there is a definable subgroup H_1 of G^* which contains H and is n-step solvable. But then by stability, $H_1 \cap G$ is definable in G and a subgroup of an n-step solvable group is n-step solvable. □

We can now establish in our context the standard fact that a normal subgroup of a semisimple group is semisimple. We need one technical fact which will be exploited several times later on.

1.15. Proposition. *Suppose H and K are normal connected $\bigwedge$-definable subgroups of G and $H \cap K$ is finite. Then H and K commute.*

Proof. Fix $h \in H$. Note that $[h, K] \subseteq H \cap K$ and so is finite. Thus, h^K is finite. This implies $C_K(h)$ has finite index in K and so by connectedness equals K. Since h was arbitrary we finish. □

1.16. Lemma. *Let H be a type-definable subgroup of a $|T|^+$-saturated stable group G. If H is not semisimple, H contains a characteristic infinite abelian subgroup which is definable from the parameters used in the definition of H.*

Proof. Since H is not semisimple, there is a normal type definable abelian subgroup H_0 of H. Let $K \subseteq H_0$ be a minimal normal infinite type-definable subgroup of H. Necessarily, K is abelian. Then every conjugate of K by an automorphism of H is also minimal type-definable infinite and abelian. Since K and each of its conjugates is connected and have finite intersection, Proposition 1.15 implies they commute in pairs. Thus the product $\tilde{K}$ of the conjugates of G is abelian. So $Z(C_H(\hat{K}))$ is characteristic in H and a definable abelian subgroup of H. Thus, it is definable from the same parameters as H. □

It is now clear that semisimplicity is a property of the theory of a stable group G rather than just of G. Formally,

1.17. Corollary. *Let G be stable and $G^* \equiv G$. If G is semisimple, so is G^*.*

Proof. Suppose not; let $\hat{G}$ be a $|T|^+$-saturated elementary extension of G^*. Theorem 1.9 implies immediately that an elementary extension of a group which is not semisimple is not semisimple. But if $\hat{G}$ is not semisimple, Lemma 1.16 implies $\hat{G}$ contains a $\emptyset$-definable infinite normal abelian subgroup. But then so does G contrary to hypothesis. □

1.18. Corollary. *Let G be a stable semisimple group and H an infinite normal $\bigwedge$-definable subgroup of G. Then H is semisimple.*

Proof. Without loss of generality G is $|T|^+$-saturated. If H is not semisimple, Lemma 1.16 implies H has a characteristic type-definable abelian subgroup and thus a normal type-definable abelian subgroup of G, contrary to hypothesis. □

We conclude this section by indicating the general form a characterization of semisimple groups should take and the various approximations to this form which have been proved. For this we need the following definition.

1.19. Definition. (i) The group G is *centerless completely reducible* (*ccr*) if G is isomorphic to a direct product of nonabelian simple groups.

(ii) The group G is *almost centerless completely reducible* (*accr*) if G/K is isomorphic to a direct product of nonabelian simple groups for some finite central subgroup K of G.

In the second case we say G is an almost direct product of the simple groups. Viewed internally this means that there exists a family H_i of normal subgroups of G whose product is G, which pairwise commute and such that the intersection of each H_i with the product of the other H_j is finite.

Now the ideal formulation of the characterization theorem asserts

1.20. Ideal result. *A connected* $\mathbb{K}$*-semisimple group is accr.*

This result in fact holds for the category of algebraic groups. It comes as close as possible to holding for finite groups (where connectedness makes no sense). If G is a finite semisimple group, then there is a ccr group $\hat{G}$ with $\hat{G} \subseteq G \subseteq \mathrm{Aut}(\hat{G})$ [11]. Here are three model-theoretic analogs of this result. The similarity of the first with the case of algebraic groups is clearer if one recalls that a simple ω-stable group of finite Morley rank is $\aleph_1$-categorical [15]. The third depends on the notion of α-semiregularity introduced in Section 3.

1.21. Theorem. *Let G be a stable semisimple group.*

Lascar [12]. *If G is a connected ω-stable group with finite rank, then G is an almost direct product of almost strongly minimal groups.*

Theorem 2.4. *If G is a $|T|^+$-saturated connected stable group, then G is contained in the algebraic closure of an $\bigwedge$-definable subgroup $\hat{G}$ of G which is an almost direct product of finitely many definably simple groups.*

Theorem 4.2. *If G is a connected superstable group with monomial U-rank $\omega^{\alpha}k$, then G is an almost direct product of α-semiregular groups.*

In the second case we can replace definably simple by simple and $\bigwedge$-definable by definable if we strengthen the hypothesis to superstable. The last two results are proved in Sections 2 and 4 respectively of this paper.

We have proved the results of this section for groups which are stable in their own right. It is not difficult to see that the arguments here generalize to type-definable subgroups of a stable group and we use this fact in Section 4.

2. Algebraic decomposition theorems

In this section we find in a stable semisimple group G a subgroup $\hat{G}$ such that $G \subseteq \operatorname{acl} \hat{G}$ and $\hat{G}$ is a direct product of (definably)-simple groups.

We begin with two model-theoretic facts which are used to prove this result. The first is a slight variant on an old remark [3].

2.1. Lemma. *Let $\langle H_i : i \in I \rangle$ be a collection of nonabelian subgroups of a group G which commute in pairs. If I is infinite, then G has the independence property (and so is unstable).*

Proof. We show that for each $n < \omega$, each $S \subseteq n$, and each $m < n$ there are elements c_m and d_S such that $[c_m, d_S] = 1$ if and only if $m \in S$. Choose, for $i < n$, a_i, $b_i \in H_i$ which do not commute. Now let $c_m = a_m$ and $d_S = \prod_{i \notin S} b_i$. This implies the formula $[x, y] = 1$ has the independence property. □

We again employ the descending condition on centralizers.

2.2. Lemma. *Suppose G is a stable group and H is a normal subgroup of G. Then $G/C_G(H)$ is contained in the definable closure of H.*

Proof. Choose a finite subset $H_0 \subseteq H$ with $C_G(H_0) = C_G(H)$. Note that a and b in G are in the same coset mod $C_G(H)$ if and only if $h^a = h^b$ for all $h \in H$ if and only if $h^a = h^b$ for all $h \in H_0$. Thus the coset $\bar{g} = gC_G(H)$ is defined by the formula $\phi_g(x)$: $(\exists y)\, y \in x \wedge \bigwedge_{h \in H_0} h^y = h^g$. The g^h for $h \in H_0$ form a finite subset of H so the parameters of this formula are from H. □

We deduce immediately

2.3. Corollary. *If G is stable, H is a normal subgroup of G, and $C_G(H)$ is finite (trivial), then $G \subseteq \operatorname{acl}(H)$ $(G \subseteq \operatorname{dcl}(H))$.*

Here is the first decomposition theorem.

2.4. Theorem. *If G is a $|T|^+$-saturated connected stable semisimple group, then G is contained in the algebraic closure of a $\bigwedge$-definable subgroup $\hat{G}$ which is an almost direct product of $\bigwedge$-definable definably simple groups.*

Proof. Let $\{H_i : i \in I\}$ be the set of type-definable minimal infinite normal subgroups of G. The minimality guarantees that the H_i are connected. Since G is semisimple, the H_i are nonabelian. Clearly if $i \neq j$, then $H_i \cap H_j$ is finite. So by Proposition 1.15, the H_i commute pairwise. Thus by Lemma 2.1, I is finite. Now $\hat{G} = \prod_{i \in I} H_i$ is a connected characteristic subgroup of G which is type-definable. By Corollary 2.3 to show $G \subseteq \operatorname{acl} \hat{G}$ we need only show $C_G(\hat{G})$ is finite. But if not, $C_G(\hat{G})$ must contain one of the H_i which is therefore abelian. But this contradicts the semisimplicity of G. □

It remains only to note that the following lemma shows each H_i is a definably simple group. The lemma extends a similar observation for ω-stable groups that is made in [16].

2.5. Lemma. *Suppose G is a $|T|^+$-saturated connected stable semisimple group and H is a minimal type-definable infinite normal subgroup of G. Then H is definably simple.*

Proof. Suppose for contradiction that H is not definably simple. By Lemma 2.1 there are a finite number $K_1, \ldots, K_n$ of minimal type-definable (in G) normal subgroups of H. Conjugation of H by an element of G induces a permutation of the K_i. Since the intersection of any two K_i is finite, by compactness there exists for each i a definable subgroup K_i^* of G containing K_i such that distinct K_i^* intersect in a finite set. Let Δ be a finite collection of formulas such that every conjugate of any of the K_i^* is definable by an instance of a formula in Δ. Now let L_i be a minimal Δ-definable subgroup containing K_i. An element $g \in G$ conjugates K_i to K_j if and only if it conjugates L_i to L_j. Thus the action of G by conjugation on H induces a definable action which permutes the finitely many L_i. Since G is connected, each L_i is normal in G. A fortiori, each K_i is normal in G. By the minimality of H, each $K_i = H$. That is, H is definably simple. □

Note that if G is superstable, the definably simple factors are actually simple. With the aid of the following lemma we can remove the saturation hypothesis in Theorem 2.4 if the group is superstable.

2.6. Lemma. *If H is a type-definable normal simple subgroup of the $|T|^+$-saturated group G, then H is definable.*

Proof. Let $\phi_m(x, y)$ assert that y is a product of at most m conjugates in G of x or x^{-1}. Since H is simple the type asserting that a and b are in H but for every m,

$\neg\phi_m(a, b)$ is inconsistent. But then, since $H \lhd G$, there is an m such that for any $a \in H$ the formula $\phi_m(a, b)$ defines H. □

We can now conclude

2.7. Corollary. *If G is a superstable semisimple group, then there is a definable subgroup $\hat{G}$ of G which is an almost direct product of simple groups such that $G \subseteq \mathrm{acl}(\hat{G})$.*

Proof. By Theorem 2.4 we find a $\hat{G}$ in a $|T|^+$-saturated elementary extension of the given G. By Lemma 2.6 $\hat{G}$ is definable. But as the product of the minimal type definable normal subgroups $\hat{G}$ is characteristic and thus 0-definable. The conclusion now transfers back to G. □

3. Monomial U rank and α-semiregularity

This section depends on some rudimentary but important consequences of the Lascar inequalities for U-rank. The first is well known. If $U(q) = \omega^\alpha k$ and $U(p) < \omega^\alpha$ then $p \perp q$. Note that in fact p is hereditarily orthogonal to $q (p \perp\!\!\perp q)$ in the sense that every extension of p is orthogonal to q. Secondly, if $U(q) = \omega^\alpha k$ then q can be decomposed as (i.e. is bidominant with) $\bigotimes p_i$ where each p_i is a regular type of rank ω^α. (This remark is contained in the proof of Proposition 5 of [13].) Finally, recall the transitivity of 'U-rank is less than ω^α': if $U(a; bB) < \omega^\alpha$ and $U(b; B) < \omega^\alpha$ then $U(a; B) < \omega^\alpha$.

It is easy to find examples of regular types which do not have U-rank ω^α or even monomial U-rank. For the first just take a single equivalence relation with infinitely many infinite classes. For the second take the theory of $\omega + 1$ refining equivalence relations with finite splitting. Note that in the second case, the type of rank $\omega + 1$ is nonorthogonal in T^{eq} to a type of rank 1 but is orthogonal to any type of rank ω.

Recall that a superstable group G is α-connected if G has no proper definable subgroups K with $U(G/K) < \omega^\alpha$. If G has monomial U-rank $\omega^\alpha k$, then G is connected if and only if G is α-connected [6, IV.4.6]. Corollary IV.2.8 of [6] asserts that any superstable simple group has monomial U-rank.

3.1. Notation. For any connected stable group G, we denote by q_G the generic type of G.

Borrowing from the practice in commutative ring theory of describing a power of a prime ideal as primary, we define the concept of a primary type.

3.2. Definition. A type q is *p-primary* if q is bidominant with a power of the regular type p. The type q is *primary* if it is p-primary for some regular p.

Note that if q is p-primary then for any regular type r, $r \not\perp q$ if and only if $r \not\perp p$. Thus, q is p-primary is really a property of the nonorthogonality class of p.

Now, we introduce the main notion of this section, an α-semiregular group. We will decompose a superstable group with monomial U-rank as an almost direct product of α-semiregular groups. This decomposition generalizes the kinds of decomposition we have described above (Theorem 1.21) since a group which is almost strongly minimal or simple (see Corollary 3.16) is α-semiregular.

3.3. Definition. (i) The type q is *α-semiregular* with respect to p where $U(p) = \omega^\alpha$ if for some k, $U(q) = \omega^\alpha k$ and $q \,\square\, p^k$.

(ii) The connected superstable group G is *α-semiregular* with respect to p if q_G is.

The demand of α-semiregularity exceeds p-primacy by demanding that the k with $q \,\square\, p^k$ is the coefficient of the U-rank. We see below that this imposes the ostensibly stronger requirement that q is p-simple.

We will apply the following technical remark on several occasions.

3.4. Proposition. *Suppose $U(a; M) = \omega^\alpha k$, $\bar{b} = \langle b_1, \ldots, b_l \rangle$ is an independent sequence over M, and $a \,\square_M\, \bar{b}$ where for each i, $U(b_i; M) = \omega^\alpha$. Then $U(a; \bar{b}M) < \omega^\alpha$ if and only if $l = k$.*

Proof. By the Lascar inequality we have

$$U(a; \bar{b}M) + U(\bar{b}; M) \leq U(\bar{b}; aM) \oplus U(a; M).$$

From this inequality and the hypotheses we have

$$U(a; \bar{b}M) + \omega^\alpha l \leq U(\bar{b}; aM) \oplus \omega^\alpha k.$$

Since each b_i depends on a over M (by bidomination), $U(\bar{b}; aM) < \omega^\alpha$. Examining the last inequality we see that $l \leq k$ (always). On the other hand, we have

$$U(\bar{b}; aM) + U(a; M) \leq U((a; \bar{b}M) \oplus U(\bar{b}; M).$$

Again since $U(\bar{b}; aM) < \omega^\alpha$, this implies

$$\omega^\alpha k \leq U(a; \bar{b}M) \oplus \omega^\alpha l.$$

So $U(a; \bar{b}M) < \omega^\alpha$ implies $k \leq l$ and so $k = l$. Conversely, if $k = l$ the last inequality implies $U(a; \bar{b}M) < \omega^\alpha$ and we finish. $\square$

α-semiregularity is a refinement of the notions of semiregularity and p-simplicity introduced by Shelah [19] and Hrushovski [9].

3.5. Definition. (i) A stationary type q is *hereditarily orthogonal* to a type p (written $q \perp p$) if every extension of q is orthogonal to p.

(ii) The stationary type q is *p-simple* (where p is regular) if there exist B, c, and I such that c realizes $q \mid B$, I is a set of independent realizations of $p \mid B$ and $\mathrm{stp}(c; B \cup I) \perp p$. The minimal cardinality of such an I is the p-weight of q, $\mathrm{wt}_p(q)$. We sometimes shorten $\mathrm{wt}_p(t(a; B))$ to $\mathrm{wt}_p(a/B)$.

(iii) The group G is *p-semiregular* (of weight n) if G is connected, q_G is p-simple, and $q_G \square p^n$.

3.6. Lemma. *If G is α-semiregular with respect to p, then G is p-semiregular.*

Proof. We must show $q = q_G$ is p-simple. Since $q \square p^k$, there is a very saturated model M, a realizing q^M and I realising the nonforking extension of p^k to M such that a and I are bidominant over M. It suffices to show that $t(a; M \cup I)$ is hereditarily orthogonal to p. Noting $U(q) = \omega^\alpha |I|$, this now follows immediately from Proposition 3.4 (using I for $\bar{b}$) and the observation which opened the section. □

When α is 0 we get a more familiar notion.

3.7. Theorem. *An ω-stable 0-semiregular group is almost strongly minimal.*

Proof. By assumption, q_G is bidominant with p^k for some type p (without loss of generality stationary) which has U-rank 1. We show in the next paragraph that p is nontrivial; by [8], p has Morley rank 1. As in the previous argument we can find a saturated model M, a realizing q^M, and I realising the nonforking extension of p^k to M such that a and I are bidominant over M. Again, by Proposition 3.4, $U(a; IM) < \omega^0 = 1$. Thus G is contained in the algebraic closure of a strongly minimal set.

Since G is a group q_G is nontrivial. Over some quite saturated model M choose a triple $\langle a_1, a_2, a_3 \rangle$ to witness the nontriviality and for each i a sequence $\bar{b}_i = \langle b_i^0, \ldots, b_i^k \rangle$ which is bidominant with a_i. Let l be minimal so that $b_3^0, \ldots, b_3^l \not\downarrow_M \bar{b}_1 {}^\frown \bar{b}_2$. Three elements from $\bar{b}_1 \cup \bar{b}_2 \cup \{b_3^0, \ldots, b_l^3\}$ form a triangle over the rest so p is nontrivial. □

3.8. Proposition. *Suppose $U(a; M) = \omega^\alpha k$ and X is a set of realizations of regular types over M, which have rank ω^α, such that $a \not\downarrow_M X$ but $a \downarrow_M Y$ for any proper subset Y of X. Then X is an independent set over M.*

Proof. For any $x \in X$, let $X_x = X - \{x\}$. Then $a \downarrow_M X_x$. If $U(x/X_x M) < \omega^\alpha$ then the initial observation of this section again yields $t(x; X_x M) \perp q$. But, $a \not\downarrow_{X_x M} x$ so for each $x \in X$, $U(x; X_x M) \geqslant \omega^\alpha$ and so since $U(p_i) = \omega^\alpha$ we conclude $x \downarrow_M X_x$. □

There are two quite distinct possible meanings for the assertion $q \perp H$ where H is a group. One might mean q is orthogonal to the generic type of H; we adopt a much stronger convention.

3.9. Definition. We write $q \perp H$ if for any sequence $\bar{c} \in H$, q is orthogonal to $t(\bar{c}; \emptyset)$.

In general a group K may be contained in the algebraic closure of a group H and yet the generic of K is orthogonal to the generic of H. (See e.g., Exercise 17 of [1].) However, q_K is certainly nonorthogonal to H in the sense defined here.

We are most interested in products of α-semiregular groups.

3.10. Notation. The subgroup H of a group G is *centerless α-completely reducible* (written α-ccr) if H is a finite almost direct product of subgroups H_i satisfying the following conditions. Each H_i is normal in G and $\bigwedge$-definable over acl($\emptyset$). For each i, the generic p_i of H_i is α-semiregular with respect to $\bar{p}_i$ for a regular type $\bar{p}_i$ with U-rank ω^α and the $\bar{p}_i$ are pairwise orthogonal.

Whenever we deal with an α-ccr group we will assume that the associated subgroups and types are denoted p_i, $\bar{p}_i$ and H_i.

3.11. Lemma. *Suppose $q \not\perp H$, H is α-ccr, and $U(q) = \omega^\alpha k$. If q is primary, then q is $\bar{p}_i$-primary for some i.*

Proof. Let M be a large saturated model and a realize q^M. Since $q \not\perp H$, for some finite $X \subseteq H(\mathfrak{M})$, $a \not\downarrow_M X$. Since every element of H is a product of generics and every realization of q_H is a product of realizations of the p_i, we may assume that each member of X realizes p_i^M for some i. Since each realization of p_i is bidominant with a sequence of realizations of $\bar{p}_i$, we may assume X is a set of realizations of the $\bar{p}_i^M$. If X has minimal cardinality, by Proposition 3.8, X is independent over M. But $q \not\perp t(x; MX_x)$ so $q \not\perp \bar{p}_i$ for some i. Since q is primary the i must be unique and q is domination equivalent to a power of $\bar{p}_i$. □

The following technical property of U-rank plays a crucial role in later arguments.

3.12. Lemma. *Let $q = \text{tp}(c; M)$ and suppose $U(q) \geq \omega^\alpha$. Suppose further that $U(c; IM) < \omega^\alpha$ and each $x \in I$ realizes a type p_x over M with U-rank ω^α.*

(i) *If I' is a basis for I over M, then $U(c; I'M) < \omega^\alpha$. Thus $t(c; M) \not\perp p_x$ for some $x \in I$.*

(ii) *If for some k, $U(q) = \omega^\alpha k$ and all $x \in I$ realize the same type $p \in S(M)$ where the U-rank of $p = \omega^\alpha$, then q is p-primary*

Proof. (i) Let I' be a basis for I. Then $U(I-I';I')<\omega^\alpha$ so by 'transitivity of U-rank less than ω^α', $U(c;I'M)<\omega^\alpha$. By the triviality of nonorthogonality $t(c;M)\not\perp p_x$ for some $x\in I'$ which is more than required.

(ii) We may choose a representative, r, of the nonorthogonality class of any regular type nonorthogonal to q, with $U(r)=\omega^\alpha$. Thus, if $d\models r^M$ and $d\not\downarrow_M c$, $U(d;cM)<\omega^\alpha$. By 'transitivity of U-rank less than ω^α', $U(d;IM)<\omega^\alpha$. By (i), $r\not\perp p$. We have shown every regular type nonorthogonal to q is nonorthogonal to p so q is p-primary. □

The class of α-semiregular type satisfies the following important closure condition.

3.13. Lemma. *Suppose that for $i<r$, $t(b_i;A)$ is α-semiregular with respect to p, $c\in\operatorname{acl}(b_0,\ldots,b_{r-1})$, and $U(c;A)=\omega^\alpha k$ (for some k). Then $t(c;A)$ is α-semiregular with respect to p.*

Proof. Without loss of generality, A is the universe of an extremely saturated model M. For $i<r$, let I_i be an independent sequence of realizations of p^M which is bidominant with b_i and let I denote the union of the I_i.

We note first that $U(c;IM)<\omega^\alpha$. Since each p_i is α-semiregular with respect to p, Proposition 3.4 implies $U(b_i;IM)<\omega^\alpha$ so $U(b_0,\ldots,b_{r-1};IM)<\omega^\alpha$. By 'transitivity of U-rank less than ω^α', $U(c;IM)<\omega^\alpha$. Lemma 3.12(ii) shows that $q=t(c;M)$ is p-primary.

Since q is p-primary, we can choose J so that $c\,\square_M J$ and $J\rhd_M cJ$ (i.e. J is a p-basis of $M[c]$). Extend J to J' a basis for JI over M. We have $U(c;JIM)<\omega^\alpha$. Applying Lemma 3.12(i) to J' and JI we see $U(c;J'M)<\omega^\alpha$. Now $(J'-J)\downarrow_M J$ so, by our special choice of J, $(J'-J)\downarrow_M cJ$ and thus $c\downarrow_{MJ}(J'-J)$. Hence $U(c;JM)<\omega^\alpha$. By 3.4, $|J|=k$ and we finish. □

Now we see that α-semiregular groups exist. We begin with a model-theoretic version and then translate to the group theoretic situation. We are just squeezing a little more information from the existence arguments of Hrushovski for p-simple types and p-semiregular groups.

We need one more property relating the U-rank of a type q with the U-rank of regular types nonorthogonal to q. If $U(q)=\omega^{\alpha_1}n_1+\cdots+\omega^{\alpha_k}n_k$ with $\alpha_1>\cdots>\alpha_k$, then there is a regular type r which is not orthogonal to q with $U(r)=\omega^{\alpha_k}$ [16, Lemma 7.5].

3.14. Lemma. *Let $q=\operatorname{tp}(a/A)$ where A is algebraically closed. If p is a regular type of least U-rank (necessarily of the form ω^α) which is nonorthogonal to q there is an $e\in\operatorname{dcl}(aA)$ such that $t(e;A)$ is α-semiregular with respect to p.*

Proof. Let p be based on Y, where without loss of generality $a \downarrow_A Y$. Choose c realizing p^Y such that $c \not\downarrow_Y a$. We will show that e, the canonical base of $\mathrm{stp}(cY; Aa)$, is α-semiregular with respect to p. By rudimentary properties of a canonical base $e \in \mathrm{acl}(aA)$. A slight modification of e will find the appropriate member of $\mathrm{dcl}(aA)$. Let $Y_1c_1, \ldots, Y_nc_n \ldots$ be an independent sequence based on $\mathrm{stp}(cY; aA)$. Now $a \downarrow_A Y$ implies $a \downarrow_A (Y_1, \ldots, Y_n \ldots)$ and thus since $e \in \mathrm{acl}(Aa)$ we conclude $e \downarrow_A (Y_1, \ldots, Y_n \ldots)$. A second basic property of canonical types yields that $e \in \mathrm{dcl}(Y_1c_1, \ldots, Y_nc_n, \ldots)$. So, for some n, $e \in \mathrm{dcl}(A \cup Y_1c_1, \ldots, Y_nc_n)$. Since $e \downarrow_A (Y_1, \ldots, Y_n)$, it suffices to show $t(e; A \cup \{Y_1, \ldots, Y_n\})$ is p-semiregular.

Let $U(a; A) = \omega^{\alpha_1}n_1 \cdots + \omega^{\alpha_k}n_k$ with $\alpha_1 > \cdots > \alpha_k$. By the minimality of α and the remark before the theorem $\alpha_k = \alpha$. On the other hand since $e \in \mathrm{dcl}(Y_1c_1, \ldots, Y_nC_n)$ and $e \downarrow_A (Y_1, \ldots, Y_n)$,

- $U(e; A \cup \{Y_1, \ldots, Y_n\}) \leq \omega^\alpha n$, and
- every regular type not orthogonal to $\mathrm{tp}(e; A \cup \{Y_1, \ldots, Y_n\})$ is not orthogonal to $\mathrm{tp}(Y_1c_1, \ldots, Y_nc_n; A)$ and so to $t(c; Y)$.

From the remark preceding this lemma we see

$$U(e; A) = U(e; A \cup \{Y_1, \ldots, Y_n\}) = \omega^\alpha m$$

for some $m \leq n$. Now by Lemma 3.12, $\mathrm{tp}(e; A)$ is p-primary and then by Lemma 3.13, $\mathrm{tp}(e; A)$ is α-semiregular with respect to p.

To get $e \in \mathrm{dcl}(aA)$ replace it by an element e' of M^{eq} that is equidefinable with the finite set of its conjugates over Aa. Each conjugate is α-semiregular and is in $\mathrm{acl}(Aa)$ so $e' \in \mathrm{acl}(Aa)$. Applying again the remark before this lemma $U(e'; A) = \omega^\alpha m$ for some m. Now Lemmas 3.12 and 3.13 yield that $t(e'; A)$ is α-semiregular and we finish. □

3.15. Theorem. *Suppose G is a connected superstable group which is normal in a group G^* and p is a regular type of at least U-rank, $\omega^\alpha > 0$, which is nonorthogonal to the generic q of G. Then there is a definable $K \lhd G$ which is normal in G^* such that G/K is infinite and α-semiregular with respect to p.*

Proof. Since q is over the empty set, Lemma 3.14 implies there is an $\mathrm{acl}(\emptyset)$-definable function f such that if a realizes q, then $r = \mathrm{tp}(f(a); \emptyset)$ is α-semiregular with respect to p. For each $b \in G$ let f_b be the germ of a map from $G^* \times G$ into r defined generically as follows. For $\langle \sigma_1, \sigma_2 \rangle$ a generic of $G^* \times G$ over b, let

$$f_b(\sigma_1, \sigma_2) = f(\sigma_1\sigma_2 b\sigma_1^{-1}).$$

Note that as an element of $\mathscr{C}^{\mathrm{eq}}$, $f_b \in \mathrm{dcl}(b)$. Let

$$K = \{b \in G : f_b(\sigma_1, \sigma_2) = f_e(\sigma_1, \sigma_2)\}$$

where $\langle \sigma_1, \sigma_2 \rangle$ is generic over b and e is the identity of G. Now, if a is a generic of G, we show f_a is interdefinable with the coset a/K. For this we need to show that for any $a_1, a_2 \in G$, $a_1 a_2^{-1} \in K$ if and only if $f_{a_1} = f_{a_2}$. Suppose first that $a_1 a_2^{-1} \in K$. Let $\langle \sigma_1, \sigma_2 \rangle$ be generic for $G^* \times G$ over $\{a_1, a_2\}$. Then $\langle \sigma_1 a_2^{-1}, a_2 \sigma_2 \rangle$ is generic for $G^* \times G$ over $\{a_1 a_2^{-1}\}$. But

$$f_{a_1}(\sigma_1, \sigma_2) = f(\sigma_1 \sigma_2 a_1 \sigma_1^{-1}) = f((\sigma_1 a_2^{-1}) a_2 \sigma_2)(a_1 a_2^{-1}) a_2 \sigma_1^{-1})).$$

So by the definition of $f_{a_1 a_2^{-1}}$ and since $a_1 a_2^{-1} \in K$,

$$f((\sigma_1 a_2^{-1})(a_2 \sigma_2)(a_1 a_2^{-1})(a_2 \sigma_1^{-1})) = f((\sigma_1 a_2^{-1})(a_2 \sigma_2) e (a_2 \sigma_1^{-1})) = f_{a_2}(\sigma_1, \sigma_2).$$

Reversing the steps of this argument shows that $f_{a_1} = f_{a_2}$ implies a_1 and a_2 are congruent mod K. A similar computation with generics shows that K is normal in G^*.

We now need Fact 2.3 of [10]: If f is the germ of a map with domain q and range r, then there is a B which is independent from f and a set I of realizations of $r \mid B$ such that f is definable over $B \cup I$. This result implies that for some B independent from f_a, f_a is definable over $B \cup I$ where I is a set of realizations of $r \mid B$. Since $U(r) = \omega^\alpha$, we conclude $U(f_a; B) = \omega^\alpha k + \gamma$ for some k and some $\gamma < \omega^\alpha$. If $\gamma \neq 0$, the remark before Lemma 3.14 implies there is an $r' \not\perp t(f_a; B)$ with $U(r') < \omega^\alpha$. Since f_a is definable from a, $r' \not\perp q$ which contradicts the choice of α. Thus $U(t(f_a; B)) = \omega^\alpha k$. By Lemma 3.13, $t(f_a; B)$ is α-semiregular. As f_a is interdefinable with a generic of G/K we finish. □

Now we can slightly improve Hrushovski's observation that a simple superstable group is semiregular. (He actually states a version for stable groups.)

3.16. Corollary. *If G is a simple superstable group, then G is α-semiregular for some α.*

The next lemma allows us to assume that any product of α-semiregular groups satisfies the orthogonality conditions in the definition of α-crr.

3.17. Lemma. *If H_1 and H_2 are connected subgroups of a superstable group G which are α-semiregular with respect to a type p and $H_1 \cap H_2$ is finite, then $H_1 H_2$ is α-semiregular.*

Proof. Let a_1 and a_2 be independent realizations of the generic types of H_1 and H_2 respectively. It is easy to see that $a_1 {}^\frown a_2$ is α-semiregular with respect to p. Note that since $H_1 \cap H_2$ is finite and H_1 commutes with H_2, $H_1 H_2$ is isomorphic to $(H_1 \times H_2)/H_1 \cap H_2$. Thus the pair $a_1 {}^\frown a_2$ is interalgebraic with the generic $a_1 a_2$ of $H_1 H_2$ and we finish. □

Since H_1 and H_2 are connected we could (with some further argument) replace the hypothesis $H_1 \cap H_2$ is finite by the hypothesis $U(H_1 \cap H_2) < \omega^\alpha$.

3.18. Lemma. *Suppose G is a connected superstable group such that $U(G) = \omega^\alpha k$, $L \lhd G$, and $U(L) < \omega^\alpha$. Then G is α-semiregular if and only if G/L is α-semiregular.*

Proof. From the rank hypotheses and Corollary 8.2 of [6] it is easy to see that $U(G) = U(G/L) = \omega^\alpha k$. Let g realize the generic type of G and $\bar{g}$, the canonical image of g, realize the generic type of G/L. Fix a large saturated model M and choose I and J to be independent realizations of regular types over M which are bidominant with g and $\bar{g}$ respectively. As noted at the beginning of the section we can assume that each member of I or J realises a type of rank ω^α. Note that $U(L) < \omega^\alpha$ implies $U(g; M\bar{g}) < \omega^\alpha$ while $U(\bar{g}; gM) = 0$. Moreover, each $x \in I$ (J) has rank $< \omega^\alpha$ over g ($\bar{g}$). By the 'transitivity of U-rank less than ω^α, we conclude that each $x \in J$ (I) depends on g ($\bar{g}$). Since $g \square I$ and $\bar{g} \square J$, we conclude that each $x \in I$ (J) depends on J (I). Thus $|J| = \mathrm{wt}\, J \geqslant \mathrm{wt}\, I = |I|$ and $|I| = \mathrm{wt}\, I \geqslant \mathrm{wt}\, J = |J|$. So $|I| = |J|$. Suppose G is α-semiregular. Then $|I| = k$ and consequently $|J| = k$. Moreover, since I can then be taken to realize some p^k and each $x \in J$ depends on I, J can be taken as an independent sequence realizing some regular type nonorthogonal to p. So G/L is also α-semiregular. Reversing the roles of G and G/L we finish. □

3.19. Corollary. *Let $G \leqslant G^*$ be a semisimple connected superstable group with monomial U-rank $\omega^\alpha k$. If G is not α-semiregular, then G has a proper definable subgroup K of rank $\geqslant \omega^\alpha$. Moreover, K is* acl($\emptyset$)*-definable and normal in G^*.*

Proof. By Theorem 3.15, there is a definable, normal in G^*, and acl($\emptyset$)-definable subgroup L of G such that G/L is α-semiregular and infinite. By Lemma 3.18, if $U(L) < \omega^\alpha$, then G is α-semiregular. □

We conclude this section by characterizing semisimple groups of monomial U-rank with 'small' coefficient.

3.20. Theorem. *Let G be a connected semisimple group with $U(G) = \omega^\alpha k$. Then $k \geqslant 3$. If $k = 3$, G is α-semiregular.*

Proof. Superstable groups with U rank $\omega^\alpha k$ and k at most 2 are solvable [4]. When $k = 3$, we finish by Corollary 3.19 unless G has proper definable normal subgroup K with $U(K) \geqslant \omega^\alpha$. Without loss of generality (by Corollary 1.17) we may assume G is $|T|^+$-saturated. But the α-connected component of K cannot have rank strictly less than $\omega^\alpha 3$ by the first part of this proof. But $U(K^0) = \omega^\alpha 3$ is impossible since K is proper and G is connected. □

4. Model-theoretic decomposition

We need the following proposition.

4.1. Lemma. *Suppose G is superstable and α-connected. If H is a normal definable subgroup of G with rank less than ω^α, then $H \subseteq Z(G)$.*

Proof. By Lemma 2.2, $G/C_G(H) \subseteq \mathrm{dcl}(H)$ and thus has rank less than ω^α. So $C_G(H) = G$ (by α-connectedness) as required.

Here is the main result of this section.

4.2. Theorem. *Let G be a semisimple connected superstable group with monomial U-rank $\omega^\alpha k$. Then G is a finite almost direct product of definable normal semiregular groups.*

Proof. We first assume G is $|L|^+$-saturated and then eliminate that assumption. We must work through a rather arcane induction hypothesis. That is, we work with a subgroup K of G and show that K can be written as a product of groups which are *normal in G*. In the end we take $K = G$ to conclude the result. But we need to consider subgroups K of G since finding subgroups normal in K would not suffice.

Now we will prove by induction on n:

(*) Let G be a semisimple connected superstable group. Let $K \lhd G$ be $\bigwedge$-definable over $\mathrm{acl}(\emptyset)$ and have monomial U-rank $\omega^\alpha n$. Then K is α-ccr.

By Theorem 3.20 we know the result holds trivially for $n \leqslant 3$. Suppose (*) holds for $k < n$.

We deduce from Corollary 3.19 that there is a $\bigwedge$-definable over $\mathrm{acl}(\emptyset)$ proper subgroup H of K which is normal in G with $U(H) \geqslant \omega^\alpha$. By taking the α-connected component we can choose H so that $U(H) = \omega^\alpha k$ and k is maximal for any such H that is a proper subgroup of K. Since the α-connected component of a group is invariant in that group, this requirement leaves H $\bigwedge$-definable over $\mathrm{acl}(\emptyset)$. By induction H is α-ccr. By Lemma 3.17 we may assume the factors H_i of H are pairwise orthogonal. We first show $C_K(H)$ is finite.

Suppose for contradiction that $U(C_K(H)) \geqslant \omega^\alpha$ and let L denote the α-connected component of $C_K(H)$. Then $L \cap H$ is normal and abelian so, G being semisimple, $L \cap H$ is finite. Therefore $U(LH) > U(H)$ so by the maximality of k, $LH = K$. Now $U(L) < U(K)$ so by induction L is α-ccr and so LH, i.e. K, is α-ccr and we finish. Thus we may assume $U(C_K(H)) < \omega^\alpha$. By Lemma 4.1, $C_K(H) \leqslant Z(G)$ and so $C_K(H)$ is finite.

If H is α-semiregular, we finish by Lemmas 3.13 and 2.3. Thus we can assume that H is a product of at least two α-semiregular factors.

In order to be sure 'quotients are safe' we replace the $\bigwedge$-definable subgroup H by a definable H^*. More precisely, choose $\hat{H}\lhd G$ with $H\subseteq H^*$ and $U(H^*)=U(H)=\omega^\alpha k$. Let $H^*=K\cap\hat{H}$. Then $K/H^*\lhd G/H^*$; $K/H^*\approx K\hat{H}/\hat{H}$ which is type-definable; and so K/H^* is $\bigwedge$-definable. (K/H might fail the last condition.)

We claim K/H^* is α-semiregular. Applying Theorem 3.15 to K/H^* there is a $K_1\lhd K/H^*$ with $(K/H^*)/K_1$ α-semiregular. Then there is an $\bar{H}$ with $H^*\leqslant\bar{H}\lhd K$ such that $\bar{H}/H^*\approx K_1$. If $U(\bar{H})\geqslant\omega^\alpha(k+1)$, the connected component of $\bar{H}$ contradicts the maximality of k. Thus $U(\bar{H})=\omega^\alpha k+\gamma$ for some $\gamma<\omega^\alpha$ and $U(\bar{H}/H^*)<\omega^\alpha$. But $K/\bar{H}\approx(K/H^*)/(\bar{H}/H^*)$. By Lemma 3.18, since $K/\bar{H}$ is α-semiregular so is K/H^*.

Now let q be the generic of K/H^*. Then q is primary. By Corollary 2.3, $K/H^*\subseteq\operatorname{acl} H$ and so $q\not\perp H$. Thus, by Lemma 3.11, q is p_i-primary for some i, say 1. Now let $\bar{H}=\prod_{i>1}H_i$. If $q\not\perp\bar{H}$, applying Lemma 3.11 again (this time with $\bar{H}$ playing the role of H) we find q is p_i-primary for some $i>2$. Since the p_i are pairwise orthogonal this is clearly impossible so $q\perp\bar{H}$.

Now consider the normal subgroup $C=C_K(\bar{H})$ of G. As usual, Corollary 2.3 implies that $K/C\subseteq\operatorname{dcl}(\bar{H})$. Since the H_i commute in pairs, $H_1\subseteq C$; the semisimplicity of G implies that $C\cap\bar{H}$ is finite. Let N be the α-connected component of C and let $U(N)=\omega^\alpha k_0$. Observe that N is a proper subgroup of K which is normal in G and contains H_1. By induction each factor of $\bar{H}$ is acl($\emptyset$)-definable. Thus both C and N are acl($\emptyset$)-definable.

The crucial remark is that N properly contains H_1. If not, we can replace the $\bigwedge$-definable $N=H_1$ by a definable N^* with $N\leqslant N^*\leqslant C$, $N^*\lhd G$, $N^*\leqslant H^*$, and $U(N^*)=U(H_1)\,(=\omega^\alpha k_1$, say). Now we will see that for every $a\in K/H^*$, $U(a,\bar{H})<\omega^\alpha$. There is a natural map from K/N^* onto K/C with kernel C/N. Since $U(C/N^*)<\omega^\alpha$, $U(a;\bar{a})<\omega^\alpha$ if a realizes the generic of K/N^* and $\bar{a}$ is its image under this map. But $\bar{a}\in\operatorname{dcl}(\bar{H})$ and so by transitivity of U rank less than ω^α, $U(a;\bar{H})<\omega^\alpha$. But this contradicts the observation above that the generic of K/H^* is orthogonal to every sequence from $\bar{H}$. We conclude that N properly contains H_1.

So we have $U(N)=\omega^\alpha k_0>\omega^\alpha k_1=U(H_1)$. Moreover, $N\cap\bar{H}$ is finite and N commutes with $\bar{H}$. So $N\bar{H}$ is $\bigwedge$-definable connected and has U-rank greater than H. By the maximality of H, $N\bar{H}=K$. N is a normal subgroup of K with rank $\omega^\alpha k_1<\omega^\alpha k$. So by induction N, as well as $\bar{N}$, is α-ccr so their product K is α-ccr as required.

In order to remove the saturation hypothesis on G we want to show that the H_i are in fact definable almost over the empty set. We know that each $H_i=\bigcap_{k<|T|}H_i^k$ and if $i\neq j$, $H_i\cap H_j$ is finite. By compactness we conclude that for each i and j there exist $H_i^{k_j}$ and $H_j^{k_i}$ containing H_i and H_j respectively that are definable over $\operatorname{acl}\emptyset$ and such that $H_i^{k_j}\cap H_j^{k_i}$ is finite. Taking the intersection of the $H_j^{k_i}$ for all $i\neq j$ we obtain a group H_j^* which is definable (over $\operatorname{acl}\emptyset$) and such that G is an almost direct product of the H_j^*. In fact $H_j^*=H_j$. $\square$

We saw in Section 1 that a semisimple group is one in which every normal abelian subgroup is 'small'. The arguments of this section suggest a different approach to small—replacing 'finite' by 'U-rank less than ω^{α}'.

4.3. Definition. If G has no normal definable abelian subgroup with U rank greater than or equal ω^{α}, then G is *α-semisimple.*

This idea leads to the following theorem which can be proved along the same line as Theorem 4.2.

4.4. Theorem. *Let G be a connected superstable group with monomial U-rank $\omega^{\alpha}k$. If G is α-semisimple, then G is a finite product of definable normal α-semiregular groups H_i such that if $i \neq j$, $U(H_i \cap H_j) < \omega^{\alpha}$.*

We combine this fact with an important result of Hrushovski to get a model-theoretic condition for α-semisimplicity.

4.5. Corollary. *Let G be a superstable group with monomial U-rank $\omega^{\alpha}k$. If a locally modular regular type p is nonorthogonal to the generic of G, then G is not α-semisimple.*

Proof. We may assume G is connected. If G is α-semisimple, Theorem 4.4 allows us to write G as $\prod H_i$ where the H_i are α-semiregular and have 'small' intersection. But then p must be nonorthogonal to $q = q_{H_i}$ for some i. As q is p semiregular and p is locally modular, by Corollary 5.4 of [10], H_i is abelian. Since each H_i has U-rank greater than or equal ω^{α} we have a contradiction. □

Note that if G has finite U-rank 0-semisimple coincides with the notions of semisimple studied in Section 1. Now we use Buechler's dichotomy between local modularity and ω-stability.

4.6. Theorem. *If G is a superstable semisimple group of finite U-rank, then G is ω-stable.*

Proof. By Theorem 4.2, we can write H as $\prod H_i$ where each H_i is p_i-semiregular for some regular type p_i, which since G has finite U-rank, has U-rank 1. By [8], p_i has ∞-rank 1. So by [7], each p_i either has Morley rank 1 or is locally modular. If any p_i is locally modular, we contradict the semisimplicity of G by Corollary 4.5. But if each p_i has Morley rank 1, it is easy to see that G is ω-stable as required. □

Combining this analysis with Section 1 we obtain

4.7. Theorem. *Let G be a superstable group of finite U-rank. Then there is a definable normal subgroup H of G such that H is solvable and G/H is ω-stable.*

Proof. Corollary 1.8 asserts that G is solvable by semisimple; by Corollary 1.14 the solvable subgroup is definable; and Theorem 4.6 asserts that the semisimple part is ω-stable. □

5. Problems

In this section we discuss some problems which arise from or are exacerbated by the results above.

Question 1. Is a stable definably simply group simple? Is simplicity preserved by elementary equivalence between stable groups?

Question 2. Can one characterize those properties which hold for stable groups if and only if they hold for type-definable subgroups of $|T|^+$-saturated stable groups?

This is relevant to a most vexing problem.

Question 3. Can the saturation hypothesis be eliminated from Theorem 2.4?

A *pure group* is a structure $\langle G, \cdot \rangle$ with no further basic relations. The next question is the semisimple version of similar problem raised for simple ω-stable groups by Lascar [12].

Question 4. Is a connected semisimple stable pure group an almost direct product of simple groups? An attempt to answer this question affirmatively on the model of algebraic groups leads to the following question. Let G be semisimple $|T|^+$-saturated and stable (or even ω-stable of finite rank). Does the group of inner automorphisms of G have finite index in the group of definable automorphisms of G?

Question 5. Any group has a unique maximal centerless completely reducible subgroup $\hat{G}$. Can one show that if G is semisimple and stable then $\hat{G} \neq 1$?

For superstable groups a positive result holds by Corollary 2.7 and the same kind of argument would extend the result to stable groups if Question 1 were answered positively. But a positive answer to the following question would provide a more direct approach.

Question 6. Can a stable group be both residually finite and semisimple?

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