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Physics Letters B 643 (2006) 354–361

PHYSICS LETTERS B

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Long-range $SL(2)$ Baxter equation in $\mathcal{N} = 4$ super-Yang–Mills theory

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Received 19 September 2006; received in revised form 9 November 2006; accepted 9 November 2006

Available online 15 November 2006

Editor: B. Grinstein

Abstract

Relying on a few lowest order perturbative calculations of anomalous dimensions of gauge invariant operators built from holomorphic scalar fields and an arbitrary number of covariant derivatives in maximally supersymmetric gauge theory, we propose an all-loop generalization of the Baxter equation which determines their spectrum. The equation does not take into account wrapping effects and is thus asymptotic in character. We develop an asymptotic expansion of the deformed Baxter equation for large values of the conformal spin and derive an integral equation for the cusp anomalous dimension.

1. Introduction

Four-dimensional non-Abelian gauge theories were found to possess integrable structures. The latter imply the existence of hidden symmetries of the dilatation operator whose eigenvalues determine anomalous dimensions of gauge invariant composite operators of elementary fields in underlying models. Integrability was revealed in one-loop anomalous dimensions of twist-L maximal-helicity Wilson operators in QCD by identifying the former with eigenenergies of the L-site XXX Heisenberg spin chain [1]. The magnet turns out to be noncompact, for the spin operators acting on its sites transform in the infinite-dimensional representation of the collinear subgroup $SL(2, \mathbb{R})$ of the conformal group $SO(4, 2)$. Since the one-loop phenomenon is spawned by gluons, invariably present in Yang–Mills theories—supersymmetric or not—they all necessarily exhibit the same, universal integrable structures. The differences arise merely due to distinct particle contents of the models: while only holomorphic sectors are integrable in QCD and its nearest supersymmetric $\mathcal{N} = 1, 2$ siblings [2], the maximal supersymmetry of the $\mathcal{N} = 4$ super-Yang–Mills theory extends integrability to all operators [3,5,2]. Recent perturbative studies build up a growing amount of direct evidence that integrability persists in certain closed compact [4,5] and noncompact [6–10] subsectors of gauge theories even in higher orders of perturbation theory. Thus, while ruled out for gauge theories with $\mathcal{N} < 4$ supercharges, it is plausible that the maximally supersymmetric Yang–Mills theory is completely integrable. An additional confirmation for this conjecture comes from studies of multi-loop multi-leg scattering amplitudes which display intriguing iterative structures [11,12]. These arguments suggest that the spectrum of all-loop anomalous dimensions in $\mathcal{N} = 4$ SYM theory is determined by a putative long-range integrable spin chain with the dilatation operator being its Hamiltonian.

In this note we probe the underlying integrable long-range magnet by proposing its multi-loop perturbative structure within the framework of the Baxter $\mathbb{Q}$-operator [13]. This approach is based on the existence of an operator $\mathbb{Q}(u)$ depending on a spectral parameter u and acting on the Hilbert space of the magnet. For different values of u it forms a family of mutually commuting operators, simultaneously commuting with the spin-chain Hamiltonian as well. Although in the present circumstances, the formalism is equivalent to the Bethe Ansatz approach, it possesses certain advantages. First, the eigenvalue $Q(u)$ of the Baxter operator $\mathbb{Q}(u)$ determines the single-particle wave function of the chain in the representation of separated variables [14]. Second, the equation for the $\mathbb{Q}$-operator—known as the Baxter equation—is polynomial, to be contrasted with a set of coupled transcendental Bethe equa-

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0370-2693
doi:10.1016/j.physletb.2006.11.008

tions. Third, it allows for a straightforward asymptotic analysis when quantum numbers of the chain are large as will be demonstrated below.

Currently we restrict our consideration to the closed [15,16] noncompact $SL(2)$ sector [1,15] of the gauge theory which is spanned by single-trace maximal R-charge Wilson operators built from the holomorphic scalar fields $X = \phi_1 + i\phi_2$ and covariant derivatives,

$$\mathcal{O}_{n_1 n_2 \dots n_L}(0) = \operatorname{tr}\{(iD_+)^{n_1} X(0)(iD_+)^{n_2} X(0) \cdots (iD_+)^{n_L} X(0)\}. \quad (1.1)$$

Here $D_+ = D_\mu n^\mu$ is projected on the light cone with a null vector n^μ, $n^2 = 0$, in order to factor out the maximal Lorentz-spin component from the operator in question. These Wilson operators mix with each other under renormalization group evolution and acquire anomalous dimensions at all orders of perturbative series in coupling constant[1]

$$\gamma(g) = \sum_{n=1}^{\infty} g^{2n} \gamma^{(n)}. \quad (1.2)$$

We find it convenient to use the expansion parameter g related to the 't Hooft coupling constant λ via

$$g = \sqrt{2\lambda} = \frac{g_{\mathrm{YM}}\sqrt{N_c}}{2\pi}. \quad (1.3)$$

The anomalous dimension $\gamma(g)$ depends on parameters characterizing the operator: its twist L, determined by the number of X-fields, and its Lorentz spin $N = n_1 + n_2 + \cdots + n_L$. Within the method of the Baxter $\mathbb{Q}$-operator, the eigenspectrum of one-loop anomalous dimensions $\gamma^{(0)}$ and the corresponding quasimomentum $\theta^{(0)}$ are determined by the leading order Baxter function $Q^{(0)}(u)$

$$\gamma^{(0)} = \frac{i}{2}\left[\ln Q^{(0)}\left(\frac{i}{2}\right)\right]' - \frac{i}{2}\left[\ln Q^{(0)}\left(-\frac{i}{2}\right)\right]',$$
$$\theta^{(0)} = \ln Q^{(0)}\left(\frac{i}{2}\right) - \ln Q^{(0)}\left(-\frac{i}{2}\right). \quad (1.4)$$

Since the Baxter function $Q^{(0)}(u)$ is related to the eigenfunction of the mixing matrix, it corresponds to a multiplicatively renormalizable Wilson operator and thus has to be polynomial in u of order N, $Q^{(0)}(u) = (u - u_1^{(0)})(u - u_2^{(0)}) \cdots (u - u_N^{(0)})$. The zeros of this polynomial are determined by the Bethe roots $u_n^{(0)}$ which take only real values for the noncompact $SL(2,\mathbb{R})$ spin chain [17]. The function $Q^{(0)}(u)$ obeys the finite-difference Baxter equation [13]

$$u_+^L Q^{(0)}(u+i) + u_-^L Q^{(0)}(u-i) = t^{(0)}(u) Q^{(0)}(u), \quad (1.5)$$

where the spectral parameter in the dressing factors $u_\pm^L$ is shifted by the conformal spin $s = \frac{1}{2}$ of the scalar field X, $u_\pm = u \pm \frac{i}{2}$ and $t^{(0)}(u)$ is an order-L polynomial in u depending on the integrals of motion.

[1] Their complete two-loop planar mixing matrix has been recently computed in Ref. [10].

2. Three-loop Baxter equation

Explicit perturbative calculations [6,7,9] of two-loop corrections to the anomalous dimensions of the scalar operators (1.1) exhibit double degeneracy of energy levels with zero quasimomentum. This hints at the existence of nontrivial odd-parity conserved charges and thus persistence of integrability at higher orders of perturbation theory.

Beyond one loop, the formalism of the Baxter operator gets modified accordingly. The Bethe roots acquire corrections in coupling constant to all orders of perturbation theory,

$$u_n(g) = \sum_{k=0}^{\infty} g^{2k} u_n^{(k)}, \quad (2.1)$$

and obey deformed Bethe Ansatz equations [18]. The reality of Bethe roots $u_k(g)$ have to be preserved to all orders since the eigenvalue $Q(u)$ of $\mathbb{Q}(u)$ is a wave function of the chain with the number of its nodes on the real u-axis coinciding with the spin N of the operator. The polynomial

$$Q(u) = \prod_{n=1}^{N} \big(u - u_n(g)\big), \quad (2.2)$$

fulfills these properties and is real $Q^*(u) = Q(u^*)$ for $u^* = u$. In Ref. [10] we found from available two- [19,20,6,7] and three-loop [21–23] diagrammatic calculations of anomalous dimensions that the Baxter equation possesses the form

$$x_+^L e^{\sigma_+(x_+)} Q(u+i) + x_-^L e^{\sigma_-(x_-)} Q(u-i) = t(u) Q(u), \quad (2.3)$$

with the dressing factors depending on the renormalized spectral parameter [24]

$$x[u] = \frac{1}{2}\left(u + \sqrt{u^2 - g^2}\right), \qquad x_\pm = x[u_\pm]. \quad (2.4)$$

The multi-loop transfer matrix[2]

$$t(u) = 2u^L + q_1(g) u^{L-1} + q_2(g) u^{L-2} + \cdots + q_L(g) \quad (2.5)$$

acquires the "missing" term $\sim u^{L-1}$ at g^2-order, i.e., $q_1(g) \sim \mathcal{O}(g^2)$, while the rest of the charges start from $\mathcal{O}(g^0)$, $q_k(g) = q_k^{(0)} + \mathcal{O}(g^2)$. The additional dressing factors $\sigma_\pm$ obey the complex conjugation condition $(\sigma_-(x_-))^* = \sigma_+(x_+)$ for $\Im\mathrm{m}\, u = 0$ and encode the renormalization of the noncompact charges $q_k(g)$ at higher orders. An analysis yielded the following result to three-loop order [10]

$$\sigma_\pm(x) = -\frac{g^2}{2x}\left[\ln Q\left(\pm\frac{i}{2}\right)\right]' - \frac{g^4}{16x^2}\left\{\left[\ln Q\left(\pm\frac{i}{2}\right)\right]'' + x\left[\ln Q\left(\pm\frac{i}{2}\right)\right]'''\right\} + \mathcal{O}(g^6). \quad (2.6)$$

[2] Note that with this transfer matrix the resulting Baxter equation breaks down already at order $\mathcal{O}(g^{2n})$ in coupling constant with $n = L$. It turns out that one can correctly incorporate order $n = L$ corrections by replacing the leading term in $t(u)$ with the following combination $2u^L \to x_+^L + x_-^L - (\frac{i}{2})^L - (-\frac{i}{2})^L$. This also allows to set $q_1(\lambda) = 0$.

While the anomalous dimension is expressed order-by-order in coupling constant g in terms of the solution to Eq. (2.3) as [10]

$$\gamma(g) = i\left\{\frac{g^2}{2}\left[\ln Q(u)\right]' + \frac{g^4}{16}\left[\ln Q(u)\right]''' + \frac{g^6}{384}\left[\ln Q(u)\right]^{(5)} + \mathcal{O}\left(g^8\right)\right\}_{u=-i/2}^{u=i/2}. \tag{2.7}$$

The anomalous dimensions found using these equations reproduced exactly available perturbative predictions. One can demonstrate that the condition of the pole-free transfer matrix at Bethe roots $u_n(g)$, $t(u_n) = 0$ immediately produces the three-loop Bethe Ansatz of Ref. [25].

3. Multi-loop conjecture

The above representation (2.6) of the dressing factors $\sigma_\pm$ can be brought to a very suggestive form. Namely, a quick inspection allows one to rewrite these terms as an expansion in terms of the Chebyshev polynomials of the second kind U_k,

$$\sigma_\pm(x) = \frac{2g}{\pi}\int_{-1}^{1} dt\,\sqrt{1-t^2}\left[\ln Q\left(\pm\frac{i}{2} - gt\right)\right]' \times \sum_{n=0}^{n_{\max}}\left(-\frac{g}{2x}\right)^{n+1}\frac{U_n(t)}{n+1}, \tag{3.1}$$

with $n_{\max} = 2$, valid to $\mathcal{O}(g^3)$ in the approximation of Eq. (2.6). Having this representation at our disposal, we may naturally extend the first few terms of the available perturbative series to all orders in coupling g, by sending $n_{\max} \to \infty$. Using the summation theorem for Chebyshev polynomials, one can sum the infinite series up into the function $-(\operatorname{arccot}\frac{t+2x/g}{\sqrt{1-t^2}})/\sqrt{1-t^2}$ and, upon a variable transformation, write $\sigma_\pm$ in the form (with $\bar{z} = 1 - z$)

$$\begin{aligned}\sigma_\pm(x) &= -\frac{g^2}{2\pi x}\int_0^1 dz\int_{-1}^{1}\frac{dt}{\sqrt{1-t^2}} \times\left[\ln Q\left(\pm\frac{i}{2} - g\sqrt{z}t + \bar{z}\frac{g^2}{4x}\right)\right]' \\ &= i\theta_\pm - \int_{-1}^{1}\frac{dt}{\pi}\frac{\ln Q(\pm\frac{i}{2} - gt)}{\sqrt{1-t^2}}\frac{\sqrt{u^2-g^2}}{u+gt}.\end{aligned} \tag{3.2}$$

Here we integrated by parts in the second line in order to separate the components $\theta_\pm$ of the spin-chain quasimomentum $\theta = \theta_+ - \theta_-$,

$$i\theta_\pm = \int_{-1}^{1}\frac{dt}{\pi}\frac{\ln Q(\pm\frac{i}{2} - gt)}{\sqrt{1-t^2}}. \tag{3.3}$$

Notice that θ reduces to the one-loop expression (1.4) upon setting $g = 0$. While the condition $t(u_n) = 0$ yields the all-order Bethe Ansatz equations suggested in Ref. [18].

The conjectured multi-loop Baxter equation (2.3) with (3.2) and the known pattern of renormalization of the conformal spin in field theories can be used to determine the all-loop analytic expression for the anomalous dimensions in terms of the Baxter function. To this end, recall that the conformal spin of Wilson operators $J^{(0)} = N + \frac{1}{2}L$ defining the quadratic Casimir $q_2^{(0)} = -J^{(0)}(J^{(0)} - 1) - \frac{1}{4}L$ gets additive renormalization by the anomalous dimensions $\gamma(g)$ of composite Wilson operators at higher orders in coupling, $J^{(0)} \to J = N + \frac{1}{2}L + \frac{1}{2}\gamma(g)$. This conclusion arises from considerations of conformal Ward identities for Green functions with conformal operator insertion [26,7]. Then a short inspection of the Baxter equation (2.3) with the dressing factors $\sigma_\pm$ in the form (3.1) demonstrates that the first term in the series of $\sigma_\pm(x) = i\gamma_\pm(g)/x + \cdots$ induces the shift of the conformal spin,

$$J^{(0)} = N + \frac{1}{2}L \to J = N + \frac{1}{2}L + \frac{1}{2}\left(\gamma_+(g) - \gamma_-(g)\right). \tag{3.4}$$

Consequently, we may naturally identify the addendum with the anomalous dimensions of a multiplicatively renormalizable composite operators, $\gamma(g) = \gamma_+(g) - \gamma_-(g)$. Making use of the explicit form of the dressing factors $\sigma_\pm$, we find the integral representation of $\gamma(g)$ in terms of the solution to the Baxter equation,

$$\gamma(g) = i\frac{g^2}{\pi}\int_{-1}^{1} dt\,\sqrt{1-t^2}\left[\ln Q\left(\frac{i}{2} - gt\right) - \ln Q\left(-\frac{i}{2} - gt\right)\right]'. \tag{3.5}$$

The Taylor expansion shows that the lowest three orders in g^2 coincide with Eq. (2.7).

The Baxter equation (2.3) can be solved analytically order-by-order in coupling constant for specific values of L and N, e.g., for $L = 4$, $N = 2$ eigenvalue with zero quasimomentum reads,[3]

$$\gamma(g) = \frac{5\pm\sqrt{5}}{2}g^2 - \frac{17\pm 5\sqrt{5}}{8}g^4 + \frac{585\pm 207\sqrt{5}}{160}g^6 - \frac{5185\pm 2039\sqrt{5}}{640}g^8 + \mathcal{O}\left(g^{10}\right). \tag{3.6}$$

However, it has a limited range of applicability being asymptotic in character: it allows to find the anomalous dimensions up to order $\mathcal{O}(g^{2n})$ only for operators of length $L \geqslant n$. This restriction arises from the breaking of its polynomiality above a boundary value of n, i.e., for $L \leqslant n$. Analogous limitations apply to the Bethe Ansatz equations of Ref. [24]. A generic dependence of $\gamma(g)$ on the parameters L and N is not known however and below we will develop an asymptotic scheme to find it in the large spin limit.

[3] This anomalous dimension, when related to Berenstein–Maldacena–Nastase operators [27], agrees with previous one-, two- and five-loop analyses of Refs. [28], [4] and [24], respectively.

4. Asymptotic expansion

The large-N behavior of anomalous dimension is of special interest in its own right since it governs the Sudakov asymptotics of scattering amplitudes [29,30], and in light of gauge/string duality, for it can be compared (at strong coupling) to energies of quasiclassical strings [31–35]. Recall at first that the anomalous dimensions of twist-L operators occupy a band of width $L-2$, with the upper and lower boundaries scaling like [1,15]

$$\gamma_{\text{lower}}(g) = 2\Gamma_{\text{cusp}}(g)\ln N,$$
$$\gamma_{\text{upper}}(g) = L\Gamma_{\text{cusp}}(g)\ln N, \qquad (4.1)$$

and the coefficient $\Gamma_{\text{cusp}}(g)$ being the cusp anomalous dimension [36,37], known to one- [36], two- [37,19,20,15] and three-loop orders [22,23,12]. The minimal anomalous dimension $\gamma_{\text{lower}}(g)$ of high-twist operators develops the asymptotic behavior identical to the one of twist-two operators [1,34,9]. Since the single-logarithmic regime is realized for $Le^L \ll N$ with $L, N \to \infty$ [34], this allows one to evade the limitation of the asymptotic character of the Baxter equation and to derive an all-loop equation for the cusp anomaly Γ_{cusp}.

Notice that although we have to solve the problem with large quantum numbers, we cannot apply traditional WKB expansion for $Q(u)$ (see, e.g., Ref. [1]) since the latter is valid for the spectral parameter which scales as $u \sim N^1$ while the energy is determined by the Baxter function $Q(u)$ evaluated at the argument $u = \pm\frac{i}{2} - gt$ which behaves as $u \sim N^0$. Therefore, we have to resort to other techniques. To this end, we will use in the following the approach developed in Refs. [38,34] for one-loop anomalous dimensions and which, as we will see momentarily, is easily generalizable beyond leading order of perturbation theory.

4.1. One-loop Baxter equation

Let us briefly review the formalism of Refs. [38,34] applied to the one-loop Baxter equation (1.5). Though we are interested only in the lowest energy curve, at the beginning we will be general enough to discuss subleading trajectories as well in order to point out approximations which have to be imposed to separate the lowest anomalous dimension only. In the regime in question, the conserved charges are large $q_k^{(0)} \sim N^k$ and, therefore, the transfer matrix is large $|t^{(0)}(u)| \gg 1$. Introducing a new function

$$\phi^{(0)}(u) = \frac{Q^{(0)}(u+i)}{Q^{(0)}(u)}, \qquad (4.2)$$

we can rewrite the Baxter equation in the form

$$u_+^L \phi^{(0)}(u) + \frac{u_-^L}{\phi^{(0)}(u-i)} = t^{(0)}(u). \qquad (4.3)$$

The solution to it is based upon different scaling behavior of the right- and left-hand sides with N. For the spectral parameter $u \sim N^0$, the solution is given by an infinite fraction. Keeping the leading terms only we come to two difference equations

$$u_+^L Q_+^{(0)}(u+i) = t^{(0)}(u) Q_+^{(0)}(u),$$
$$u_-^L Q_-^{(0)}(u-i) = t^{(0)}(u) Q_-^{(0)}(u). \qquad (4.4)$$

The additive corrections to their right-hand sides go as $\mathcal{O}(1/q_n^{(0)})$, where $q_n^{(0)}$ is a conserved charge which scales with the maximal power of N. For cyclically symmetric states $\theta = 0$, the asymptotic solution to (1.5) reads

$$Q^{(0)}(u) = Q_+^{(0)}(u) Q_-^{(0)}\left(-\frac{i}{2}\right) + Q_-^{(0)}(u) Q_+^{(0)}\left(\frac{i}{2}\right), \qquad (4.5)$$

in terms of the solution to the two-term recursion relations (4.4) written with the help of the roots δ_k of the transfer matrix $t^{(0)}(u) = 2(u-\delta_1)(u-\delta_2)\cdots(u-\delta_L)$ [34],

$$Q_\mp^{(0)}(u) = 2^{\pm iu} \prod_{k=1}^{L} \frac{\Gamma(\pm iu + i\delta_k)}{\Gamma(\pm iu + \frac{1}{2})}. \qquad (4.6)$$

Now recall that we are interested only in the trajectory with the lowest energy only. The latter does not depend on the twist of the operator, i.e., it is L-independent. The reason for this being that for the corresponding state only the quadratic Casimir $q_2^{(0)}$ is large while all other integrals of motion become anomalously small. For the roots of the transfer matrix this is translated into the statement that just two roots $\delta_1 = \delta_L$ are much larger than the rest of δ's which are negligible [34], yielding the relation

$$\delta_1^2 \simeq -q_2^{(0)}/2. \qquad (4.7)$$

In this case the genus-$(L-2)$ hyperelliptic Riemann surface parameterizing the magnet, with its moduli determined by the conserved charges $q_k^{(0)}$, degenerates into a sphere, i.e., the spectral curve of twist-two operators [34]. This implies that all zones but one of allowed classical motion in separated variables collapse into points. In this limit the transfer matrix reduces to $t^{(0)}(u) \simeq u^L \tau^{(0)}(u) = u^L(2 - N^2/u^2)$ and the solutions to the recursion relations (4.4) becomes symmetric under the interchange $u \to -u$ and equal, $Q_+^{(0)}(u) = Q_-^{(0)}(u)$. In the infinite-spin limit, we then find that the leading behavior of the Baxter function is

$$\left(i \ln Q_\pm^{(0)}(u)\right)' = \psi(-iu + i\delta_1) + \psi(-iu - i\delta_1) + \cdots$$
$$\simeq 2\ln N + \cdots, \qquad (4.8)$$

where in the last step we imposed the condition that the evaluation of the anomalous dimensions (1.4) requires $u \sim N^0$ and thus it can be neglected compared to N. This consideration immediately suggests that for the minimal-energy trajectory in the single-logarithmic asymptotics the dressing factors $u_\pm^L$ in the left-hand side of Eq. (4.4) are irrelevant. Thus they can be reduced to $u_\pm^L \to u^L$ and canceled with the factor extracted from the transfer matrix $t^{(0)}(u)$, making the equation L-independent, as expected. The latter is clearly seen in the quasiclassical approach when one assumes the spectral parameter to scale with N, i.e., $u = N\hat{u}$ and $\hat{u} \sim 1$. We will use the same argument below to write the all-loop Baxter equation for the lowest trajectory.

4.2. Beyond one loop

Let us find the equation for the minimal trajectory starting from the multi-loop Baxter equation (2.3). Again, we have to separate only terms which generate leading behavior in the large-spin limit. The transfer matrix degenerates on the minimal trajectory to the one of twist-two operators, i.e., $t(u) \simeq u^{L-2}(2u^2 + q_1 u + q_2)$. Notice however that only $\mathcal{O}(g^0)$ contributions to the charges $q_{1,2}(g)$ can induce the leading effect in the large-N limit since the quantum corrections grow at most logarithmically with $N \to \infty$. Therefore, we can replace $t(u) \simeq t^{(0)}(u)$ in the right-hand side of (2.3). Hence the reduced Baxter equation admits the form

$$e^{\sigma_+(x_+)} Q(u+i) + e^{\sigma_-(x_-)} Q(u-i) = \tau^{(0)}(u) Q(u). \tag{4.9}$$

Introducing again the ratio of the Baxter functions Q analogous to Eq. (4.2), we can write again two asymptotic equations for the two components of Q. However, since we are interested solely in the lowest trajectory, both equations generate the same contributions to the anomalous dimension. Therefore, we may consider only one of the resulting equations, e.g.,

$$e^{\sigma_+(x_+)} Q(u+i) = \tau^{(0)}(u) Q(u). \tag{4.10}$$

Next, introducing the one- and all-loop Hamilton–Jacobi functions,

$$S^{(0)}(u) = \ln Q^{(0)}(u), \qquad S(u) = \ln Q(u), \tag{4.11}$$

Eq. (4.10) can be rewritten by virtue of the one-loop degenerate Baxter equation (4.4) for the lowest trajectory as follows

$$S(u+i) - S^{(0)}(u+i) - S(u) + S^{(0)}(u) + \sigma_+(x_+) = 2\pi i m. \tag{4.12}$$

Here m displays the ambiguity in choosing the branch of the logarithm. Since the anomalous dimension (3.5) is expressed in terms of the derivative of the Hamilton–Jacobi function, it is instructive to differentiate both side of Eq. (4.12) with respect to u. Using the perturbative decomposition of the all-order Hamilton–Jacobi function

$$S(u) = S^{(0)}(u) + g^2 S_h(u), \quad \text{with } S_h(u) = \sum_{n=1}^{\infty} g^{2(n-1)} S^{(n)}(u), \tag{4.13}$$

and rescaling S_h' by extracting its single logarithmic behavior

$$i S_h'(u) = \Sigma(u) \ln N, \tag{4.14}$$

we finally arrive at the equation for the cusp anomaly

$$\Sigma(u+i) - \Sigma(u) + \frac{1}{\sqrt{u_+^2 - g^2}} \times \int_{-1}^{1} \frac{dt}{\pi} \frac{\sqrt{1-t^2}}{u_+ + gt} \left[2 + g^2 \Sigma\left(\frac{i}{2} - gt\right)\right] = 0. \tag{4.15}$$

The cusp anomalous dimension is then found in terms of Σ making use of Eq. (3.5) as

$$\Gamma_{\text{cusp}}(g) = g^2 + g^4 \int_{-1}^{1} \frac{dt}{\pi} \sqrt{1-t^2}\, \Sigma'\left(\frac{i}{2} - gt\right). \tag{4.16}$$

As we will demonstrate below, there exists yet another expression for the cusp anomalous dimension in terms of the rescaled Hamilton–Jacobi function Σ which leads to realization of an iterative perturbative structure of Γ_{cusp} in gauge theory. Eqs. (4.15) and (4.16) are the main results of this sections. If one shifts the spectral parameter as $u \to u - \frac{i}{2}$, one immediately realizes that the first two terms give the imaginary part of Σ for real u. Then the use of a dispersion relation for the rescaled Hamilton–Jacobi function in the last term allows us to bring the equation into the form of a Fredholm equation of the second kind. Then the large-x asymptotics of the solution to this integral equation yields the cusp anomaly $2x[u]\, \Im\text{m}\, S_h(u + \frac{i}{2})|_{x[u]\to\infty} = -[\Gamma_{\text{cusp}}(g)/g^2] \ln N$. However below we choose a slightly different route to solve Eq. (4.15) at weak coupling.

5. Weak-coupling expansion

We will seek the solution to the cusp equation (4.15) in the form [17]

$$\Sigma(u) = \int_0^1 d\omega\, \omega^{iu-1} \bar{\omega}^{-iu-1} \hat{\Sigma}\left(\ln\frac{\omega}{\bar{\omega}}\right), \tag{5.1}$$

with $\bar{\omega} = 1 - \omega$. This integral representation immediately diagonalizes the difference terms. The change of variables to $p = \ln \omega/\bar{\omega}$ brings Eq. (5.1) into the form of a Fourier transform. However, before we proceed with the above transformation of Eq. (4.15), we will manipulate it at first. We notice that the first term in the infinite series expansion in Chebyshev polynomials in Eq. (3.1) is determined by the all-order anomalous dimension. Therefore, we can separate it from the kernel and rewrite the equation for the cusp anomaly Γ_{cusp} in a form which immediately suggests yet another relation of the Hamilton–Jacobi function to the cusp anomalous dimension. Performing these steps, we find

$$\sinh\left(\frac{p}{2}\right) \hat{\Sigma}(p) + \frac{\Gamma_{\text{cusp}}(g)}{g^3} J_1(gp) + \frac{gp}{2} \int_0^{\infty} dp' \frac{e^{-p'/2}}{p - p'} \mathbb{U}(gp, gp') \hat{\Sigma}(p') = 0, \tag{5.2}$$

where the kernel $\mathbb{U}$ is expressed in terms of the Bessel functions,

$$\mathbb{U}(p, p') = J_1(p)\left[J_0(p') - \frac{2}{p'} J_1(p')\right] - J_1(p')\left[J_0(p) - \frac{2}{p} J_1(p)\right]. \tag{5.3}$$

An examination of Eq. (5.2) immediately suggests that the last term dies out for $p \to 0$ much faster than the first two, which scale linearly with p. Therefore, we deduce yet another representation for Γ_{cusp} in terms of the solution $\hat{\Sigma}$ to the cusp equation (5.2), namely,

$$\hat{\Sigma}(0) = -\frac{\Gamma_{\text{cusp}}(g)}{g^2}. \tag{5.4}$$

At the same time, we can use Eq. (4.16) for the anomalous dimension in terms of the Hamilton–Jacobi function, such that we get

$$\hat{\Sigma}(0) = -1 - g \int_0^\infty \frac{dp}{p} \mathrm{e}^{-p/2} J_1(gp) \hat{\Sigma}(p). \tag{5.5}$$

This expression clearly displays the mixing of orders and thus exhibits an iterative structure of the perturbative series in coupling constant, i.e., the cusp anomaly at higher orders can be determined in terms of $\Sigma(p)$ at lower orders. Combining Eqs. (5.2)–(5.5) together we reproduce the cusp equation derived in Ref. [9].

Finally, let us solve the cusp equation perturbatively. Writing the expansion in coupling constant as

$$\hat{\Sigma}(p) = \frac{p/2}{\sinh p/2} \sum_{n=0}^{\infty} g^{2n} \hat{\Sigma}_n(p), \tag{5.6}$$

where the prefactor is extracted for the latter convenience, and substituting it into the cusp equation (5.2), we find for the few lowest order functions

$$\begin{aligned}
\hat{\Sigma}_0(p) &= -1, \\
\hat{\Sigma}_1(p) &= \frac{\pi^2}{12} + \frac{1}{8} p^2, \\
\hat{\Sigma}_2(p) &= -\frac{11}{720}\pi^4 + \frac{1}{8}\zeta(3) p - \frac{\pi^2}{96} p^2 - \frac{1}{192} p^4, \\
\hat{\Sigma}_3(p) &= \frac{73\pi^6}{20160} - \frac{\zeta(3)^2}{8} - \left(\frac{5}{16}\zeta(5) + \frac{\pi^2}{96}\zeta(3)\right) p \\
&\quad + \frac{\pi^4}{480} p^2 - \frac{1}{96}\zeta(3) p^3 + \frac{\pi^2 p^4}{2304} + \frac{p^6}{9216}, \\
&\ldots.
\end{aligned}$$

The p-independent term in these expressions determines the cusp anomaly according to Eq. (5.4). The lowest six orders of Γ_{cusp} read

$$\begin{aligned}
\Gamma_{\text{cusp}}(g) &= g^2 - \frac{\pi^2}{12} g^4 + \frac{11\pi^4}{720} g^6 - \left(\frac{73\pi^6}{20160} - \frac{\zeta(3)^2}{8}\right) g^8 \\
&\quad + \left(\frac{887\pi^8}{907200} - \frac{\pi^2}{48}\zeta(3)^2 - \frac{5}{8}\zeta(3)\zeta(5)\right) g^{10} \\
&\quad - \left(\frac{136883\pi^{10}}{479001600} - \frac{\pi^4}{240}\zeta(3)^2 - \frac{5\pi^2}{48}\zeta(3)\zeta(5) \right. \\
&\quad \left. - \frac{51}{64}\zeta(5)^2 - \frac{105}{64}\zeta(3)\zeta(7)\right) g^{12} \\
&\quad + \left(\frac{7680089\pi^{12}}{87178291200} - \frac{47\pi^6}{48384}\zeta(3)^2 + \frac{\zeta(3)^4}{64} \right. \\
&\quad - \frac{41\pi^4}{1920}\zeta(3)\zeta(5) - \frac{17\pi^2}{128}\zeta(5)^2 - \frac{35\pi^2}{128}\zeta(3)\zeta(7) \\
&\quad \left. - \frac{273}{64}\zeta(5)\zeta(7) - \frac{147}{32}\zeta(3)\zeta(9)\right) g^{14} + \cdots.
\end{aligned} \tag{5.7}$$

The two- and three-loop coefficients agree with Feynman diagram calculations of Refs. [19,20,15] and [22,23,12], respectively, and the rest with available predictions of Ref. [9]. The calculation can be extended to few dozens of terms in the series (5.6), but the results are too cumbersome to display here.

6. Outlook

In this note we proposed a multi-loop asymptotic Baxter equation for anomalous dimensions of arbitrary twist-L, spin-N single-trace holomorphic Wilson operators in maximally supersymmetric Yang–Mills theory. We developed an approach for the asymptotic solution of the resulting equation for large values of spin N and derived an all-order equation for the cusp anomaly which governs the Sudakov asymptotics of anomalous dimensions. The problem with the asymptotic nature of the equation was overcome by studying the lowest-energy trajectory which is insensitive to the twist of the operator in the single logarithmic regime $L\mathrm{e}^L \ll N$, $L, N \to \infty$.

There are many questions which remain to be addressed. One has to constrain the amount of ambiguity left in restoration of higher loop effects from the lowest few terms of perturbative series for the dressing factors. The analysis of the strong-coupling expansion of Γ_{cusp} is of special interest in light of available predictions for it from string theory [31]. A preliminary analysis reveals however that $g = \infty$ is an essential singularity of the cusp equation. Next, one has to understand how to incorporate wrapping effects to the Baxter equation (2.3) and to identify a putative microscopic spin chain standing behind it. An ultimate goal would be to generalize the all-order Baxter equation to all sectors of $\mathcal{N} = 4$ super-Yang–Mills theory which is conceivably described by a long-range graded magnet.

Note added

Recently a new calculation was published of the four-loop cusp anomalous dimension using the unitarity technique [39]. Their numerical finding explicitly demonstrates that the prediction (5.7) based on the Baxter equation (2.3) with the dressing factor (3.1) is incorrect starting from four loops. In a companion paper [40], a modified form of the cusp equation was proposed which takes into account a nontrivial dressing factor in Bethe equations of Ref. [18].

Presently we use the result of Ref. [39] in order to fix the form of the four-loop correction to the Baxter equation (2.3) and find anomalous dimensions of local Wilson operators. It was suggested [39], that to reconcile within error bars the result of their numerical calculation with the one coming from the cusp equation, the sign of the $\zeta^2(3)$ in four-loop contribution of Eq. (5.7) has to be flipped. This requires the following additive modification of the four-loop cusp anomaly (5.7),

$$\Gamma_{\text{cusp}}(g) = -\frac{\zeta(3)^2}{4} g^8 + \mathcal{O}(g^{10}). \tag{6.1}$$

In order to generate it from the cusp equation, one has to add the following term[4] to the left-hand side of Eq. (5.2)

$$\cdots + 2\alpha g^2 J_2(gp)\left(1 + g\int_0^\infty dp'\, \mathrm{e}^{-p'/2}\hat{\Sigma}(p')\frac{J_1(gp')}{p'}\right) + \mathcal{O}(g^3), \tag{6.2}$$

in agreement with Ref. [18]. Here the favored value of the constant is $\alpha = \frac{1}{2}\zeta(3)$ [39,40]. This translates into a modification of the integrand in Eq. (4.15),

$$\frac{1}{u_+ + gt} \to \frac{1}{u_+ + gt} + \frac{i\alpha g^4}{x_+^2} + \mathcal{O}(g^6). \tag{6.3}$$

A simple analysis allows to unambiguously restore the correction term to the dressing factors (3.1) of the Baxter equation (2.3). Namely, the former get shifted as

$$\sigma_\pm(x) \to \sigma_\pm(x) + \Delta_\pm(x), \tag{6.4}$$

with

$$\Delta_\pm(x) = \mp\frac{i\alpha g^6}{2x^2}\int_{-1}^{1}\frac{dt}{\pi}\sqrt{1-t^2}\left[\ln Q\left(\pm\frac{i}{2} - gt\right)\right]' + \mathcal{O}(g^7). \tag{6.5}$$

Taking into account this extra term, the anomalous dimensions of Wilson operators acquire additional contributions. For instance, the four-loop term in Eq. (3.6) gets corrected by

$$\gamma(g) = \cdots - \alpha\frac{5 \pm \sqrt{5}}{8}g^8 + \mathcal{O}(g^{10}). \tag{6.6}$$

This explicitly demonstrates that the attempt to rescue the principle of maximal transcendentality [23] in the cusp anomalous dimension with $\alpha = \frac{1}{2}\zeta(3)$ results in breaking of the rational form of anomalous dimensions of local Wilson operators, i.e., they acquire transcendental addenda (6.6) in addition to rational terms (3.6). At the current state-of-the-art of higher loop calculations such terms are not ruled out yet. Within Mueller's cut vertex technique [41], the main sources of transcendental constants in local anomalous dimensions comes from virtual self-energy and vertex corrections with rational terms being generated by real cuts. The finiteness of maximally supersymmetric Yang–Mills theory, especially transparent in the light-cone gauge where Ward identities imply equality of the vanishing beta function with all field renormalization constants, seems to suggests the absence of transcendental constants in local anomalous dimensions. This question deserves however a thorough study.

[4] In the unnumbered equations above (5.2), it yields corrections to the right-hand side the equations, i.e., $\hat{\Sigma}_2(p) = \cdots + \frac{1}{2}\alpha p$, $\hat{\Sigma}_3(p) = \cdots + \frac{1}{2}\alpha\zeta(3) + \frac{1}{24}\alpha p(\pi^2 + p)$.

Acknowledgement

This work was supported by the US National Science Foundation under grant No. PHY-0456520.

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