ELSEVIER
Available online at www.sciencedirect.com
SCIENCE DIRECT®
European Journal of Combinatorics 26 (2005) 995–1008
European Journal of Combinatorics
www.elsevier.com/locate/ejc

Combinatorics and geometry of Littlewood–Richardson cones

Igor Pak[a], Ernesto Vallejo[b]

[a] *Massachusetts Institute of Technology, Cambridge, MA 02138, USA*
[b] *Instituto de Matemáticas, Unidad Morelia, UNAM, Apartado Postal 61-3, Xangari, 58089 Morelia, Mich., Mexico*

Received 24 March 2003; received in revised form 22 September 2003; accepted 13 November 2003
Available online 24 July 2004

Abstract

We present several direct bijections between different combinatorial interpretations of the Littlewood–Richardson coefficients. The bijections are defined by explicit linear maps which have other applications.

Keywords: Young tableaux; Littlewood–Richardson rule; Berenstein–Zelevinsky triangles; Knutson–Tao hives

1. Introduction

In the past decade the Littlewood–Richardson rule (LR rule) has moved into center stage in the combinatorics of Young tableaux. Classical applications (to representation theory of the symmetric and the full linear group, to the symmetric functions, etc.) as well as more recent developments (Schubert calculus, eigenvalues of Hermitian matrices, etc.) have received much attention. While various combinatorial interpretations of the Littlewood–Richardson coefficients have been discovered, there seems to be little understanding of how they are related to each other, and little order among them. This paper makes a new step in this direction.

E-mail addresses: pak@math.mit.edu (I. Pak), vallejo@matmor.unam.mx (E. Vallejo).

0195-6698/$ - see front matter © 2004 Published by Elsevier Ltd
doi:10.1016/j.ejc.2004.06.008

We start with three major combinatorial interpretations of the LR coefficients which we view as integer points in certain cones. We present simple linear maps between the cones which produce explicit bijections for all triples of partitions involved in the LR rule. These bijections are quite natural in this setting and in a certain sense can be shown to be unique. Below we further emphasize the importance of the linear maps.

A classical version of the LR rule, in terms of certain Young tableaux, is now well understood, and its proof has been perfected for decades. We refer to [14] for a beautifully written survey of the "classical" approach, with a historical overview and connections to the jeu-de-taquin, Schützenberger involution, etc. Unfortunately, the language of Young tableaux is often too rigid to be able to demonstrate the inherent symmetries of the LR coefficients.

A radically different combinatorial interpretation is due to Berenstein and Zelevinsky, in terms of the so called BZ triangles, which makes explicit all but one symmetry of the LR coefficients.[1] The authors' proof in [6] relies on a series of previous papers [10,4, 5], a situation that is hardly satisfactory. A paper [8] establishes a technically involved bijection with the contratableaux associated with certain Yamanouchi words, which gives another combinatorial interpretation of the LR rule. This combinatorial interpretation is in fact different from the one given by LR tableaux, which makes the matter even more confusing.

In a subsequent development, Knutson and Tao introduced [13] the so called honeycombs, which are related to BZ triangles by a bijection that they sketch at the end. The paper [11] uses a related construction of "web diagrams" for a different purpose. The appendix in [13] also introduces a different language of *hives*, which proved to be more flexible to restate the Knutson–Tao proof of saturation conjecture [7].

In the appendix to [7], Fulton described in a simple language a bijection with a set of certain contratableaux, similar to that of Carré [8]. As mentioned at the end of the appendix (cf. also [9]), the latter are in a well known bijection with the classical LR tableaux. Unfortunately, this bijection is based on the Schützenberger involution, which is in fact quite involved and goes beyond the scope of this paper.

Now, let us return to the linear maps establishing the bijections. First, these maps show that the LR cones have the same combinatorial structure. Despite a visual difference between definitions of LR tableaux, hives, and BZ triangles, these combinatorial objects are essentially the same and should be treated as equivalent. In a sense, this varying nature of these combinatorial interpretations of the LR coefficients makes them "more fundamental" than others.

Let us mention here a "local" nature of the bijections we present. In general, computing the action linear maps $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ requires $O(d^2)$ arithmetic operations (multiplications and additions) to perform. In this case, however, the local nature of bijections allows a $O(d)$ computation, where $d = \binom{k}{2}$, and k is the number of rows in LR tableaux. This is especially striking when comparing with other Young tableau bijections, which require $O(d^{3/2})$ operations. We refer to a forthcoming paper [17] for references and details, and for a new theory explaining this phenomenon. As observed previously, the bijections in this

[1] We should warn the reader that the BZ triangles presented in [18] are different, albeit strongly related.

paper combined with the symmetries of BZ triangles give nearly all the symmetries[2] of the LR coefficients, except for one: $c^{\lambda}_{\mu,\nu} = c^{\lambda}_{\nu,\mu}$. The latter again requires $O(d^{3/2})$ operations and is in the same class as other Young tableau bijections (ibid).

The idea of using integer points in cones is a direct descendant of the earlier papers [10, 5] and most recently has appeared in the context of integer partitions [16]. While the fact that the linear maps between cones exist at all may seem surprising, we do not claim to be the first to establish that. It is perhaps surprising that the resulting linear maps are so simple and natural in this language. We believe that this approach is perhaps more direct and fruitful when compared to other more traditional combinatorial techniques employed earlier (see above).

To conclude, let us describe the structure of the paper. We present in separate sections the LR tableaux, the hives of Knutson and Tao, and the BZ triangles. Along the way we establish the bijections between these combinatorial interpretations. While the linear maps which produce these bijections are easy to define, their proofs are not straightforward and are delayed until the end of the paper. We conclude with the final remarks.

2. Littlewood–Richardson tableaux

Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of a positive integer n, that is, a sequence of integers whose sum is n and satisfy $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$. Its diagram is the set of pairs of positive integers $\{(i, j) \mid 1 \leq i \leq k, 1 \leq j \leq \lambda_i\}$, which we also denote by λ. If μ is another partition and the diagram of μ is a subset of the diagram of λ, in symbols $\mu \subseteq \lambda$, we denote by λ/μ the *skew diagram* consisting of the points in λ that are not in μ, and by $|\lambda/\mu|$ its cardinality. It is customary to represent diagrams pictorially as a collection of boxes [9,15,18]. Any filling T of a skew diagram λ/μ with positive integers, formally a map $T : \lambda/\mu \longrightarrow \mathbb{N}$, will be called a *Young tableau* or just a *tableau* of *shape* λ/μ. A Young tableau T is called *semistandard* if its rows are weakly increasing from left to right and its columns are strictly increasing from top to bottom. The *content* of T is the composition $\gamma(T) = (\gamma_1, \ldots, \gamma_c)$, where γ_i is the number of is in T. The *word* of T, denoted by $w(T)$, is obtained from T by reading its entries from right to left, in successive rows, starting with the top row and moving down. For example, let

$$D = \begin{array}{|c|c|c|c|c|c|} \hline & & & \square & \square & \square \\ \hline & & \square & \square & & \\ \hline \square & \square & \square & \square & & \\ \hline \square & \square & \square & & & \\ \hline \end{array} \qquad \text{and} \qquad T = \begin{array}{cccccc} & & & 1 & 1 & 7 \\ & & 1 & 4 & & \\ 1 & 4 & 5 & 7 & & \\ 3 & 5 & 7 & & & \end{array}$$

then D is a diagram of shape $(6, 4, 4, 3)/(3, 2)$ and T is a semistandard tableaux of this shape, has content $(4, 0, 1, 2, 2, 0, 3)$ and its word is $w(T) = 711417541753$. Finally, a word $w = w_1 \cdots w_k$ in the alphabet $1, \ldots, n$ is called a *lattice permutation* if for all $1 \leq j \leq k$ and all $1 \leq i \leq n-1$ the number of occurrences of i in $w_1 \cdots w_j$ is not less than the number of occurrences of $i + 1$ in $w_1 \cdots w_j$. A semistandard tableau T of skew

[2] There is one additional "symmetry" $c^{\lambda}_{\mu,\nu} = c^{\lambda'}_{\mu',\nu'}$; which does not seem to have a "geometric interpretation". For a combinatorial proof see [12].

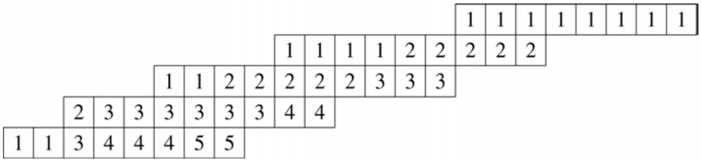

Fig. 1. Littlewood–Richardson tableau.

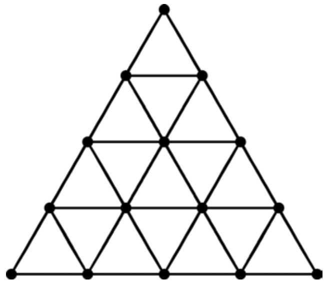

Fig. 2. Hive graph Δ_4.

shape is called a *Littlewood–Richardson* tableau if its word $w(T)$ is a lattice permutation. Note that the content of a Littlewood–Richardson tableau is always a partition. Given three partitions λ, μ, ν such that $\mu \subseteq \lambda$ and $|\lambda| = |\mu| + |\nu|$, we denote by $c^{\lambda}_{\mu\nu}$ the number of Littlewood–Richardson tableaux of shape λ/μ and content ν. We will use the following example throughout the paper. Let

$$\lambda = (23, 18, 15, 11, 8), \quad \mu = (15, 9, 5, 2, 0) \quad \text{and} \quad \nu = (16, 11, 10, 5, 2), \qquad (1)$$

then the tableau in Fig. 1 is an example of a Littlewood–Richardson tableau of shape λ/μ and content ν.

3. Littlewood–Richardson triangles

The *hive graph Δ_k of size k* is a graph in the plane with $\binom{k+2}{2}$ vertices arranged in a triangular grid consisting of k^2 small equilateral triangles, as shown in Fig. 2. Let T_k denote the vector space of all labelings $A = (a_{ij})_{0 \le i \le j \le k}$ of the vertices of Δ_k with real numbers such that $a_{00} = 0$. We will write such labelings as triangular arrays of real numbers in the way shown in Fig. 3. The dimension of T_k is clearly $\binom{k+2}{2} - 1$.

We now proceed to explain how Littlewood–Richardson tableaux can be coded in a simple way as elements in T_k satisfying certain inequalities. A *Littlewood–Richardson triangle of size k* is an element $A = (a_{ij}) \in T_k$ that satisfies the following conditions:

(P) $a_{ij} \ge 0$, for all $1 \le i < j \le k$.
(CS) $\sum_{p=0}^{i-1} a_{pj} \ge \sum_{p=0}^{i} a_{pj+1}$, for all $1 \le i \le j < k$.
(LR) $\sum_{q=i}^{j} a_{iq} \ge \sum_{q=i+1}^{j+1} a_{i+1q}$, for all $1 \le i \le j < k$.

$$\begin{array}{ccccccc} & & & a_{00} & & & \\ & & a_{01} & & a_{11} & & \\ & a_{02} & & a_{12} & & a_{22} & \\ a_{03} & & a_{13} & & a_{23} & & a_{33} \end{array}$$

Fig. 3. Triangular array of size 3.

Note that the inequality

$$\sum_{p=0}^{j} a_{pj} \geq \sum_{p=0}^{j+1} a_{pj+1}, \qquad \text{for } 1 \leq j < k \tag{2}$$

follows from (CS) with $i = j$ and (LR) with $i = j$; also note that a_{0j} and a_{jj} could be negative. We denote by LR_k the cone of all Littlewood–Richardson triangles in T_k, and call it a *Littlewood–Richardson cone*; this has the same dimension as T_k. Also let D_k denote the set of all k-tuples $\lambda = (\lambda_1, \ldots, \lambda_k)$ of real numbers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$, and $|\lambda|$ the sum of its entries, that is, $|\lambda| = \sum_{i=1}^{k} \lambda_i$. With each $A = (a_{ij}) \in \mathsf{LR}_k$ we associate the following numbers:

(B1) $\mu_j = a_{0j}$, for all $1 \leq j \leq k$.
(B2) $\lambda_j = \sum_{p=0}^{j} a_{pj}$, for all $1 \leq j \leq k$.
(B3) $\nu_i = \sum_{q=i}^{k} a_{iq}$, for all $1 \leq i \leq k$.

Then it follows from (P), (CS), and (LR) that the vectors $\lambda = (\lambda_1, \ldots, \lambda_k)$, $\mu = (\mu_1, \ldots, \mu_k)$, and $\nu = (\nu_1, \ldots, \nu_k)$ are in D_k and that $|\lambda| = |\mu| + |\nu|$. We call (λ, μ, ν) the *type* of A, and denote by $\mathsf{LR}_k(\lambda, \mu, \nu)$ the set of all Littlewood–Richardson triangles of type (λ, μ, ν); this is a convex polytope. For example, let λ, μ, ν be as in (1), then the triangle in Fig. 4 is in $\mathsf{LR}_5(\lambda, \mu, \nu)$.

Let $\lambda, \mu, \nu \in \mathsf{D}_k$ be partitions, that is λ, μ, and ν have non-negative integer coefficients, and suppose that $|\lambda| = |\mu| + |\nu|$. With each Littlewood–Richardson tableau T of shape λ/μ and content ν we associate a triangular array $A_T = (a_{ij}) \in T_k$ by defining

(i) $a_{00} = 0$, $a_{0j} = \mu_j$ for $1 \leq j \leq k$, and
(ii) a_{ij} equal to the number of is in row j of T for $1 \leq i \leq j \leq k$.

Note that the Littlewood–Richardson triangle in Fig. 4 corresponds to the Littlewood–Richardson tableau in Fig. 1.

Lemma 3.1. *Let $\lambda, \mu, \nu \in \mathsf{D}_k$ be partitions such that $|\lambda| = |\mu| + |\nu|$. Then the correspondence $T \longmapsto A_T$ is a bijection between the set of all Littlewood–Richardson tableaux of shape λ/μ and content ν and the set of all Littlewood–Richardson triangles of type (λ, μ, ν) with integer entries. In particular $\mathsf{LR}_k(\lambda, \mu, \nu)$ has $c^{\lambda}_{\mu\nu}$ integer points.*

0

15 8

9 4 5

5 2 5 3

2 0 1 6 2

0 2 0 1 3 2

Fig. 4. Littlewood–Richardson triangle of size 5.

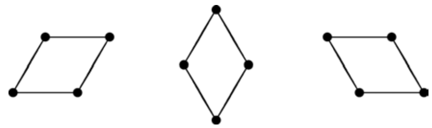

Fig. 5. Types of rhombus in a hive graph.

In effect, Lemma 3.1 translates combinatorics of Littlewood–Richardson tableaux into the language of integer points in polyhedra. Various other translations of this kind appear in the literature and are more or less equivalent to ours. A short “verification style” proof is given in Section 6.

4. Hives

The hive graph Δ_k of size k is divided into k^2 small equilateral triangles. Each two adjacent such triangles form a rhombus with two obtuse angles and two acute angles. There are three types of rhombus: tilted to the right, vertical and tilted to the left. They are shown in Fig. 5.

A *hive of size k* is a labeling $H = (h_{ij})_{0\le i\le j\le k}$ of the vertices of the hive graph Δ_k with real numbers such that for each rhombus the sum of the labels at obtuse vertices is bigger than or equal to the sum of the labels at acute vertices; equivalently, $H = (h_{ij})$ satisfies the following inequalities:

(R) $h_{ij} - h_{ij-1} \ge h_{i-1j} - h_{i-1j-1}$, for $1 \le i < j \le k$.
(V) $h_{i-1j} - h_{i-1\,j-1} \ge h_{ij+1} - h_{ij}$, for $1 \le i \le j < k$.
(L) $h_{ij} - h_{i-1j} \ge h_{i+1j+1} - h_{ij+1}$, for $1 \le i \le j < k$.

We denote by H_k the cone of all hives of size k that satisfy the extra condition $h_{00} = 0$, and call it a *hive cone*. As we did for Littlewood–Richardson triangles, we associate with each hive $H = (h_{ij}) \in \mathsf{H}_k$ numbers:

0

15 23

24 36 41

29 43 53 56

31 45 56 65 67

31 47 58 68 73 75

Fig. 6. Hive of size 5.

(B1′) $\mu_j = h_{0j} - h_{0j-1}$, for $1 \le j \le k$.
(B2′) $\lambda_j = h_{jj} - h_{j-1j-1}$, for $1 \le j \le k$.
(B3′) $\nu_i = h_{ik} - h_{i-1k}$, for $1 \le i \le k$.

Then it follows from (R), (V), and (L) that the vectors $\lambda = (\lambda_1, \ldots, \lambda_k)$, $\mu = (\mu_1, \ldots, \mu_k)$, and $\nu = (\nu_1, \ldots, \nu_k)$ are in D_k and that $|\lambda| = |\mu| + |\nu|$. For example,

$$\mu_j = h_{0j} - h_{0j-1} \ge h_{1j+1} - h_{1j} \ge h_{0j+1} - h_{0j} = \mu_{j+1}.$$

We call (λ, μ, ν) the *type* of A, and denote by $\mathsf{H}_k(\lambda, \mu, \nu)$ the set of all hives of type (λ, μ, ν); this is a convex polytope. For example, let λ, μ and ν be as in (1), then the triangle in Fig. 6 is in $\mathsf{H}_5(\lambda, \mu, \nu)$.

For any positive integer k, we define a linear map $\Phi_k : T_k \longrightarrow T_k$ by

$$\Phi_k(a_{ij}) = (h_{ij}), \qquad \text{where } h_{ij} = \sum_{p=0}^{i} \sum_{q=p}^{j} a_{pq}.$$

Note that the hive in Fig. 6 is the image under Φ_5 of the Littlewood–Richardson triangle in Fig. 4. We have the following theorem.

Theorem 4.1. *The map Φ_k defined above is a volume preserving linear operator which maps LR_k bijectively onto H_k, and $\mathsf{LR}_k(\lambda, \mu, \nu)$ onto $\mathsf{H}_k(\lambda, \mu, \nu)$, for all $\lambda, \mu, \nu \in \mathsf{D}_k$.*

As mentioned in the Introduction, the proof can be found in Section 6. Let us mention here two important corollaries. For any polytope P let $e(P)$ denote the number of integer points in P.

Corollary 4.2. $e(\mathsf{H}_k(\lambda, \mu, \nu)) = c^{\lambda}_{\mu\nu}$, *for all $\lambda, \mu, \nu \in \mathsf{D}_k$ with non-negative integer coefficients.*

Corollary 4.3. $\mathrm{Vol}(\mathsf{H}_k(\lambda, \mu, \nu)) = \mathrm{Vol}(\mathsf{LR}_k(\lambda, \mu, \nu))$, *for all $\lambda, \mu, \nu \in \mathsf{D}_k$.*

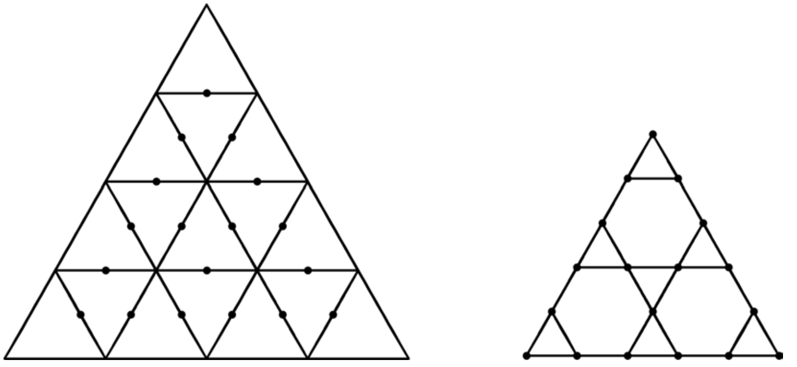

Fig. 7. Hive graph Δ_4 and the corresponding Berenstein–Zelevinsky graph Γ_3.

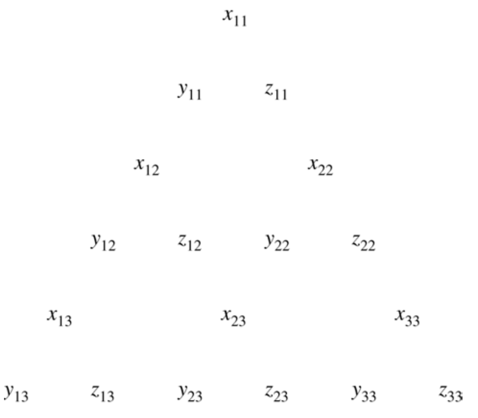

Fig. 8. Labeling of Γ_3.

5. Berenstein–Zelevinsky triangles

For any integer $k \geq 1$ we construct a graph Γ_k from the hive graph Δ_{k+1} in the following way: its vertices are the middle points of the edges of the hive graph that do not lie on the boundary, and their edges are those joining pairs of middle points on edges lying on small triangles of Δ_{k+1}, see Fig. 7. We call Γ_k the Berenstein–Zelevinsky graph of size k. The vertices of the Berenstein–Zelevinsky graph are partitioned into disjoint blocks of cardinality three, each block corresponding to a small equilateral triangle; these triangles are distributed in the graph: one on the first (top) level, two on the second level, three on the third level, and so on. Let V_k denote the vector space of all labelings $X = (x_{ij}, y_{ij}, z_{ij})_{1 \leq i \leq j \leq k}$ of Γ_k with real numbers. The labelings are carried out in such a way that the vertices of the i-th triangle on the j-th level are labeled with x_{ij}, y_{ij}, z_{ij} as indicated in Fig. 8. The dimension of V_k is $3\binom{k+1}{2}$. Note that the labels $y_{ij}, z_{ij}, x_{i+1j+1}, y_{i+1j+1}, z_{ij+1}, x_{ij+1}$ form an hexagon for each $1 \leq i \leq j < k$ and hence there are $\binom{k}{2}$ hexagons in Γ_k. We will be interested in the subspace W_k of V_k consisting

2

4 *3*

2 1

2 *2* **5** *2*

3 4 3

0 *3* **1** *1* **6** *1*

0 0 0 3

2 *3* **0** *1* **1** *4* **3** *0*

Fig. 9. Berenstein–Zelevinsky triangle of size 4.

of all labelings such that for each hexagon in Γ_k the sum of the labels in each edge equals the sum of the labels of the diametrically opposite edge, that is

(BZ1) $y_{ij} + z_{ij} = y_{i+1j+1} + z_{ij+1}$, for all $1 \le i \le j < k$.
(BZ2) $x_{ij+1} + y_{ij} = x_{i+1j+1} + y_{i+1j+1}$, for all $1 \le i \le j < k$.
(BZ3) $x_{ij+1} + z_{ij+1} = x_{i+1j+1} + z_{ij}$, for all $1 \le i \le j < k$.

Observe that any of these three equalities follows from the other two.

Lemma 5.1. *The vector space W_k has dimension $\frac{1}{2}k(k+5) = \dim T_{k+1} - 2$.*

A *Berenstein–Zelevinsky triangle of size k* is any labeling of Γ_k in W_k with non-negative entries. Let BZ_k denote the cone of all Berenstein–Zelevinsky triangles of size k. Let $\lambda, \mu, \nu \in \mathsf{D}_{k+1}$, then we say that a Berenstein–Zelevinsky triangle is of type (λ, μ, ν) if it satisfies the following conditions:

(B1″) $x_{1j} + y_{1j} = \mu_j - \mu_{j+1}$, for $1 \le j \le k$.
(B2″) $x_{jj} + z_{jj} = \lambda_j - \lambda_{j+1}$, for $1 \le j \le k$.
(B3″) $y_{ik} + z_{ik} = \nu_i - \nu_{i+1}$, for $1 \le i \le k$.

Note that, in contrast to Littlewood–Richardson triangles and hives, a Berenstein–Zelevinsky triangle has many different types. Let $\mathsf{BZ}_k(\lambda, \mu, \nu)$ denote the set of all Berenstein–Zelevinsky triangles of type (λ, μ, ν); this is a convex polytope. For example, let λ, μ and ν be as in (1), then the triangle in Fig. 9 is in $\mathsf{BZ}_4(\lambda, \mu, \nu)$. Here the x_{ij}s are written with roman numerals, the y_{ij}s by **boldface** numerals, and the z_{ij}s by *italic* numerals.

For any integer $k \ge 2$, we define a linear map $\Psi_k : T_k \longrightarrow W_{k-1}$ by setting $\Psi_k(h_{ij}) = (x_{ij}, y_{ij}, z_{ij})$ where

$$x_{ij} = h_{ij} + h_{i-1j} - h_{i-1j-1} - h_{ij+1},$$

$$y_{ij} = h_{i-1j} + h_{ij+1} - h_{ij} - h_{i-1j+1},$$
$$z_{ij} = h_{ij} + h_{ij+1} - h_{i-1j} - h_{i+1j+1},$$

for all $1 \le i \le j < k$. Note that the values of the x_{ij}s, y_{ij}s, and z_{ij}s are obtained by taking, respectively, the differences of the inequalities (V), (R), and (L) used to define hives. It should be remarked that the y_{ij}s are obtained from (R) by adding one to j. It is straightforward to check that the image of Φ_k is contained in W_{k-1}. The composition $\Psi_k \circ \Phi_k : T_k \longrightarrow W_{k-1}$ has also a nice description: $\Psi_k \circ \Phi_k(a_{ij}) = (x_{ij}, y_{ij}, z_{ij})$ with

$$\begin{aligned} x_{ij} &= \sum_{p=0}^{i-1} a_{pj} - \sum_{p=0}^{i} a_{pj+1}, \\ y_{ij} &= a_{ij} + 1, \\ z_{ij} &= \sum_{q=i}^{j} a_{iq} - \sum_{q=i+1}^{j+1} a_{i+1q}, \end{aligned} \tag{3}$$

for all $1 \le i \le j < k$. Again, the values of the x_{ij}s, y_{ij}s, and z_{ij}s are obtained by taking, respectively, the differences of the left and right hand sides in the inequalities (CS), (P), and (LR) used to define Littlewood–Richardson triangles. For example, the Berenstein–Zelevinsky triangle in Fig. 9 is the image under Ψ_5 of the hive in Fig. 6 and the image under $\Psi_5 \circ \Phi_5$ of the Littlewood–Richardson triangle in Fig. 4. Note that the boldface numerals in Fig. 9 are contained in the Littlewood–Richardson triangle from Fig. 4.

Theorem 5.2. *The linear operator $\Psi_k \circ \Phi_k$ maps LR_k surjectively onto BZ_{k-1}, and $\mathsf{LR}_k(\lambda, \mu, \nu)$ bijectively onto $\mathsf{BZ}_{k-1}(\lambda, \mu, \nu)$, for any $\lambda, \mu, \nu \in \mathsf{D}_k$.*

Corollary 5.3. *The linear operator Ψ_k maps H_k surjectively onto BZ_{k-1}, and $\mathsf{H}_k(\lambda, \mu, \nu)$ bijectively onto $\mathsf{BZ}_{k-1}(\lambda, \mu, \nu)$, for any $\lambda, \mu, \nu \in \mathsf{D}_k$.*

Corollary 5.4. *$e(\mathsf{BZ}_{k-1}(\lambda, \mu, \nu)) = c^{\lambda}_{\mu\nu}$, for any $\lambda, \mu, \nu \in \mathsf{D}_k$ with non-negative integer coefficients.*

It will follow from Lemma 6.1 and the proof of Theorem 5.2 that the cones LR_k and H_k are isomorphic to $\mathsf{BZ}_{k-1} \times \mathbb{R}^2$. One can embed the cone BZ_{k-1} into LR_k in the following way: for any $k \ge 2$, let $\Omega_k : W_{k-1} \longrightarrow T_k$ be the linear operator defined by $\Omega_k(x_{ij}, y_{ij}, z_{ij}) = (a_{ij})$ where

$$\begin{aligned} a_{0j} &= \sum_{l=j}^{k-1} x_{1l} + y_{1l} \qquad \text{for } 1 \le j < k, \text{ and } a_{0k} = 0, \\ a_{ij} &= y_{ij-1}, \qquad \text{for } 1 \le i < j \le k, \\ a_{jj} &= \sum_{l=j}^{k-1} z_{ll} \qquad \text{for } 1 \le j < k, \text{ and } a_{kk} = 0. \end{aligned} \tag{4}$$

Then we have:

Theorem 5.5. *The linear operator Ω_k defined above maps BZ_{k-1} injectively into LR_k, and $\mathsf{BZ}_{k-1}(\lambda, \mu, \nu)$ bijectively onto $\mathsf{LR}_k(\lambda, \mu, \nu)$ for any $\lambda, \mu, \nu \in \mathsf{D}_k$ such that $\mu_k = 0$ and $\nu_k = 0$.*

6. Proof of results

Proof of Lemma 3.1. Let T be a Littlewood–Richardson tableau of shape λ/μ and content ν, then A_T satisfies (P) by definition. Since T has strictly increasing columns (CS) follows, and since $w(T)$ is a lattice permutation, A_T satisfies (LR). It is also clear that A_T is of type (λ, μ, ν). Conversely, for any Littlewood–Richardson triangle $A = (a_{ij})$ in $\mathsf{LR}_k(\lambda, \mu, \nu)$ with integer entries, we define a tableau T_A of shape λ/μ by placing in row j, in weakly increasing order, a_{ij} is for each i and j. It is routine to check that T is a Littlewood–Richardson tableau of shape λ/μ and content ν, and that both constructions are inverses of each other. Here we use that (2) follows from (CS) and (LR). □

Proof of Theorem 4.1. Let $\{E_{ij}\}$ be the canonical basis of T_k, that is $E_{ij} = (e^{ij}_{pq})$, where

$$e^{ij}_{pq} = \begin{cases} 1, & \text{if } p = i \text{ and } q = j; \\ 0, & \text{otherwise.} \end{cases}$$

We order it according to the lexicographic order of the subindices, that is,

$$\mathcal{B} = \{E_{01}, E_{02}, \ldots, E_{0k}, E_{11}, \ldots, E_{1k}, \ldots, E_{kk}\}.$$

The matrix of Φ_k with respect to $\mathcal{B}$ is lower triangular with ones on the main diagonal, therefore it has determinant one, is volume preserving, and maps $\mathbb{Z}^{\binom{k+2}{2}-1}$ bijectively onto $\mathbb{Z}^{\binom{k+2}{2}-1}$. The inverse of Φ_k is given by $\Phi_k^{-1}(h_{ij}) = (a_{ij})$ where

$$a_{ij} = \begin{cases} h_{0j} - h_{0j-1}, & \text{if } i = 0 \text{ and } 1 \le j \le k. \\ h_{jj} - h_{j-1j}, & \text{if } 1 \le i = j \le k. \\ h_{ij} - h_{ij-1} - h_{i-1j} + h_{i-1j-1}, & \text{if } 1 \le i < j \le k. \end{cases}$$

Let $(a_{ij}) \in \mathsf{LR}_k$ and $(h_{ij}) = \Phi_k(a_{ij})$, then we have

$$h_{st} - h_{st-1} = \sum_{p=0}^{s} a_{pt} \qquad \text{and} \qquad h_{s+1t} - h_{st} = \sum_{q=s+1}^{t} a_{s+1q},$$

for $0 \le s < t \le k$. It is straightforward, using these two identities, to check that (a_{ij}) satisfies (P), (CS), or (LR), respectively, if and only if (h_{ij}) satisfies (R), (V), or (L), respectively; therefore $\Phi_k(\mathsf{LR}_k) = \mathsf{H}_k$. Also, it is straightforward to check that (a_{ij}) and (h_{ij}) have the same type; therefore $\Phi_k(\mathsf{LR}_k(\lambda, \mu, \nu)) = \mathsf{H}_k(\lambda, \mu, \nu)$, for all λ, μ and $\nu \in \mathsf{D}_k$. □

Proof of Lemma 5.1. We form a system of linear equations by taking, for each $1 \le i \le j < k$, that is, for each hexagon in Γ_k, equations (BZ2) and (BZ3). Then, after arranging the variables in the order $x_{11}, y_{11}, z_{11}, x_{12}, y_{12}, z_{12}, x_{22}, \ldots, z_{kk}$, we easily check that the matrix of coefficients of the system is in echelon form and has rank $2\binom{k}{2}$. Thus $\dim W_k = 3\binom{k+1}{2} - 2\binom{k}{2} = \frac{1}{2}k(k+5)$. □

Before we prove Theorem 5.2, let us prove the following lemma.

Lemma 6.1. *The linear operators Ψ_k and $\Psi_k \circ \Phi_k$ are surjective. Moreover, Eq. (5) give a full description of $(\Psi_k \circ \Phi_k)^{-1}(X)$ for any $X \in W_{k-1}$.*

Proof. It is enough to show that $\Psi_k \circ \Phi_k$ is surjective. Let $X = (x_{ij}, y_{ij}, z_{ij}) \in W_{k-1}$. For each $s, t \in \mathbb{R}$ we define an element $A_{st} = (a_{ij}) \in T_k$ by

$$
\begin{aligned}
a_{0j} &= s + \sum_{l=j}^{k-1} x_{1l} + y_{1l} \qquad \text{for } 1 \le j < k, \text{ and } a_{0k} = s, \\
a_{ij} &= y_{ij-1}, \qquad \text{for } 1 \le i < j \le k, \\
a_{jj} &= t + \sum_{l=j}^{k-1} z_{ll} \qquad \text{for } 1 \le j < k, \text{ and } a_{kk} = t.
\end{aligned} \tag{5}
$$

Let $X' = (x'_{ij}, y'_{ij}, z'_{ij}) = \Psi_k \circ \Phi_k(A_{st})$. We claim that $X' = X$. By definition, x'_{ij}, y'_{ij} and z'_{ij} satisfy Eq. (3). Combining (3) and (5) we get that $y'_{ij} = a_{ij+1} = y_{ij}$ for all $1 \le i \le j < k$. Again, combining (3) and (5), we obtain

$$
\begin{aligned}
x'_{ij} &= (x_{1j} + y_{1j}) + \sum_{p=1}^{i-1} y_{pj-1} - \sum_{p=1}^{i} y_{pj} \\
&= (x_{1j} + y_{1j-1} - y_{2j}) + \sum_{p=2}^{i-1} y_{pj-1} - \sum_{p=3}^{i} y_{pj}.
\end{aligned}
$$

Condition (BZ2) implies that $x_{1j} + y_{1j-1} - y_{2j} = x_{2j}$; and repeated application of (BZ2) yields $x'_{ij} = x_{ij}$ for all $1 \le i \le j < k$. Finally, the equality $z'_{ij} = z_{ij}$, is obtained in a similar way from (BZ1). Thus $\Psi_k \circ \Phi_k$ is surjective. The last statement follows from the identity $\dim T_k = \dim W_{k-1} + 2$.

Proof of Theorem 5.2. It follows from (3) that $\Psi_k \circ \Phi_k(\mathsf{LR}_k) = \mathsf{BZ}_{k-1}$; and it follows from (3) and (B1)–(B3) that A and $\Psi_k \circ \Phi_k(A)$ have the same type, for any $A \in \mathsf{LR}_k$, thus $\Psi_k \circ \Phi_k(\mathsf{LR}_k(\lambda, \mu, \nu)) = \mathsf{BZ}_k(\lambda, \mu, \nu)$. The last claim follows from the remark that different elements in the preimage of an $X \in \mathsf{BZ}_k(\lambda, \mu, \nu)$ have different types. □

Proof of Theorem 5.5. It follows from (3), (4), and the proof of Lemma 6.1 that $\Psi_k \circ \Phi_k \circ \Omega_k$ is the identity map on W_{k-1}, and that $\Omega(\mathsf{BZ}_{k-1}) \subseteq \mathsf{LR}_k$. The last statement follows from Theorem 5.2. □

7. Final remarks

Let us start with the complexity issues. Recall that the LR triangles, hives, and BZ triangles, all of size k, are given by $\theta(k^2)$ entries. As defined, maps Φ^{-1} and Ψ require only a constant number of arithmetic operations per entry, and thus have $O(k^2)$ complexity. It is an easy exercise in dynamic programming to show that Φ and Ψ^{-1} have the same complexity, linear in the input.

The complexity $O(k^2)$ is in stark contrast with the $O(k^3)$ complexity required by the jeu-de-taquin and Schützenberger involution (cf. [9,18,17]). This explains why Fulton's map in [7] has the same complexity. In fact, Fulton reworks the bijection of Carré [8] which establishes a combinatorial map $\Upsilon : e(\mathsf{LR}_k(\lambda, \mu, \nu)) \to e(\mathsf{H}_k(\lambda, \nu, \mu))$. As we mentioned in the Introduction and will reiterate below, there is no linear map establishing the symmetry $\mathsf{H}_k(\lambda, \nu, \mu) \to \mathsf{H}_k(\lambda, \mu, \nu)$. One can use a more complicated map called tableaux switching to demonstrate this symmetry [3] (see also [14,17]).

Now, the symmetries of the LR coefficients are quite intriguing in a sense that most of them can be established by simple means. If one operates with LR tableaux, one simply has to map them into BZ triangles (which takes $O(k^2)$ steps), perform the symmetry, and return back to LR tableaux (which takes $O(k^2)$ steps again). For the remaining $\mu \leftrightarrow \nu$ symmetry several authors found an explicit map (in different languages) [1,2,14,19] but all of them use $\theta(k^3)$ steps (see [17] for the theory and the explanation). It would be interesting to prove the lower bound $\Omega(k^3)$ but we are doubtful such a result is feasible at the moment. What one can show, however, is that this "last" symmetry cannot be performed by a linear map already for $k = 4$. We leave this statement as an interesting exercise to the reader, and refer to a sequel paper [17] for further results in this direction.

Acknowledgements

We are grateful to Oleg Gleizer, Michael Kleber, Alex Postnikov, and Terry Tao for interesting conversations and helpful remarks. We thank Olga Azenhas and Christophe Carré for the help with the literature.

The first author was supported by NSA and NSF. The second author is grateful to Richard Stanley for his support in organization of the sabbatical visit to MIT. The second author's work was done during a sabbatical stay at MIT Mathematics Department. I would like to thank CONACYT and DGAPA-UNAM for financial support.

References

[1] O. Azenhas, Littlewood–Richardson fillings and their symmetries, Textos Mat. Sér. B 19 (1999) 81–92.

[2] O. Azenhas, On an involution on the set of Littlewood–Richardson tableaux and the hidden commutativity, Pré-publicações do Departamento de Matemática da Universidade de Coimbra 00-27, 2000, Available from http://www.mat.uc.pt/preprints/2000.html.

[3] G. Benkart, F. Sottile, J. Stroomer, Tableau switching: algorithms and applications, J. Combin. Theory Ser. A 76 (1996) 11–43.

[4] A.D. Berenstein, A.V. Zelevinsky, Involutions on Gelfand–Tsetlin schemes and multiplicities in skew GL_n-modules, Dokl. Akad. Nauk SSSR 300 (1988) 1291–1294 (in Russian).

[5] A.D. Berenstein, A.V. Zelevinsky, Tensor product multiplicities and convex polytopes in partition space, J. Geom. Phys. 5 (3) (1988) 453–472.

[6] A.D. Berenstein, A. Zelevinsky, Triple multiplicities for $sl(r + 1)$ and the spectrum of the exterior algebra of the adjoint representation, J. Algebraic Combin. 1 (1992) 7–22.

[7] A. Buch, The saturation conjecture (after A. Knutson and T. Tao), Enseign. Math. 46 (2000) 43–60.

[8] C. Carré, The rule of Littlewood–Richardson in a construction of Berenstein–Zelevinsky, Internat. J. Algebra Comput. 1 (1991) 473–491.
[9] W. Fulton, Young Tableaux, London Math. Soc. Student Texts, vol. 35, Cambridge University Press, 1997.
[10] I.M. Gelfand, A.V. Zelevinsky, Multiplicities and regular bases for gl_n, in: Group-Theoretic Methods in Physics (Jūrmala, 1985), vol. 2, Nauka, Moscow, 1986, pp. 22–31 (in Russian).
[11] O. Gleizer, A. Postnikov, Littlewood–Richardson coefficients via Yang-Baxter equation, Internat. Math. Res. Notices (14) (2000) 741–774.
[12] P. Hanlon, S. Sundaram, On a bijection between Littlewood–Richardson fillings of conjugate shape, J. Combin. Theory Ser. A 60 (1992) 1–18.
[13] A. Knutson, T. Tao, The honeycomb model of $\mathrm{GL}_n(\mathbb{C})$ tensor products I: proof of the saturation conjecture, J. Amer. Math. Soc. 12 (1999) 1055–1090.
[14] M.A.A. van Leeuwen, The Littlewood–Richardson rule, and related combinatorics, in: Interaction of Combinatorics and Representation Theory, MSJ Mem., vol. 11, Math. Soc. Japan, Tokyo, 2001, pp. 95–145.
[15] I.G. Macdonald, Symmetric functions and Hall polynomials, 2nd edition, Oxford University Press, 1995.
[16] I. Pak, Partition identities and geometric bijections, Proc. Amer. Math. Soc. (2002) in press.
[17] I. Pak, E. Vallejo, Reductions of Young tableau bijections, preprint, 2004, Available from http://www-math.mit.edu/~pak/research.html.
[18] R.P. Stanley, Enumerative Combinatorics, Cambridge Studies in Advanced Mathematics 62, vol. 2, Cambridge University Press, 1999.
[19] T. Tao, personal communication.