Abstract

Operator polynomials (i.e., polynomials over endomorphism algebras of vector spaces) are studied in the framework of polynomial modules and their comodule duals, i.e., the comodules of linearly recursive sequences. The subcomodules, which replace the concept of Jordan matrix pairs, contain enough information to recover the polynomials up to unit factors.

0. Introduction

Our main objective is to present a coordinate independent approach to the study of operator polynomials (i.e., polynomials over endomorphism algebras of vector spaces). It works for finite dimensional vector spaces $V$ over any perfect field $k$ (e.g., char $k = 0$ or $k$ finite) and is based on the observation that the coordinate free version of a Jordan matrix pair in the sense of [4, 7] is that of a finite dimensional subcomodule of the comodule $V[x]^0$ of linearly recursive sequences in the linear dual $V^*$ of $V$. The subcomodule associated with an operator polynomial $P$ contains all the information required to recover $P$ up to unit factors. Our approach also
provides an effective method to construct divisors of operator polynomials.

The exposition is self-contained in the sense that no advance knowledge of the theory of coalgebras and comodules is needed. General definitions are given in Section 1, and whatever special knowledge is required will be developed in Sections 3 and 4.

A basic fact, established in more general form in Section 1, is the inclusion reversing bijective correspondence between submodules of finite codimension $M$ of the polynomial module $V[x]$ and finite dimensional subcomodules $M'$ of the comodule of linearly recursive sequences $V[x]^0$ (for the scalar case see also [8]). Together with some results of [2, Chapter 1; 3], which are recalled in Section 2 for an arbitrary principal ideal domain, this gives bijective correspondences between equivalence classes of regular operator polynomials (i.e., polynomials $P$ with $\det P \neq 0$), submodules of finite codimension of $V[x]$, and finite dimensional subcomodules of $V[x]^0$. These bijective correspondences lead to a useful characterization of monic (the leading coefficient is the identity endomorphism) and of comonic (the constant term is the identity) operator polynomials in terms of the corresponding subcomodules (which are called monic and comonic, respectively). A monic operator polynomial $P$ and its monic divisors are completely determined by (and can be constructed from) the corresponding subcomodules.

The standard method of avoiding the eigenvalue zero reduces the general divisor problem (i.e., that of finding divisors) to that of comonic divisors of comonic polynomials. An intrinsic "extension to monies" process in Section 6 establishes a bijective correspondence between comonic subcomodules of $V[x]^0$ and monic subcomodules not containing all of $V^*$, which reflects the correspondence $Q = x^{\deg P} P(1/x)$ between comonic polynomials $P$ and monic polynomials $Q$ not divisible by $x$. It allows, together with the results for the monic case, the explicit construction of a comonic operator polynomial of least degree associated with a comonic subcomodule of $V[x]^0$.

1. COALGEBRAS AND COMODULES

1.1 A cocommutative coalgebra $C$ is a vector space with a structure "dual" to that of a commutative algebra, i.e., a counit $\epsilon : C \to k$ and a comultiplication $\delta : C \to C \otimes C$ satisfying

(a) the counit law: $(\epsilon \otimes C)\delta = C = (C \otimes \epsilon)\delta$,
(b) coassociativity: $(\delta \otimes C)\delta = (C \otimes \delta)\delta$,
(c) cocommutativity: $\sigma\delta = \delta$, where $\sigma$ switches tensor factors.
Note that the symbol $C$ is used in the dual role of the underlying vector space and of its identity automorphism. Every coalgebra has a canonical decomposition into a direct sum (coproduct) of irreducible (indecomposable) subcoalgebras. This is easy to see in the finite dimensional case and follows in the general case from the fact that every element of a coalgebra is contained in a finite dimensional subcoalgebra [1, 5, 9].

A $C$-comodule $N$ is a vector space together with a coaction of $C$ on $N$, that is, a linear map $\alpha : N \to N \otimes C$ such that

\begin{enumerate}
\item[(d)] $(N \otimes \epsilon)\alpha = N$
\item[(e)] $(N \otimes \delta)\alpha = (\alpha \otimes C)\alpha$
\end{enumerate}

1.2

If $V$ and $W$ are vector spaces, then we have a canonical linear map $V^* \otimes W^* \to (V \otimes W)^*$, which is an isomorphism if and only if one of the factors $V$ or $W$ is finite dimensional. From this we see that the linear dual $C^*$ of a coalgebra $C$ is an algebra and that of a $C$-comodule $N$ is a $C^*$-module. The linear dual of a finite dimensional algebra $(A, \kappa, \mu)$ is a coalgebra $(A^*, \kappa^*, \mu^*)$, where $\kappa^* : A^* \to k$ and $\mu^* : A^* \to (A \otimes A)^* \cong A^* \otimes A^*$ are the duals of the unit map $\kappa : k \to A$ and of the multiplication map $\mu : A \otimes A \to A$, respectively. If $A$ is infinite dimensional, then the linear map $A^* \otimes A \to (A \otimes A)^*$ is not bijective and $A^*$ is not a coalgebra yet. But the subspaces of representable functionals $A^0$ is a coalgebra in a natural way.

A functional $f \in A^*$ is called \textit{representative} if its kernel $\ker f$ contains an ideal $J$ of finite codimension [so that $f$ can be viewed as an element of the finite dimensional coalgebra $(A/J)^*$]. The coalgebra $A^0$ is then the direct limit $A^0 = \lim J(A/J)^*$ of the directed system of finite dimensional coalgebras $(A/J)^*$, where $J$ runs through the ideals of finite codimension of $A$. Similar considerations apply to $A$-modules $M$ and their $A^0$-comodules of representative functionals $M^0 = \lim J(M/L)^*.$

The $A$-module structure on $M^*$ defined by $(af)(m) = f(am)$ clearly restricts to $M^0$, and the $A^0$-comodule structure is its "adjoint" in the following sense. If $L$ is a submodule of finite codimension of $M$, then $J = \text{ann}(M/L) = \{a | aM \subseteq L\}$ is an ideal of finite codimension in $A$. The $A$-module structure on $(M/L)^*$ defined by $(af)(m + L) = f(am + L)$ is actually an $A/J$-module structure. The $(A/J)^*$-comodule structure on $(M/L)^*$ is then given by the adjoint map

$$\alpha : (M/L)^* \to \text{Vect}(A/J, (M/L)^*) \cong (A/J)^* \otimes (M/L)^*.$$. 
Thus $M^0$ is the sum of the finite dimensional submodules of $M^*$, and a finite dimensional subspace of $M^*$ is an $A$-submodule of $M^0$ if and only if it is an $A^0$-subcomodule. This applies in particular to $A$ itself.

1.3

The annihilator maps $' : \text{Sub}(M^0) \to \text{Sub}(M)$ and $' : \text{Sub}(M) \to \text{Sub}(M^0)$ between the lattice of subcomodules $N$ of the $A^0$-comodule $M^0$ and the lattice of submodules $L$ of the $A$-module $M$ are defined by $N' = \{ m \mid Nm = 0 \} = \ker(M \to N^*)$ and $L' = \{ f \mid f(L) = 0 \} = \ker(M^0 \to L^0)$, respectively.

**Lemma 1.3.** The annihilator map $' : \text{Sub}(M^0) \to \text{Sub}(M)$ is inclusion reversing and sends sums to intersections. It restricts to a bijection between finite dimensional subcomodules of $M^0$ and submodules of finite codimension of $M$.

**Proof.** The first assertion follows directly from the definitions. The annihilator $N'$ of a finite dimensional subcomodule of $M^0$ is a submodule of finite codimension in $M$, and $N = N'$, since $N^*$ is a finite dimensional $A$-module and $N \cong N^{**}$ as $A^0$-comodules. Conversely, if $L$ is a submodule of finite codimension in $M$, then $M/L \cong (M/L)^*$; hence $L' = \ker(M^0 \to L^0) \cong (M/L)^*$ is finite dimensional and $L = L''$. 

1.4

A commutative and cocommutative Hopf algebra $H$ is a cocommutative coalgebra with

1. a commutative algebra structure such that counit and comultiplication are algebra maps, and

2. something called an antipode $\omega : H \to H$, which is an automorphism of order two for the coalgebra structure as well as the algebra structure of $H$ and which satisfies the equation $\mu(C \otimes \omega)\delta = \kappa \epsilon = \mu(\omega \otimes C)\delta$.

The representative functionals on a Hopf algebra $H$ form again a Hopf algebra $H^*$, since the product of representative functionals in the algebra $H^*$ is again representative. A Hopf algebra whose underlying coalgebra is irreducible is called connected. The coalgebras and comodules considered in this paper are subcoalgebras of the Hopf algebra of linearly recursive sequences $k[x]^0$ and subcomodules of its comodules of linearly recursive sequen-
ces $V[x]^0$, respectively. These are constructed explicitly in Sections 3 and 4. The reader who wants to learn more about coalgebras and their applications may consult [9, 1, 6, 5].

2. MODULES OVER PRINCIPAL IDEAL DOMAINS

2.1

Our main interest lies with finitely generated modules over the polynomial algebra $k[x]$, which is a principal ideal domain. Among commutative rings with identity the principal ideal domains are characterized by the following property:

Every submodule $M$ of a free module $F$ is free.

Moreover, rank $M \leq \text{rank } F$. The results of this section depend on this fact alone and are formulated for a general principal ideal domain $A$, postponing everything that is specific to $k[x]$ to Section 3. Here is a description of the relationship between submodules of a free $A$-module $F$ and endomorphisms of $F$.

2.2

PROPOSITION 2.2. If $M$ is a submodule of a free module $F$ of finite rank over a principal ideal domain $A$, then:

(a) $M = PF$ for some endomorphism $P \in \text{End}(F)$.

(b) $M = PF \subset QF$ if and only if $P = QE$ for some $E \in \text{End}(F)$.

(c) $F/M$ is torsion ($M = PF$ is of maximal rank in $F$) if and only if $\det P \neq 0$ (i.e., $P$ is regular).

Proof. Let $\{f_1, f_2, \ldots, f_n\}$ be a basis of $F$. Then $M$, as a submodule of $F$, is free, and $r = \text{rank } M \leq \text{rank } F = n$.

(a): If $\{m_1, m_2, \ldots, m_r\}$ is a basis of $M$, then $r \leq n$. Define $P \in \text{End}(F)$ by $Pf_i = m_i$ for $1 \leq i \leq r$ and $Pf_i = 0$ for $r < i < n$. Then $M = PF$.

(b): If $P = QE$, then clearly $M = PF = QEF \subset QF$. Conversely, if $M \subset QF$, then for $1 \leq i \leq r$ choose $g_i \in F$ such that $Qg_i = m_i$. Define $E \in \text{End}(F)$ by $Ef_i = g_i$ for $1 \leq i \leq r$ and $Ef_i = 0$ for $r < i < n$. Then $Pf_i = QEf_i$ for $1 \leq i \leq n$.

(c): If $F/M$ is torsion, then its annihilator ideal in $A$ is $aA$ for some $0 \neq a \in A$. Hence, $aF \subset M = PF$, and by (b) we have $aI = PE$ for some $E \in \text{End}(F)$, where $I$ is the identity map. This implies that $0 \neq a^n = \det aI = \det PE = \det P \det E$ and thus $\det P \neq 0$. Conversely, if $0 \neq I \det P =$
If \( P \) adjoins \( P \), then (b) implies that \((\det P)F \subseteq PF\). Thus, the \( A \)-module \( F/\text{PF} = F/M \), as a quotient of the torsion module \( F/(\det P)F \), is itself torsion.

\[\text{2.3}
\]

**Corollary 2.3.** Let \( M = PF \) be a submodule of maximal rank in the free \( A \)-module \( F \), i.e., a submodule such that \( F/M \) is torsion. Then \( M = QF \) if and only if \( P = QU \) for some automorphism \( U \in \text{Aut}(F) \).

**Proof.** If \( PF = M = QF \), then by Proposition 2.2(b), \( P = QE \) and \( Q = PG \) for some \( E \) and \( G \) in \( \text{End}(F) \). Thus, \( P = PEG \), and from \( 0 \neq \det P = \det P \det E \det G \) in \( A \) we see that \( E \) is invertible. The converse is obvious.

\[\text{2.4. Remarks}
\]

(1) The automorphism group \( \text{Aut}(F) \) of the free \( A \)-module \( F \) acts by right multiplication on the submonoid of regular endomorphisms of \( \text{End}(F) \). [After a choice of basis in \( F \), this can be described as the action by right multiplication of the group \( \text{GL}_n(A) \) on the regular matrices in \( M_n(A) \).] Corollary 2.3 establishes a bijective correspondence between the orbits of this action (equivalence classes of regular endos of \( F \)) and submodules of maximal rank in \( F \).

(2) It follows from Proposition 2.2 that greatest common left divisors and least common left multiples of a finite subset of \( \text{End}(F) \) exist and that they can be characterized by sums and intersections of submodules of \( F \). In particular,

\[\text{gcd}(P, Q) F = PF + QF, \quad \text{lcm}(P, Q) F = PF \cap QF.\]

This can be used to show that the map which sends \( P \) to \( P \text{End}(F) \) establishes a bijective correspondence between the orbits under the above action and the right ideals of maximal rank in \( \text{End}(F) \). As a consequence of this fact we get the following analog to Proposition 2.2.

(3) Let \( J \) be a right ideal of \( \text{End}(F) \). Then:

(a) \( J = P \text{End}(F) \) for some \( P \in \text{End}(F) \), i.e., the right ideals of \( \text{End}(F) \) are principal.

(b) \( J \subseteq Q \text{End}(F) \) if and only if \( P = QE \) for some \( E \in \text{End}(F) \).

(c) \( \text{End}(F)/J \) is torsion exactly when \( \det P \neq 0 \), i.e., when \( P \) is regular.
3. THE HOPF ALGEBRA OF LINEARLY RECURSIVE SEQUENCES

3.1

The polynomial algebra $k[x]$ is not only a principal ideal domain but also a Hopf algebra with counit $\epsilon : k[x] \to k$, diagonal $\delta : k[x] \to k[x] \otimes k[x]$, and antipode $\omega : k[x] \to k[x]$ given by $\epsilon x^n = 0^n$, $\delta x^n = (x \otimes 1 + 1 \otimes x)^n$, and $\omega x^n = (-x)^n$, respectively. In fact, $k[x]$ is the only connected Hopf algebra which is also a principal ideal domain. The linear dual $k[x]^*$ is a topological algebra (product topology with $k$ discrete) with unit $\epsilon$ and with convolution $f \ast g = (f \otimes g)\delta$ as multiplication. In terms of the “topological basis” $\{z_m|m \geq 0\}$, where $z_m(x^n) = \delta^n_m$, multiplication is given by

$$z_m \ast z_n = \binom{n + m}{m} z_{n+m}.$$ 

The Hopf algebra dual $k[x]^0$ (see Section 1) is the subalgebra of representative functionals in $k[x]^*$. These are functionals $f \in k[x]^*$ whose kernel contains an ideal $J$ of finite codimension (i.e., an ideal $J \neq 0$), so that $f$ can be viewed as an element of the finite dimensional coalgebra $(k[x]/J)^*$ and hence

$$k[x]^0 = \lim_{\longrightarrow J} (k[x]/J)^*,$$

where the direct limit is taken over all nonzero ideals of $k[x]$. Since $J = pk[x]$ for some polynomial $p = \sum_{0 \leq i \leq r} p_i x^i$, we see that each $f \in k[x]^0$ is represented by an (infinite) series $f = \sum_{m \geq 0} a_m z_m$, whose coefficients form a linearly recursive sequence in $k$, i.e., satisfy $\sum_{0 \leq i \leq r} p_i a_{i+j} = 0$ for all $j$ (see also [8]). Along with $k[x]^0$ we shall also use the Hopf dual $k[\mathbb{X}]^0$ of the algebra of formal power series $k[\mathbb{X}]$. Note that every finite dimensional homomorphic image of $k[\mathbb{X}]$ has the form $k[\mathbb{X}]/(x^n)$; hence comparing the direct limits defining $k[\mathbb{X}]^0$ and $k[x]^0$, we see that we can identify the former with a subcoalgebra of the latter.

The coalgebra structures on $k[x]^0$ and $k[\mathbb{X}]^0$ are induced by the algebra structures on $k[x]$ and $k[\mathbb{X}]$, respectively. In particular, $\delta z_n = \sum_{r+s = n} z_r \otimes z_s$ and $\epsilon z_n = \delta_{0n}$, which means that $z_n(pq) = \sum_{r+s = n} z_r(p)z_s(q)$ and $z_n(1) = \delta_{0n}$. The corresponding $k[x]$-module structure on $k[x]^0$ (see Section 1.2) is given by $(xf)(p) = f(xp)$; in particular $xz_n = z_{n-1}$. Multiplication by $x$ is a derivation, i.e., $x(f \ast g) = (xf) \ast g + f \ast (xg)$. To simplify notation let $B_{n-1}^q = (k[x]/q^nk[x])^*$ for $n > 0$. 
3.2

**Theorem 3.2** (The coalgebra of linearly recursive sequences).

(a) As a coalgebra $k[x]^0 = \bigoplus_q B_q$, where the direct sum is over all monic irreducible polynomials $q \in k[x]$. The subcoalgebra $B_q = \lim_n (k[x]/q^n k[x])^*$ of $k[x]^0$ is irreducible and will be called the irreducible component of $q$.

(b) $B_x = \lim_n (k[x]/x^n k[x])^* = k[[x]]^0$ is a connected Hopf algebra with countable basis $\{z_m \mid m \geq 0\}$.

(c) The coalgebra map $\gamma : B_x \otimes (k[x]/qk[x])^* \to k[x]^0$, defined by $\gamma(z_n \otimes g) = z_n * g$, takes values in $B_q$. It maps isomorphically onto $B_q$ if $q$ is separable (always the case if $k$ is a perfect field).

**Proof:** If $p = q_1^{a_1}q_2^{a_2} \cdots q_s^{a_s}$ is the primary decomposition of the polynomial $p \in k[x]$, then it follows from the Chinese remainder theorem that $k[x]/pk[x] \cong \bigoplus_{1 \leq i \leq s} k[x]/q_i^n k[x]$ as an algebra, and since these summands are irreducible (local) algebras, we have

$$(k[x]/pk[x])^* \cong \bigoplus_{1 \leq i \leq s} (k[x]/q_i^n k[x])^* = \bigoplus_{1 \leq i \leq s} B_{q_i^{(a_i-1)}},$$

where the summands are irreducible coalgebras. This gives the decomposition

$$k[x]^0 = \lim_j (k[x]/J)^* \cong \bigoplus_q B_q$$

into the maximal irreducible components $B_q$, where the sum is over all monic irreducible polynomials in $k[x]$.

The irreducible component $B_x = k[[x]]^0$ (see Section 3.1 for notation) is a connected Hopf algebra with countable basis $\{z_m \mid m \geq 0\}$, unit $z_0$, multiplication

$$z_m * z_n = \binom{m+n}{m} z_{m+n},$$

unit counit $\varepsilon z_m = \delta_m 0$, comultiplication $\delta z_m = \sum_{r+s=m} z_r \otimes z_s$, and antipode $\omega z_m = (-1)^m z_m$.

If $q \in k[x]$ is irreducible then $\dim(k[x]/q^n k[x]) = \dim(k[x]/q^n k[x])^* = n \deg q$. To prove assertion (c) it therefore suffices to show that the restrictions $\gamma_n : B_x^{(n)} \otimes B_q^{(0)} \to k[x]^0$ map into $B_q^{(n)}$ for $n \geq 0$, and that they
are all injective exactly when \( q \) is separable. Clearly, \( \gamma_0 \) maps bijectively onto \( B_q^{(0)} \), and we proceed by induction. Notice first that \( \mu_n : k[x]^\otimes n \to k[x] \), the \( n \)-fold iteration of multiplication, is a coalgebra map (\( k[x]^\otimes n \) has the usual tensor product coalgebra structure), so that \( \delta \mu_n = (\mu_n \otimes \mu_n) \delta \), and 

\[
(z_n + g) \mu_n - (z_n \otimes g) \delta \mu_n = (z_n \mu_n \otimes g \mu_n) \delta \quad \text{for} \quad g \in k[x]^0. 
\]

But \( z_n \mu_n - \sum_{i_1 + \cdots + i_n = n} z_{i_1} \otimes \cdots \otimes z_{i_n} \) and thus \( (z_n \ast g) \mu_n = g \mu_n (\sum_{i_1 + \cdots + i_n = n} (z_{i_1} \otimes 1) \delta \otimes \cdots \otimes (z_{i_n} \otimes 1) \delta) \). This is summarized in the commutative diagram

\[
\begin{array}{c}
k[x]^\otimes n \xrightarrow{\delta} k[x]^\otimes n \otimes k[x]^\otimes n \xrightarrow{\gamma_n \otimes \mu_n} k \otimes k[x] \\
\downarrow \mu_n \downarrow \mu_n \otimes \mu_n \downarrow k \otimes g \\
k[x] \xrightarrow{\delta} k[x] \otimes k[x] \xrightarrow{z_n \otimes g} k \otimes k. 
\end{array}
\]

Now if \( g(qk[x]) = 0 \), i.e., \( g \in B_q^{(0)} \), then \( (z_n \ast g)q^{n+1} = 0 \) and \( (z_n \ast g)q^n p = g((q')^p)p \). This is because \( (z_n \otimes 1) \delta (qp) = qp \) and \( (z_n \otimes 1) \delta (qp) = qp + qp' \). We conclude that \( z_n \ast g \in B_q^{(n)} \), and also that \( z_n \ast g \in B_q^{(n-1)} \) if and only if \( g((q')^p k[x]) = 0 \). Since \( \gcd(q, (q')^p) = 1 \) if \( q \) is separable and \( q \) otherwise, it follows that \( z_n \ast g \in B_q^{(n-1)} \) if and only if \( g = 0 \) or if \( q \) is inseparable.

3.3. Remarks

(1) If \( q \) is irreducible and separable, then \( B_q^{(n)} / B_q^{(n-1)} \cong B_q^{(0)} \) by Theorem 3.2(c). Multiplication by \( q \), i.e., the map \( q^n : k[x]^0 \to k[x]^0 \) defined by \( (q^n f)(p) = f(pq) \), restricts to \( B_q \) and the resulting maps \( q^n : B_q^{(j)} \to B_q^{(j-1)} \) give rise to isomorphisms \( q : B_q^{(0)} \to B_q^{(n-1)} / B_q^{(n)} \) if and only if \( g((q')^p k[x]) = 0 \). Since \( \gcd(q, (q')^p) = 1 \) is 1 if \( q \) is separable and \( q \) otherwise, it follows that \( z_n \ast g \in B_q^{(n-1)} \) if and only if \( g = 0 \) or if \( q \) is inseparable.

(2) If \( \text{char } k = 0 \) then \( B_x = k[[x]]^0 \) is isomorphic to the polynomial algebra \( k[y] \) via the identification \( z_n = y^n / n! \). Note also that \( q^n : k[x]^0 \to k[x]^0 \) is given by \( q^n f z_r = \Sigma r+s=f (x q(r) / r)! z_s \), where \( q(r) \) is the \( r \)-th derivative of \( q \). As an algebra \( k[x]^0 \) is isomorphic to the algebra of rational functions without poles at zero under ordinary multiplication. (3) If \( \text{char } k = p \neq 0 \), then \( B_x \) has zero divisors, since \( z_r z_s = 0 \) whenever \( r + s = p^n \) with \( n > 0 \). If \( F_p \) is the prime field with \( p \) elements, then \( q = x^p - t \) is an inseparable irreducible polynomial over the field of rational functions \( k = F_p(t) \). Then \( z_r \ast g \in B_q^{(0)} \) whenever \( g \in B_q^{(0)} \), so that \( \gamma : B_x \otimes B_q^{(0)} \to B_q \) is not injective.

(4) A point of \( k[x]^0 \) is a representative functional \( f = \Sigma_{n \geq 0} f(x^n) z_n \) with \( f(1) = 1 \) and \( \delta(f) = f \otimes f \). This means that \( f(x^n) = f(x)^n \), so that \( f : k[x] \to k \) is an algebra map determined by the element \( f(x) = a \in k \).
Thus \( Gk[x]^0 = \text{Alg}(k[x], k) \equiv k \) is the set of points (group-like elements in the terminology of [9]) of \( k[x]^0 \). Each \( B_q \), \( q \) irreducible, has at most one point, and it has one exactly when \( q \) is linear, i.e., \( q = x - a \). The corresponding point is \( e(a) = \sum_{m \geq 0} a^m z_m \in k[x]^0 \). A primitive over the point \( f \) is a functional \( g = \sum_{m \geq 0} g(x^m) z_m \) satisfying \( g(1) = 0 \) and \( \delta g = g \otimes f + f \otimes g \). This implies that \( g(x^{r+s}) = g(x^r)f(x^s) + f(x^r)g(x^s) \), so that \( g : k[x] \rightarrow k \) is an \( f \)-derivation and the vector space of primitives over the point \( f \) is \( P_f k[x]^0 = \text{Der}_f(k[x], k) \equiv k \). Notice that

\[
g = g(x) \sum_{m \geq 0} m a^{m-1} z_m = g(x) \frac{d}{dx} e(x) \bigg|_{x=a}.
\]

4. COMODULES OF LINEARLY RECURSIVE SEQUENCES

4.1

Every free \( k[x] \)-module is of the form \( V[x] = V \otimes k[x] \) for some vector space \( V \). If \( V \) is finite dimensional, as we shall assume from here on, then the linear dual of the polynomial module \( V[x] \) is \( V[x]^* = V^* \otimes k[x]^* \). It has the topological basis \( \{v^i \otimes z_j | 1 \leq i \leq n, 0 \leq j \} \) if \( \{v^i | 1 \leq i \leq n \} \) is the dual basis of the basis \( \{v_i | 1 \leq i \leq n \} \) of \( V \), i.e., \( v^i(v_j) = \delta^i_j \). The comodule dual \( V[x]^0 \) is the subspace of representative functionals in \( V[x]^* \). These are the functionals \( g \) for which \( \ker g \) contains a submodule \( M \) of finite codimension, so that \( g \) can be viewed as an element of \( (V[x]/M)^* \). Since \( M = PV[x] \) for some \( P \in \text{End}(V[x]) \) by Proposition 2.2, we see that \( g \) is represented by a series \( g = \sum_{s \geq 0} g_s \otimes z_s \), whose coefficients form a linearly recursive sequence in \( V^* \), i.e., \( \sum_{0 \leq i < s} A_s^i g_{s+j} = 0 \) for every \( j \geq 0 \) if \( P = \sum_{0 \leq i < r} A_i x^i \) and if \( A_r^s \) is the transpose of \( A_s^r \).

The \( k[x] \)-module structure of \( V[x]^0 \) is given by \( (xg)(vx^m) = g(vx^{m+1}) \), and a subspace \( N \) of \( V[x]^0 \) is a \( k[x] \)-submodule if and only if it is a \( k[x]^0 \)-subcomodule (see Section 1.2). The next results describe the structure of \( V[x]^0 \) and of its subcomodules.

4.2

**Lemma 4.2.** \( V[x]^0 \) is a cofree \( k[x]^0 \)-comodule. More precisely, \( V[x]^0 \equiv V^* \otimes k[x]^0 \). The coaction \( \alpha : V[x]^0 \rightarrow V[x]^0 \otimes k[x]^0 \) is given by \( \alpha g = \sum_{s \geq 0} D^s g \otimes z_s \), where \( (D^s g)(vx^r) = g(vx^{r+s}) \).

**Proof.** By Proposition 2.2, the submodule \( M = PV[x] \) of \( V[x] \) is of finite codimension if and only if \( \det P \neq 0 \). Moreover, \( \det P = P \text{ adj } P \)
implies that \((\det P)V[x] \subseteq PV[x] = M\). This determinant argument shows that the set of submodules \(pV[x]\) with \(p \in k[x] \setminus \{0\}\) is cofinal in the partially ordered set of submodules \(PV[x]\) with \(P \in \text{End}_{\text{reg}}(V)[x]\). The canonical projection

\[ V \otimes k[x]/(\det P)V[x] \rightarrow V[x]/PV[x] \]

induces a canonical injection

\[ (V[x]/PV[x])^* \hookrightarrow (V[x]/(\det P)V[x])^* \]

\[ \cong V^* \otimes (k[x]/(\det P)k[x])^*. \]

Since this happens for every submodule \(M = PV[x]\) of finite codimension in \(V[x]\), we conclude that \(V[x]^0 = V^* \otimes k[x]^0\). Now, if \(g \in V[x]^0\) then by definition \((\alpha g)(vx^r \otimes x^s) = g(vx^{r+s})\), so that \(\alpha g = \sum_{s \geq 0} D^s g \otimes z_s\).

4.3

**THEOREM 4.3.** Let \(N\) be a finite dimensional subcomodule of \(V[x]^0\). Then:

(a) \(N = \bigoplus q N_q\), where the sum is over all monic irreducible polynomials of \(k[x]\) and where \(N_q = N \cap (V^* \otimes B_q)\) is the “largest” \(B_q\)-subcomodule of \(N\);

(b) each \(N_q\) has a finite filtration \(N_q^{(0)} \subseteq N_q^{(1)} \subseteq \cdots \subseteq N_q^{(m-1)} = N_q\) such that \(N_q^{(j)}/N_q^{(j-1)} \cong W_j \otimes B_q^{(0)}\) for some subspaces \(W_j\) of \(V^*\);

(c) if \(q\) is separable, then \(N_q = \bigoplus \bigwedge_{0 < j < m} W_j \otimes z_j B_q^{(0)}\) for a finite chain of subspaces \(W_{m-1} \subseteq W_{m-2} \subseteq \cdots \subseteq W_1 \subseteq W_0\) of \(V^*\).

**Proof.** Consider the inclusion \(N \subseteq V[x]^0 = V^* \otimes k[x]^0 = \bigoplus q (V^* \otimes B_q)\). The first assertion can be proved in several ways. A direct sum decomposition of a coalgebra induces a direct sum decomposition of any module over that coalgebra [5]. In our case the decomposition of \(N\) also follows from the fundamental theorem of finitely generated torsion modules over a principal ideal domain, since \(N\) is a finitely generated \(k[x]\)-module.

If \(N_q^{(j)} = N \cap (V^* \otimes B_q^{(j)})\) for \(j \geq 0\), where \(B_q^{(j-1)} = (k[x]/q^i k[x])^*\) as in Theorem 3.2, then \(N_q^{(j)}/N_q^{(j-1)}\) is a cofree \(B_q^{(0)}\)-subcomodule of \(V^* \otimes B_q^{(0)}\) (cofree because \(k[x]/q k[x]\) is a field and thus \(B_q^{(0)}\) is a simple coalgebra), whence (b).

If \(q\) is separable, then we get [as in remark (1) of Section 3.3] injective
maps

\[ q^1 : W_{j+1} \otimes B_q^{(0)} \cong N_q^{(j+1)}/N_q^{(j)} \hookrightarrow N_q^{(j)}/N_q^{(j-1)} \cong W_j \otimes B_q^{(0)} \]

for \( j \geq 0 \) and some subspaces \( W_j \) of \( V^* \), where \( q^1(w \otimes e) = w \otimes e^q \). This proves (c), since \( B_q^{(0)} = (k[x]/qk[x])^* \) is simple.

5. POLYNOMIALS ASSOCIATED WITH SUBCOMODULES

5.1

In order to avoid any chance of confusion, in the sequel we shall use the operator \( D \) to express the action of \( k[x] \) on \( V[x] \) (see Section 3.1), so that \((Dg) kx^m = g(kx^{m+1})\). The rank of the free \( k[x] \)-module \( V[x] \) is equal to the dimension of \( V \) over \( k \). Since the submodules of maximal rank in \( V[x] \) are precisely the submodules of finite codimension, we see that the results of Sections 2.3 and 2.4 together with Lemma 1.3 establish a bijective correspondence between equivalence classes of regular operator polynomials, submodules of finite codimension of \( V[x] \), and finite dimensional subcomodules of \( V[x]^0 \).

If \( \{f_i\}_{1 \leq i \leq m} \) is a \( k \)-basis of the subcomodule \( N \) of \( V[x]^0 \), then any subset of \( V[x] \) of the form \( \{p_j\}_{1 \leq j \leq m} \), where \( f_i(p_j) = \delta_{ij} \), is linearly independent. The linear map \( r = \sum_{1 \leq i \leq m} f_i \otimes p_i : V[x]^0 \otimes V[x] \to k \), given by \( r(g \otimes q) = \sum_{1 \leq i \leq m} g(p_i)f_i(q) \), defines linear projections \( r_1 : V[x]^0 \to V[x]^0 \) and \( r_2 : V[x] \to V[x] \). If \( q_1 = I - r_1 \) and \( q_2 = I - r_3 \) are the complementary projections, then \( N = \ker r_1 = \ker q_1 \) and \( N' = \ker r_2 = \ker q_2 \).

5.2

\textbf{Lemma 5.2.} \textit{Let \( N \) be a finite dimensional subcomodule of \( V[x]^0 \) with basis \( \{f_i\}_{1 \leq i \leq m} \), and let \( p_j \in V[x] \) be of minimal degree subject to \( f_i(p_j) = \delta_{ij} \). If } \( l - 1 = \max \{\deg p_j\}_{1 \leq i \leq m} \text{, then:} \)

\begin{enumerate}
\item \( r_2 V[x] \subseteq V[x] \), the subspace of polynomials of degree less than \( l \) in \( V[x] \), and \( \dim N < l \dim V \);
\item \( r_2 V[x] = V[x] \) if and only if \( N' = q_2 V[x] \) is equal to \( q_2 x^l V[x] \), in which case \( \dim N = l \dim V \).
\end{enumerate}
Proof. (a): If \( p \in r_2 V[x] \) then \( p = r_2 p = \Sigma_{1 \leq i \leq m} p_i f_i(p) \) and \( \deg p < l \), whence the inclusion. Now, \( \dim N = \dim N^* = \dim V(x)/N' = \dim r_2 V[x] \leq \dim V(x) = l \dim V \).

The “only if” part of (b) is obvious, since \( r_2 V[x] = \ker q_2 = V[x]_l \) clearly implies that \( q_2 V[x] = q_2 x^l V[x] \) and \( \dim N = l \dim V \). To prove the “if” part, suppose that \( q_2 V[x] = q_2 x^l V[x] \). This means that for each \( v \in V[x] \) there is a \( w \in V[x] \) such that \( q_2 v = q_2 x^l w \), i.e., \( q_2 (v - x^l w) = 0 \), so that \( v - x^l w \in \ker q_2 = r_2 V[x] \subseteq V[x]_l \). If \( v \) is in \( V[x]_l \) then \( \deg(v - x^l w) < l \) implies that \( w = 0 \) and hence that \( v \in r_2 V[x] \).

5.3

**Theorem 5.3.** A finite dimensional subcomodule \( N \) of \( V[x]^0 \) is monic (i.e., \( N' = PV[x] \)) for a monic polynomial \( P \in \text{End}(V[x]) \) if and only if the linear map \( T_i : N \rightarrow V[x]_l^* \), \((T_i f)(v) = f(v)\), is bijective. Moreover, if \( \{f_j\}_{1 \leq j \leq m} \) is a basis of \( N \) and \( \{p_j\}_{1 \leq j \leq m} \) in \( V[x] \) is such that \( p_j \) has minimal degree subject to the condition that \( f_i(p_j) = \delta_{ij} \), then \( l - 1 = \max_{1 \leq j \leq m}(\deg p_j) \) and \( P = x^l I - \Sigma_{1 \leq i \leq m} p_i \otimes (D^j f_i)(1) \).

Proof. If \( N' = PV[x] \) for some monic polynomial \( P \) of degree \( l \) in \( \text{End}(V[x]) \), then \( \dim(V[x]/PV[x]) = l \dim V \), \( N = (V[x]/PV[x])^* \), and \( V[x] = V[x]_l \oplus PV[x] \). Hence, \( \dim N = l \dim V \) and \( T_i(f) = 0 \) for \( f \in N \) if and only if \( f = 0 \). Conversely, if \( T_i : N \rightarrow V[x]_l^* \) is bijective, \( \{f_j\}_{1 \leq j \leq m} \) is a basis of \( N \), and \( p_j \in V[x] \) has minimal degree subject to the condition that \( f_i(p_j) = \delta_{ij} \), then \( \dim N = \dim V[x]_l = l \dim V \), \( l - 1 = \max(\deg p_j) \), and \( \{p_j\}_{1 \leq j \leq m} \) is a basis of \( V[x]_l \). But then \( r_2 V[x] = V[x]_l \) and thus \( N' = q_2 V[x] = q_2 x^l V[x] \) by Lemma 5.2(b), where \( q_2 = I - r_2 \) and \( r_2 = \Sigma_i p_i \otimes f_i : V[x] \rightarrow V[x]_l \) as described above. Restrict the map \( q_2 x^l = I x^l - \Sigma_i p_i \otimes D^j f_i : V[x] \rightarrow V[x] \) to \( V \), and then extend the result to a \( k[x] \)-endomorphism \( P \in \text{End}(V[x]) \). It follows that \( PV[x] \subseteq q_2 x^l V[x] = N' \), since whenever \( v = \Sigma_i v_i x^i \in V[x] \) then \( P v = \Sigma_j x^j(v_j) - \Sigma_i p_i f_i(x^i v_j) = \Sigma_j x^j q_2 x^l(v_j) \), and since \( N' = \ker r_2 = q_2 V[x] = q_2 x^l V[x] \) is a \( k[x] \)-submodule of \( V[x] \). Moreover, \( P \) is clearly monic and regular and \( \deg P = l \), so that \( \dim V[x]/PV[x] = l \dim V = \dim V[x]/N' \). Hence, \( PV[x] = N' \).

5.4. Remarks

(a) Among polynomials of minimal degree in their orbits (under the action described in Sections 2.4 and 5.1) a monic operator polynomial \( P \) is unique up to a constant invertible operator factor. This is because the leading coefficient of a nonconstant invertible operator polynomial \( U \) must be singu-
lar, since its determinant is the leading coefficient of \( \det U \). The last result not only gives a characterization of monic operator polynomials and its monic left divisors in terms of subcomodules of \( V[x]^0 \), but also gives a procedure to compute them.

(b) A polynomial \( P \in \text{End}(V[x]) \) is called \textit{comonic} if its constant term is the identity automorphism \( I \) of \( V \). It is not in general uniquely determined by the associated subcomodule \( (P V[x])' \) of \( V[x]^0 \), since it will not be the unique comonic polynomial in its orbit under the right action of \( \text{Aut}(V[x]) \). For any regular polynomial \( P \in \text{End}(V[x]) \) there is an element \( a \in k \) such that \( \det P(a) \neq 0 \). It is then clear that \( P = Q E \) precisely if \( P(x + a)P(a)^{-1} = Q(x + a)Q(a)^{-1}Q(a)E(x + a)P(a)^{-1} \). Since \( P(x + a)P(a)^{-1} \) and \( Q(x + a)Q(a)^{-1} \) are comonic, we see that the problem of finding divisors of regular polynomials reduces to that of finding comonic divisors of comonic polynomials.

(c) The map \( \beta : \text{End}(V[x]) \to \text{End}(V[x]), \beta P = x^{\deg P} P(1/x) \), gives a bijection between comonic polynomials and monic polynomials not divisible by \( x \), hence between comonic subcomodules and the monic subcomodules not containing all of \( V^* \). This fact will be crucial in the analysis of the comonic case. Let us first give a characterization of comonic subcomodules of \( V[x]^0 \).

5.5

**Theorem 5.5.** A subcomodule \( N \) of \( V[x]^0 \) is comonic \( (N^* = PV[x]) \) if and only if \( N \cap V[x]^0 = 0 \), i.e., if and only if \( D : N \to N \) is bijective.

**Proof.** If \( P = P_0 + xQ \) then \( \det P = \det P_0 + xq \). Let \( p \in k[x] \) be monic of minimal degree such that \( (\det P)V[x] \subseteq pV[x] \subseteq PV[x] \), i.e., such that \( N \subseteq V^* \otimes (k[x]/pk[x])* \). Then \( p \) divides \( \det P \). Moreover, \( Ip = PE \) by Proposition 2.2, so that \( p^n = \det P \det E \) and \( \det P \) divides \( p^n \). Hence, \( x \) divides \( \det P \) if and only if it divides \( p \). Thus, \( \det P_0 \neq 0 \) precisely if \( N \cap V[x]^0 = N \cap (V^* \otimes k[x]^0) = 0 \). \( \blacksquare \)

6. **MONIC EXTENSION AND REPRESENTATION OF COMONICS**

6.1

The inclusion map \( T : V[x]^0 \to V[x]^* \), \( T(f)v = f(v) \), takes the form

\[
T(f) = \sum_{j \geq 0} f(x^j)z_j = \sum_{j \geq 0} (D^j f)(1) z_j
\]
in terms of the topological basis \( \{z_j | j \geq 0\} \) of \( V[x]^* \). Composition with the restriction to the subspace \( V[x]_i \) of polynomials of degree less than \( i \) gives a linear map

\[
T_i : V[x]^0 \to V[x]^* \to V[x]^* ,
\]

\[
T_i f = \sum_{0 \leq j < i} f(x^j) z_j = \sum_{0 \leq j < i} (D^j f)(1) z_j , \quad \text{i.e., } \ T_i f(v) = f(v) \quad \text{for } v \in V[x] .
\]

The restriction of \( T_i \) to a finite dimensional submodule \( N \) of \( V[x]^0 \) is of course injective for large integers \( i \). The index of \( N \) is the least positive integer \( \text{ind } N = I \) for which \( T_i|_N : N \to V[x]^*_i \) is injective, i.e., for which \( \ker T_i|_N = \cap_{0 \leq i < I} \ker \chi_j|_N = 0 \), where \( \chi_j : V[x]^0 \to V^* \) is given by \( \chi_j(f) = (D^j f)(1) = f(x^j) \). It will turn out in Theorem 6.2 that the index of \( N \) is equal to the minimal degree of a polynomial \( P \in \text{End}(V[x]) \) such that \( N = (PV[x])' \).

Thus, Theorem 5.3 says that \( N \) is monic of index \( l \), i.e., \( N' = PV[x] \) for a monic polynomial \( P \) of degree \( l \), if and only if \( T_l : N \to V[x]^*_l \) is bijective. Our representation theorem for comonic operator polynomials will grow out of an “extension to monics” process.

We are now ready to explicitly construct from a basis of any comonic subcomodule \( N \) of \( V[x]^0 \) a comonic polynomial \( P \in \text{End}(V[x]) \) for which \( N = (PV[x])' \). If \( N \) is comonic of index \( l \) in \( V[x]^0 \), then \( D : N \to N \) is bijective by Theorem 5.5 and \( T_l : N \to V[x]^*_l \), \( (T_l f) u = f(u) \), is injective. It follows that \( l \) is also the least positive integer such that the map \( T_{-l} : N \to V[x]^*_l \), defined by \( T_{-l} f = \sum_{0 \leq j < l} (D^{-j} f)(1) z_j^* \), is injective. Thus, if \( \rho : N \to V[x]^* \) is the linear map defined by \( \rho(f)(x^m) = (D^{-m} f)(1) \), then \( \tilde{N} = \rho(N) \) is a subspace of \( V[x]^* \) with \( \dim \tilde{N} = \dim N \). It is a comonic subcomodule of index \( l \) in \( V[x]^0 \) by Lemma 4.2 and Theorem 5.5, since it is \( D \)-invariant, i.e., \( D \rho(f) = \rho(D^{-1} f) \), since \( 0 = D \rho(f) = \rho(D^{-1} f) \) implies \( D^{-1} f = 0 \) and hence \( f = 0 \), and since \( T_l : \tilde{N} \to V[x]^*_l \) is injective.

6.2

**Theorem 6.2.** Let \( N = (PV[x])' \) be a comonic subcomodule of index \( l \) in \( V[x]^0 \), and let the linear map \( \rho : N \to V[x]^* \) be defined by \( \rho(f)(x^m) = (D^{-m} f)(1) \). Then:

(a) \( \tilde{N} = \rho(N) \) is a comonic subcomodule of index \( l \) in \( V[x]^0 \), and \( \tilde{N} = N \).

(b) \( \text{deg } P = l \), and \( (QV[x])' = \tilde{N} \oplus \tilde{W} \) for a subcomodule \( \tilde{W} = \bigoplus_{0 \leq j < l} W_j z_j \) of index \( \leq l \) in \( V[[x]] \), where \( Q(x) = x^l P(1/x) \).

(c) \( P = I - \sum_{1 \leq i \leq m} x P_i \otimes (D^{-1} f_i)(1) \) for any basis \( \{f_i\}_{1 \leq i \leq m} \) of \( N \) and any polynomials \( P_i \) of minimal degree in \( V[x]_l \) for which \( f_i(p_j) = \delta_{ij} \) and \( W p_j = 0 \), where \( W = \bigoplus_{r+l=-l} W_r z_r \).
Proof. (a): By Section 6.1, $\tilde{N} = \rho(N)$ is a comonic subcomodule of index 1 in $V[x]^0$. It follows from $D^{-1}p(f) = D^{-1}\rho(D^{-1}Df) = D^{-1}D\rho(Df)$ that $\rho^2(f(x^n)) = (D^{-n}\rho(f))(1) = \rho(D^n(f))(1) = f(x^n)$. This clearly implies that $\rho^2(f) = f$ and $\tilde{N} = N$.

(b): The structure Theorem 4.3 will be useful here. Suppose the polynomial $P$ such that $N = (PV[x])'$ is comonic of minimal degree $s$; thus we may write $P = I - \Sigma_{1 \leq i \leq s} x_iA_i$. If $f \in N$, i.e., if $fP = f - \Sigma_{1 \leq i \leq s} A_i^*D_i f = 0$, then $D^{-s-j}f - \Sigma_{1 \leq i \leq s} A_i^*D_i^{-s-j}f = 0$ for $j \geq 0$. Thus, $gQ = D^jg - \Sigma_{1 \leq i \leq s} A_i^*D^{-j}g = 0$ for the corresponding $g = \rho(f) \in \tilde{N}$. Hence, $N \subseteq (QV[x])' \subseteq V[x]^0$. Now, $L = (QV[x])' = M \oplus W$ by Theorem 4.3, where $M$ is the maximal comonic subcomodule of $L$ and $W = L \cap V[x]^0$. Thus, $\tilde{N} \subseteq M$. Since $M$ is comonic and since $Q = x^*P(1/x)$, arguments as above show that the subspace $\tilde{M} = \rho(M)$ of $V[x]^*$ is a subcomodule of $(PV[x])'$ = $N$ with $\text{dim } M = \text{dim } M$: if $g \in M$ and $f = \rho(g) \in \tilde{M}$ then $0 = D^{-s-j}gQ = D^{-j}g - \Sigma_{1 \leq i \leq s} D_i^{-1}A_i^*g; \text{ in particular } 0 = (D^{-j}g)(1) - \Sigma_{1 \leq i \leq s}(D^{-j}A_i^*g)(1) = f(x^j) - \Sigma_{1 \leq i \leq s} A_i^*f(x^j) = (f - \Sigma_{1 \leq i \leq s} D_i A_i^*f)(x^j) = P^0f(x^j)$ for every $j \geq 0$, and hence $f = \rho(g) \in N = (PV[x])'$. Now dim $N \leq \text{dim } M = \text{dim } \tilde{M} \leq \text{dim } N$ shows that $\tilde{M} \subseteq N$ and $\tilde{N} = M$. By Theorem 4.3 and a top-down inductive argument we may assume that $W = \bigoplus_{0 \leq j < s} W_{s-j}$ for some chain of subspaces $0 = W_s \subseteq W_{s-1} \subseteq \cdots \subseteq W_1 \subseteq W_0$ in $V[*]$. The minimality of $s = - \text{deg } P$ implies that $s = - \text{deg } P = \text{deg } Q = \text{ind } L = \text{ind } M = \text{ind } N = l$.

(c): The above construction of $\tilde{N}$ from $N$ gives a bijective correspondence between bases of $N$ and bases of $\tilde{N}$. Thus, if $\{h_i\}_{1 \leq i \leq m}$ is a basis of $N$ then $\{g_i = \rho(h_i)\}_{1 \leq i \leq m}$ is a basis of $\tilde{N}$. Let $q_i \in V[x]^0$ be of minimal degree subject to the conditions $g_i(q_j) = \delta_{ij}$ and $Wq_j = 0$. Then Theorem 5.3 gives $Q = x^iI - \Sigma_{1 \leq i \leq m} q_i \otimes (D_i^*g_i)(1)$, since $D_i^*W = 0$, and hence $P = x^iQ(1/x) = I - \Sigma_{1 \leq i \leq m} x_iq_i(1/x) \otimes (D_i^*h_i)(1)$. Now let $p_i = x^{i-1}q_i(1/x)$ and $h_i = D_i^{-1}f_i$; then $\{f_i\}_{1 \leq i \leq m}$ is a basis of $N$, and $f_i(p_i) = f_i(x^{i-1}q_i(1/x)) = \Sigma(D_i^{-1}f_i)(q_{ik}) = \Sigma(D_i^{-1}h_i)(q_{ik}) = g_i(q_{ik}) = \delta_{ik}$. Moreover, if $W = \bigoplus_{r+s = i-1} W_{r+s}$, then $Wp_j = Wq_j = 0$. Thus, $P = x^iQ(1/x) = I - \Sigma_{1 \leq i \leq m} x_iq_i(1/x) \otimes (D_i^{-1}f_i)(1)$.

6.3

**Corollary 6.3.** If $N$ is a comonic subcomodule of index 1 in $V[x]^0$, then there is a subcomodule $W$ of index $\leq l$ in $V[x]^0$ such that $N \cap W = 0$ and $N \oplus W$ is monic of index 1.

**Proof.** By Theorem 6.2(a), $N$ is comonic if and only $\tilde{N}$ is. Now apply Theorem 6.2(b) to $\tilde{N}$.
6.4

The construction of the polynomial \( P \) for a comonic subcomodule \( N \) of \( V[x]^0 \) amounts to the choice of a left inverse \( L_i : V[x]^*_{l_0} \rightarrow N \) for \( T_i : N \rightarrow V[x]^*_{l_0} \) for which \( L_i W = 0 \). This corresponds to the choice of a special left inverse in [4, Chapter 2]. The subspace \( W = \oplus_{r+s=-1} W_z \) of \( V[x]^* \) is not a subcomodule in general. Given a basis \( \{ f_i \}_{1 \leq i \leq m} \) of \( N \), we must have

\[
L_i(g) = \sum_{1 \leq j \leq m} g(p_j) f_j
\]

for some \( p_j \in V[x]_l \cong V[x]^*_{j} \). In particular, \( L_i T_i(f) = \sum_{1 \leq j \leq m} T_i(f)(p_j) f_j = \sum_{1 \leq j \leq m} f_i(p_j) f_j \) for \( f \in V[x]^0 \), since \( T_i(f)(p) = f(p) \) for every \( p \in V[x]_l \). Thus, the conditions \( L_i T_i|_N = 1 \) and \( L_i|_W = 0 \) are equivalent to \( f_i(p_j) = \delta_{ij} \) and \( W(p_j) = 0 \). Then the linear map \( r_i = \sum_{1 \leq j \leq m} p_j \otimes f_i = L_i T_i : V[x]^0 \rightarrow V[x]^0 \) is a projection onto \( N \) (as in Section 5.1), and since \( (D^{-1} f_i)(x p_j) = f_i(p_j) = \delta_{ij} \), so is \( R_i = \sum x p_i \otimes D^{-1} f_i = D^{-1} L_i T_i D : V[x]^0 \rightarrow V[x]^0 \). If \( P = I - \sum x p_i \otimes (D^{-1} f_i)(1) = I - R_i(1) \) then clearly \( PV[x] \subseteq N' \), and we have seen in Theorem 6.2 that in fact \( PV[x] = N' \), i.e., \( (PV[x])' = N \).

6.5

Let us illustrate the ideas of this section with some examples. An operator polynomial \( P \) of degree \( l \) determines a subcomodule of \( V[x]^0 \), regarded as the space of sequences of elements of \( V^* \), by giving a system of linear equations to be satisfied by every block of \( l + 1 \) successive terms of such a sequence. \( P \) is manic (up to an invertible scalar factor) if and only if this system of relations describes the last term of every such block as a function of those that precede. \( P \) is comonic if and only if the system describes the first term of every such block in terms of those that follow.

If \( V = k \), then by Remark (4) of Section 3.3 the subcoalgebra associated with \( q = x - 2 \) is \( B_q^{(0)} = ke(2) \subset k[x]^0 \), where \( e(2) = \sum n \geq 0 2^n z_n \), represents the geometric progression \( \{2^n n \geq 0 \} \). The subcoalgebra associated with \( q^2 = (x - 2)^2 \) is \( B_q^{(1)} = ke(2) \oplus ke(2), \) where \( e(2) = z_n e(2) \) represents \( \{2^2 n - 1 \} n \geq 0 \). General element \( be(2) + ce(2) = (b z_0 + c z_1) e(2) \) of \( B_q^{(1)} \) represents the "mixed arithmetic geometric" progression \( \{2^{n-1}(2b + nc) n \geq 0 \} \).

If \( V = k \oplus k \), then the space of sequences of pairs \((u 2^n, 2^{n-1}(2b + nc) n \geq 0 \} \) corresponds to the subcomodule \( N = k(e(2), 0) \oplus k(0, e(2)) \oplus k(0, e(2)) \subset V[x]^0 \), which is comonic and not manic, with polynomial

\[
P = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} x^2 - \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

We can get a monic (but not comonic) subcomodule \( M = W \oplus N \), where
$W = k(1,0)z_0$, with monic polynomial

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x^2 - \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}.$$

To get $N\hat{=}q(N)$ we have to "reverse" $e(2)$ and $e_1(2)$:

$$\rho(e(2)) = \sum_{m \geq 0} (D^{-m}e(2))(1)z_m = e(\frac{1}{2}),$$

$$\rho(e_1(2)) = \sum_{m \geq 0} (D^{-m}e_1(2))(1)z_m = -z_1* e(\frac{1}{2}) = -e_1(\frac{1}{2}),$$

since $(D^{-m}e(2))(1) = 1/2^m$ and $(D^{-m}e_1(2))(1) = -m/2^{m+1}$. Thus

$$N\hat{=} = k(1,0)\mathbf{e}(\frac{1}{2}) \oplus k(0,1)\mathbf{e}(\frac{1}{2}) \oplus k(0,1)e_1(\frac{1}{2}),$$

with comonic polynomial

$$\tilde{P} = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} x^2 - \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

To extend $N\hat{=}$ to a monic subcomodule we add the simple subcomodule $\tilde{W} = k(1,0)z_0$. The resulting subcomodule $N\hat{=} \oplus \tilde{W}$ is monic, with polynomial

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x^2 - \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{4} \end{pmatrix},$$

and $P = x^2Q(1/x)$ as predicted by Theorem 6.2.

6.6

As another example let us consider a trigonometric comodule. If the polynomial $q = x^2 + 1$ is irreducible over $k$, then $k[x]/qk[x]$ is a field and $B_q^{(0)} = (k[x]/qk[x])^*, \text{ is a simple coalgebra with basis } \{c, s\}$, where $c(x^{2i}) = (-1)^i = s(x^{2i+1})$ and $c(x^{2i+1}) = 0 = s(x^{2i})$. Thus, $Dc = -s$ and $Ds = c$, $\delta A = c \otimes c - s \otimes s$ and $\delta S = s \otimes c + c \otimes s$, $\epsilon c = 1$ and $\epsilon s = 0$. By Theorem 3.2(c) the irreducible component $B_q$ of $k[x]^*$ has basis $\{c*z_j, s*z_j\}_{0 \leq j}$. If $\dim V = 3$ and $\{v^0, v^1, v^2\}$ is a basis of $V^*$, then by Theorem 5.5 the subspace $N$ of $V[x]^0$ spanned by $\{f_0 = v^0c - v^1s, f_1 = v^1c + v^0s, f_2 = \}$
\( v^2 c + f_0^* z_1, f_3 = v^2 s + f_1^* z_1 \) is a comonic subcomodule, since \( Df_0 = -f_1, \)
\( Df_1 = f_0, Df_2 = f_0 - f_3, Df_3 = f_1 + f_2. \) Moreover, \( T_2 : N \to V[x]^2 \) is injective, since \( T_2 f_0 = v^0 z_0 - v^1 z_1, T_2 f_1 = v^1 z_0 + v^0 z_1, T_2 f_2 = v^2 z_0 + v^1 z_1, T_2 f_3 = (v^1 + v^2) z_1, \) while \( \ker T_1 = k z_1, \) so that \( \text{ind} N = 2. \) It is now easy to see that \( W_1 = 0 \) and \( W_0 = kv^0 + kv^1; \) hence \( W = (kv^0 + kv^1) z_1. \) Let \( v_j \in V \) be such that \( v_j(v) = \delta_j^1. \) Then \( f_j(p_j) = \delta_j^i \) and \( W p_j = 0 \) if \( p_j = v_j \) for \( 0 < j < 2 \) and \( p_3 = xv_2. \) Thus, identifying \( \text{End}(V) \) with \( V \otimes V^*, \) the comonic polynomial associated with \( N \) is

\[
P = I - xv_0 \otimes v^1 + xv_1 \otimes v^0 - xv_2 \otimes v^0 - x^2 v_2 \otimes (v^1 - v^2),
\]
since \( D^{-1}f_0 = f_1, \) \( D^{-1}f_1 = -f_0, \) \( D^{-1}f_2 = f_0 + f_3, \) \( D^{-1}f_3 = f_1 - f_2. \)

As a monic example consider the subcomodule \( N \) of \( V[x]^2 \) spanned by \( f_0 = v^0 c, f_1 = v^0 s, f_2 = v^1 c + f_0^* z_1, f_3 = v^1 s + f_1^* z_1, \) where \( \text{dim} V = 2 \) and where the pair \( \{v^0, v^1\} \) is a basis of \( V^*. \) Then \( N \) is monic and comonic, since \( T_2 : N \to V_2[x]^* \) and \( D : N \to N \) are both bijective. Use either Theorem 5.3 or 6.2 to find the associated polynomial \( P. \) If \( v_j(v) = \delta_j^1, \) then \( p_0 = u_0, p_1 = xu_0 - u_1, p_2 = u_1, p_3 = xv_1 \) satisfy \( f_j(p_j) = \delta_j^i. \) This gives

\[
P = x^2 I + uv_0 \otimes v^0 + u_1 \otimes v^1 - xv_1 \otimes 2v^0 = (x^2 + 1) I - 2xv_1 \otimes v^0.
\]

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