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A TOTAL-CHROMATIC NUMBER ANALOGUE OF PLANTHOLT'S THEOREM

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The total chromatic number $\chi_T(G)$ of a graph G is the least number of colours needed to colour the edges and vertices of G so that no two adjacent vertices receive the same colour, no two edges incident with the same vertex receive the same colour, and no edge receives the same colour as either of the vertices it is incident with.

Let $n \geqslant 1$, let J be a subgraph of K_{2n}, let $e = |E(J)|$ and let $j(J)$ be the maximum size of a matching in J. Then

$$\chi_T(K_{2n} \setminus E(J)) = 2n + 1$$

if and only if $e + j \leqslant n - 1$.

1. Introduction

An *edge-colouring* of a graph G is a map $\phi: E(G) \to \mathscr{C}$, where $\mathscr{C}$ is a set of colours, such that no two adjacent edges receive the same colour. The *chromatic index* (or edge-chromatic number) $\chi'(G)$ of G is the least value of $|\mathscr{C}|$ for which G has an edge-colouring. A famous theorem of Vizing [5] states that $\Delta(G) \leqslant \chi'(G) \leqslant \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G, G being a simple graph.

A *total-colouring* of G is a map $\theta: E(G) \cup V(G) \to \mathscr{C}$ such that no incident or adjacent pair of elements of $E(G) \cup V(G)$ receive the same colour. The *total-chromatic number* $\chi_T(G)$ is the least value of $|\mathscr{C}|$ for which G has a total colouring. A long-standing conjecture of Behzad [1] and, independently, of Vizing [6] is that $\Delta(G) + 1 \leqslant \chi_T(G) \leqslant \Delta(G) + 2$ if G is a simple graph. The lower bound here is trivial but whether the upper bound is true is today still a fascinating mystery.

It is well-known that $\chi'(K_{2n+1}) = \Delta(K_{2n+1}) + 1 = 2n + 1$ and that $\chi'(K_{2n}) = \Delta(K_{2n}) = 2n - 1$. Plantholt [4] proved the following result about the chromatic index of the graph obtained from K_{2n+1} by removing just a few edges.

Theorem 1. *Let $n \geqslant 1$, let J be a subgraph of K_{2n+1} and let $e = |E(J)|$. Then*

$$\chi'(K_{2n+1} \setminus E(J)) = 2n + 1$$

if and only if $e \leqslant n - 1$.

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It is also well-known that $\chi_T(K_{2n}) = \Delta(K_{2n}) + 2 = 2n + 1$ and that $\chi_T(K_{2n+1}) = \Delta(K_{2n+1}) + 1 = 2n + 1$. Our object is to prove an analogue of Plantholt's theorem about the total chromatic number of complete graphs with a few edges removed; complete graphs of odd order present no problems, and it is complete graphs of even order that we have to consider.

Theorem 2. *Let $n \geq 1$, let J be a subgraph of K_{2n}, let $e = |E(J)|$ and let j be the maximum size (i.e. number of edges) of a matching in J. Then*

$$\chi_T(K_{2n} \setminus E(J)) = 2n + 1$$

if and only if $e + j \leq n - 1$.

2. Proof of Theorem 2

We first prove the necessity.

Proof of necessity. It is well-known that $\chi_T(K_{2n}) = 2n + 1$, and so it follows that $\chi_T(K_{2n} \setminus E(J)) \leq 2n + 1$. We show that if $e + j \leq n - 1$ then $\chi_T(K_{2n} \setminus E(J)) = 2n + 1$. We do this by assuming instead that $\chi_T(K_{2n} \setminus E(J)) = 2n$ and showing that then $e + j \geq n$. So suppose that $K_{2n} \setminus E(J)$ is totally-coloured with $2n$ colours.

If a colour is used to colour an odd number of vertices, then there is one vertex at which it does occur at all (neither on the vertex itself, nor on an edge incident with the vertex). We show first that the number of colours which are used to colour an odd number of vertices is at least $2n - 2j$. Let the colours be $c_1, \ldots, c_{2n}$ and let the number of vertices which are coloured c_i in a total-colouring of $K_{2n} \setminus E(J)$ be x_i $(1 \leq i \leq 2n)$. Then

$$x_1 + x_2 + \cdots + x_{2n} = 2n.$$

We may suppose that

$$x_1 \geq x_2 \geq \cdots \geq x_r \geq 2 > x_{r+1} \geq \cdots \geq x_{2n} \geq 0$$

for some r. In $K_{2n} \setminus E(J)$ there are no edges between any of the x_i vertices of colour c_i, and so it follows that

$$\left\lfloor \frac{x_1}{2} \right\rfloor + \cdots + \left\lfloor \frac{x_r}{2} \right\rfloor \leq j.$$

Suppose there are z odd numbers in $\{x_1, \ldots, x_r\}$ and so $r - z$ even numbers. Then it follows that

$$x_1 + \cdots + x_r \leq 2j + z.$$

Therefore

$$x_{r+1} + \cdots + x_{2n} \geq 2n - 2j - z.$$

Since the values of $x_{r+1}, \ldots, x_{2n}$ are either 0 or 1, there are at least $2n-2j-z$ odd numbers x_r amongst $x_{r+1}, \ldots, x_{2n}$. Therefore at least $2n-2j$ of $x_1, \ldots, x_{2n}$ are odd.

Call a pair (c, v), where c is a colour, v is a vertex, and c is used either to colour the vertex or to colour an edge incident with the vertex, a *colour-vertex pair*. Then there are $2n-2j$ colours which are each associated with at most $2n-1$ colour-vertex pairs, and so altogether these $2n-2j$ colours are associated with at most $(2n-2j)$ $(2n-1)$ colour-vertex pairs. The remaining $2j$ colours are each associated with at most $2n$ colour-vertex pairs, and thus altogether they are associated with at most $2j \cdot 2n$ colour vertex pairs. Thus there are at most

$$(2n-2j)(n-1) + 2n \cdot 2j = 4n^2 - 2n + 2j$$

colour-vertex pairs.

However since $K_{2n} \backslash E(J)$ is totally-coloured using these $2n$ colours, there are altogether $(2n)^2 - 2e$ colour-vertex pairs.

Therefore

$$(2n)^2 - 2e \leq (2n)^2 - 2n + 2j,$$

so that

$$n \leq e + j,$$

as required. □

Proof of sufficiency. In order to prove the sufficiency, by adding in edges if necessary, we may without loss of generality suppose that either $e+j=n$ or n is odd and J consists of $\frac{1}{2}(n+1)$ independent edges (so that $e+j=n+1$), and then show that $K_{2n} \backslash E(J)$ can be totally-coloured with $2n$ colours.

Suppose for the moment that $e+j=n$. We may suppose that J has no isolated vertices. Let $\{v_1, \ldots, v_y\} = V(J)$ and $\{v_{y+1}, \ldots, v_{2n}\} = V(K_{2n}) \backslash V(J)$. We associate with $K_{2n} \backslash E(J)$ a multigraph H^{**} on the $2n+2$ vertices v^{**}, v^*, $v_1, \ldots, v_{2n}$. Let M be a maximum matching in J and let the vertices of M be $v_1, \ldots, v_{2j}$; let the edges of M be $e_1, \ldots, e_j$ and, for $1 \leq i \leq j$, let e_i join v_i to v_{j+i}. Let H^* be the graph formed from $((K_{2n} \backslash (E(J) \backslash M)) \cup \{v^*\}$ by joining v^* to each of $v_{2j+1}, \ldots, v_{2n}$. Then $\Delta(H^*) = 2n$; since $2n - d_{H^*}(v^*) = 2j$ and $\sum_{v \in V(J)} (2n - d_{H^*}(v)) = 2e$ (recall the edges of M are in H^*) it follows that

$$\sum_{v \in \{v^*, v_1, \ldots, v_{2n}\}} (2n - d_{H^*}(v)) = 2(e+j) = 2n.$$

Finally form H^{**} by joining v^{**} to each $v \in \{v^*, v_1, \ldots, v_{2n}\}$ by $2n - d_{H^*}(v)$ edges (so some of these edges will be multiple); then $d_{H^{**}}(v) = 2n (\forall v \in \{v^*, v_1, \ldots, v_{2n}\})$. Then in H^{**}, v^* and v^{**} are joined by $2j$ edges. For $y \leq x \leq 2n$, let H_x^{**} denote the subgraph of H^{**} induced by $\{v^*, v^{**}, v_1, \ldots, v_y\}$. Then each vertex of $\{v^*, v_1, \ldots, v_y\}$ is joined in H^{**} to each of the $2n-y$ vertices of $V(H^{**}) \backslash \{v^*, v^{**}, v_1, \ldots, v_y\}$, and so in H_y^{**}, each vertex of $V(H_y^{**})\setminus\{v^{**}\}$ has degree y. Finally observe that each of $v_1, \ldots, v_y$ is not joined to v^* but is joined to v^{**}.

If $e+j=n+1$ and J consists of $\frac{1}{2}(n+1)$ independent edges, we vary this construction slightly. We let v^{**} be joined to v_i and v_{i+j} $(2 \leqslant i \leqslant j)$ as before, and to v^*, but not to v_1 or v_{j+1}. v_1 and v_{1+j} are joined by two edges, e_1 and e_1', instead of by just one. It again follows that $d_{H^{**}}(v)=2n(\forall v \in \{v^{**}, v^*, v_1, \ldots, v_{2n}\})$.

We shall show that H^{**}, and therefore H^*, is edge-colourable with $2n$ colours, with the edges of M and the edges $v^*v_{2j+1}, \ldots, v^*v_{2n}$ all receiving different colours. From this edge-colouring of H^* we obtain a total-colouring of $K_{2n}\setminus E(J)$ with the $2n$ colours by retaining the colours on all the edges of $K_{2n}\setminus E(J)$, colouring the vertex v_i with the colour of the edge v^*v_i $(2j+1 \leqslant i \leqslant n)$, and colouring the vertices v_i and v_{i+j} with the colour of the edge v_iv_{i+j} $(1 \leqslant i \leqslant j)$.

Suppose first that $e+j=n$. Then we edge-colour H_y^{**} as follows. We first colour v_iv_{i+j} with colour c_i $(1 \leqslant i \leqslant j)$. We colour the $2j$ edges joining v^* to v^{**} with colours $c_1, \ldots, c_{2j}$ and the remaining $2n-2j$ edges on v_y^{**} are coloured $c_{2j+1}, \ldots, c_{2n}$. After that the remaining edges of H_y^{**} are coloured one by one, greedily: if an edge e^* is uncoloured, then at most $2(y-1)$ colours are used on edges incident with the vertices at each end. Since $e+j=n$, J consists of j independent edges on $2j$ vertices, and a further $e-j=n-2j$ edges, each of which is incident with at least one of the $2j$ vertices above. Therefore the further edges are on a further at most $e-j=n-2j$ vertices, and thus J has at most $(n-2j)+2j=n$ vertices altogether. Therefore $y=|V(J)| \leqslant n$. Therefore $2(y-1) \leqslant 2(n-1) < 2n$, so there is a colour available to colour e^* with.

In the case when n is odd and J consists of $\frac{1}{2}(n+1)$ independent edges, then $y=n+1$ and the argument above does not work. However, in that case, H_y^{**} contains a 1-factor F which does not include any edge of M but does include an edge from v^* to v^{**}. We colour the edges of F with the colour c_{2j}. Apart from this, we proceed as before; this time there are at most $2(y-1)-1=2n-1<2n$ colours used on edges incident with the vertices at each end of e^*, so again there is a colour available to colour e^* with.

We now prove the following lemma.

Lemma 3. *For $x \geqslant y$, an edge-colouring of H_x^{**} with colours $c_1, \ldots, c_{2n}$ can be extended to an edge-colouring of H^{**} with the same colours if and only if each colour is used on at least $x-n+1$ edges of H_x^{**}.*

We resume the proof of the sufficiency in Theorem 2 after proving Lemma 3.

Proof of Lemma 3.

1. Necessity in Lemma 3. Suppose that an edge-colouring of H_x^{**} with colours $c_1, \ldots, c_{2n}$ can be extended to an edge-colouring of H^{**} with the same colours.

Then for each i, $1 \leq i \leq 2n$, the number of edges not in H_x^{**} which are coloured c_i is at most $2n-x$. The total number of edges in H^{**} which are coloured c_i is $n+1$. Therefore at least $n+1-(2n-x)=x-n+1$ edges of H_x^{**} are coloured c_i.

2. Sufficiency in Lemma 3. Suppose H_x^{**} is edge-coloured with $c_1, \ldots, c_{2n}$, and that, for $1 \leq i \leq 2n$, colour c_i occurs on at least $x-n+1$ edges of H_x^{**}. We shall extend this to an edge-colouring of H_{x+1}^{**} with the same colours in such a way that each colour will occur on at least $(x+1)-n+1=x-n+2$ edges. Iterating this will eventually give the required edge-colouring of H^{**}.

In order to extend H_x^{**} we construct a bipartite graph B as follows. The vertex sets of B are $\{v^{*\prime}, v_1', \ldots, v_x'\}$ and $\{c_1', \ldots, c_{2n}'\}$. A vertex v' is joined in B to a vertex c' by an edge if in H_x^{**} there is no edge coloured c incident with v. Each vertex of H_x^{**} (except v^{**}) has degree x, and so in B each v'-vertex has degree $2n-x$. Each colour c_i is used in H_x^{**} on at least $x-n+1$ edges, and so there are at least $2(x-n+1)$ vertices in H_x^{**} which are incident with an edge coloured c_i. Therefore c_i fails to be on any edges incident with at most $x+2-2(x-n+1)=2n-x$ vertices of H^* (it is incident with v^{**}). Therefore in B each c'-vertex has degree at most $2n-x$.

By König's theorem [3], B can be edge-coloured with $2n-x$ colours. Let α be one colour in such an edge-colouring. Then α occurs on every vertex of degree $2n-x$. If an edge $c'v'$ is coloured α, we colour the edge $v_{x+1}v$ in H_{x+1}^{**} with the colour c. It is easy to check that we obtain this way a proper edge-colouring of H_{x+1}^{**}. In this edge-colouring, any colour c which occurred on only $x-n+1$ edges of H_x^{**} gives rise in B to a vertex c' of degree $2n-x$, and so c' has an edge coloured α on it; therefore c is assigned to some edge incident with v_{x+1}. It follows that in H_x^{**}, each colour does occur on at least $x-n+2$ edges, as required.

The sufficiency in Lemma 3 now follows. □

We now resume the proof of the sufficiency in Theorem 2.

In the case when $e+j=n$, we have $y \leq n$, and so the condition that each colour occurs on at least $x-n+1$ edges reduces in this case to the condition that each colour occurs on some edge of H_y^{**}. But each colour occurs on an edge of H_y^{**} incident with v^{**}. By Lemma 3, therefore, H^{**} can be edge-coloured with $2n$ colours. The colours used on M and the colours used on the edges v^*v_i $(2j<i\leq 2n)$ are all different, and so this edge-colouring corresponds to the required total-colouring of $K_{2n}\setminus E(J)$, as described earlier.

If $e+j=n+1$ and J consists of $\frac{1}{2}(n+1)$ independent edges, we have $y=n+1$, and so the condition that each colour occurs on at least $x-n+1$ edges reduces in this case to the condition that each colour occurs on at least two edges of H^{**}. Each colour occurs on an edge incident with v^{**}. It may be necessary to modify our original edge-colouring of H_y^{**} if some colour, say c, is not used on any edge of $H_y^{**}\setminus\{v^{**}\}$. In that case, we find an edge e not in M whose colour occurs on more than one edge of $H_y^{**}\setminus\{v^{**}\}$ and which is not adjacent to the edge coloured c incident with v^{**}, and we colour e with c. We repeat this as necessary. It is easy to check that this can always be done. The argument now proceeds as above.

This proves the sufficiency in Theorem 2. □

3. Concluding remarks

We have the following corollary of Theorem 2.

Corollary 4. *Under the hypothesis of Theorem* 2,

$$\chi_T(H) = \begin{cases} \Delta(H)+2 & \text{if } e+j \leqslant n-1, \\ \Delta(H)+1 & \text{if } 2n-1 \geqslant e+j \geqslant n, \end{cases}$$

where $H = K_{2n}\setminus E(J)$.

Proof. We may assume that J has no isolated vertices. If $2n-1 \geqslant e+j$ then J consists of j independent edges on $2j$ vertices, and a further $e-j \leqslant 2n-1-2j$ edges, each of which is incident with at least one of the $2j$ vertices above. Therefore the further edges are on a further at most $2n-1-2j$ vertices, and so J has at most $2n-1$ vertices altogether. Therefore $\Delta(H) = 2n-1$. Corollary 4 now follows from Theorem 2. □

We remark that our proof of Theorem 2 is very like the proof of Theorem 1 given by Chetwynd and Hilton in [2].

The following conjecture may describe the 'next step' after Theorem 2.

Conjecture. *Let* $n \geqslant 1$ *and let* J *be a spanning subgraph of* K_{2n} *such that* $d_J(v) \geqslant 1 (\forall v \in V(K_{2n}))$. *Let* j_1 *be the maximum size of a 'submatching'* M *of* J (M *is a submatching of* J *if* M *is a matching and* $d_{J\setminus M}(v) \geqslant 1$ $(\forall v \in V(K_{2n}))$. *Then*

$$\chi_T(K_{2n}\setminus E(J)) \leqslant 2n$$

if and only if

$$e + j_1 \geqslant 2n-1.$$

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