Perron’s method for quasilinear hyperbolic systems, Part II

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Abstract

In part I (P. Smith, Perron’s method for quasilinear hyperbolic systems, part I, J. Math. Anal., in press) of this paper we defined a notion of viscosity solution (sub- (super-)solution) for these systems, proved a comparison principle for viscosity sub- and supersolutions. Here, in part II, we prove existence of viscosity solutions to the Cauchy problem, using a Perron-like method, for long time, and for all time.

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1. Introduction

Here in part II, continue with the program initiated in part I of this paper. At this point we are ready to develop the Perron method for our systems.

2. Perron’s method

Having laid the preliminaries, we now establish the existence of a viscosity solution of the Cauchy problem for our quasi-linear symmetric hyperbolic system.
Theorem 1. Let \( \epsilon_0 \in (0, 1) \). Let \( D_T \) be a slab domain. Let \( w_0 \in C^{2+\epsilon_0}(D_T, \mathbb{R}^N) \cap H^{3,2}(D_T, \mathbb{R}^N) \). Let \( v^\# \in C^2(D_T, \mathbb{R}^N) \) be a strict supersolution in \( D_T^0 \) of \( L(v^\#) > 0 \) with \( v^\#(0, x) = w_0(x) \). Let \( u^\# \in C^2(D_T, \mathbb{R}^N) \) be a strict subsolution in \( D_T^0 \) of \( L(u^\#) < 0 \) with \( u^\#(0, x) = w_0(x) \). Then there exists a continuous viscosity solution \( U \) of \( \mathcal{L}(U) = 0 \) in \( D_T^0 \), that is equal to \( w_0 \) when \( t = 0 \).

Definition 2. Let \( A_1 \) be a family of \( \mathbb{R}^N \)-valued functions. \( A_1 \) is "up directed" iff whenever two functions belong to \( A_1 \), then their max (in our ordering) belongs to \( A_1 \). Similarly, \( A_1 \) is down directed iff whenever two functions belong to \( A_1 \), then their min (in our ordering) belongs to \( A_1 \).

We need some classes of viscosity sub- (super-) solutions.

Definition 3. Let
\[
S := \{ w : D_T \to \mathbb{R}^N \text{ \( w \) finite with bounded jump, } \ w(0, x) = w_0 \}.
\]

Definition 4. \( S^* := S \cap \{ \text{upper semicontinuous functions} \} \).

Definition 5. \( S_* := S \cap \{ \text{lower semicontinuous functions} \} \).

Definition 6. \( S^+(D_T, w_0) := \{ v \in S^* \mid v \leq v^\#, v \text{ is a viscosity supersolution of } \mathcal{L}^+(v) \geq 0, \text{ in } D_T^0 \} \).

Definition 7. \( S^-(D_T, w_0) := \{ u \in S_* \mid u \geq u^\#, u \text{ is a viscosity subsolution of } \mathcal{L}^-(u) \leq 0, \text{ in } D_T^0 \} \).

Remark 8. Note that it follows from the comparison principle for semicontinuous super and subsolutions that any element \( u \) of \( S^{-}(D_T, w_0) \) is less than or equal to any element \( v \) of \( S^+(D_T, w_0) \).

Lemma 9. The sets \( S^+(D_T, w_0) \) and \( S^-(D_T, w_0) \) have uncountable cardinality.

Proof. Essentially a trivial vector-valued modification of the proof of [5, Lemma 8, p. 565]. □

Lemma 10. \( S^-(D_T, w_0) \) is an up directed family.

Proof. Let \( u_1, u_2 \in S^-(D_T, w_0) \). Then, on Base(\( D_T \)), we have \( \min(u_1, u_2) = w_0 \). It follows from the difference criterion for viscosity subsolutions that \( \varphi - u_1 \geq 0, \varphi - u_2 \geq 0 \) \( \forall \varphi \in C^{2+\epsilon_0}(D_T, \mathbb{R}^N) \cap H^{2,2}(D_T, \mathbb{R}^N) \) with initial data \( \varphi(0, x) = w_0 \). This implies that for such \( \varphi \) we have \( \varphi - \max(u_1, u_2) \geq 0 \). Thus by applying the reverse implication of the difference criterion for viscosity subsolutions, we see that \( \max(u_1, u_2) \) is also a viscosity subsolution. □

Lemma 11. \( S^+(D_T, w_0) \) is a down directed family.
Proof. The same proof by “mirror reflection” as the previous lemma—mutis mutandis.

2.1. Proof of Theorem 1

Let $A$ be an index set for the possibly uncountable set $S^-(D_T, w_0)$. Define $U_{up} := \sup_{\alpha \in A} \{ u_\alpha \}$, for each $u_\alpha \in S^-(D_T, w_0)$. Here, this is the usual supremum of a family of vector-valued functions, with the supremum taken pointwise. We note that, by definition, for each $\alpha$, the function $u_\alpha$ is lower semicontinuous on $D_T$. Thus, $U_{up}$ is lower semicontinuous on $D_T$. We prove the theorem by establishing nine claims.

Claim 1. $U_{up}(0, x) = w_0(x)$.

Proof. Obvious. □

Claim 2. $U_{up} \leq v^#$ on $D_T$.

Proof. It follows from the semicontinuous comparison principle that $u_\alpha \leq v^#$ on $D_T$. Taking supremum over the index set $A$, we see that $U_{up} \leq v^#$ on $D_T$. □

Remark 12. Thus, we see that $v^# \geq U_{up} \geq u^#$ on $D_T$ and thus $U_{up}$ has bounded jump, and is finite, and thus $U_{up} \in S$.

Claim 3. $U_{up}$ is a viscosity subsolution on $D^0_T$.

Proof. Since, for each $\alpha$, $u_\alpha$ is a viscosity subsolution of $-\langle u_\alpha \rangle \leq 0$ on $D^0_T$. We see from the difference criterion for viscosity subsolutions that $\varphi - u_\alpha \geq 0, \forall \varphi \in C^{2+\epsilon_0}(D_T, R^N) \cap H^{2,2}(D_T, R^N)$ with $\varphi(0, x) = w_0(x)$. But, at any fixed point $X \in D_T$, we have $U_{up} = \sup_{\alpha \in A}(u_\alpha(X))$, and this implies that $\varphi - U_{up} \geq 0$ for all such $\varphi$. Since this holds at all points of $D_T$, we see by applying the difference criterion for viscosity subsolutions, in the opposite direction of implication, that $U_{up}$ is a viscosity subsolution in $D^0_T$. □

Let $B$ be an index set for $S^+(D_T, w_0)$.

Definition 13. $V_{down} := \inf_{\tilde{u} \in B} \{ u_\tilde{u} \}$, for each $u_\tilde{u} \in S^+(D_T, w_0)$. Here the infimum is the usual inf of a family of vector-valued functions taken pointwise in each component.

Claim 4. $V_{down}$ satisfies $\mathcal{L}^- V_{down} \leq 0$ in $D^0_T$.

Proof. If not, there exists at least one point $Y_0 \in D^0_T$, and some $i \in \{1, 2, \ldots, N\}$ with

$$\mathcal{L}^- V_{down}^i(Y_0) > 0, \quad \mathcal{L}^- V_{down}^j(Y_0) \geq 0 \quad \text{for } j \neq i. \quad (2)$$

We have used that $\mathcal{L}^- V_{down}(Y_0) \geq \mathcal{L}^- V_{down}(Y_0) \geq 0$, from Claim 3. As in the proof of Theorem 15 we construct a $C^{2+\epsilon_0}(D_T, R^N) \cap H^{2,2}(D_T, R^N)$ supersolution $\tilde{\varphi}$ in $D^0_T$, 

...
with $\tilde{\phi}(0,x) = w_0(x)$ on $\text{Base}(D_T)$, but with $\tilde{\phi}(Y_0) < V_{\text{down}}(Y_0)$. Note that, although we have
\[
\Psi^-(V_{\text{down}}(Y_0)) := \lim_{\gamma \downarrow 0} \lim_{\sigma \downarrow 0} L(V_{\text{down},\sigma}^\gamma)(Y_0)
\]
by definition, and not liminfs, we still obtain
\[
\exists \phi(Y_0,\sigma) \text{ by definition, and not liminfs, we still obtain } \exists \gamma_0, \sigma_0, R_0 \in R^+, \forall Y \in B_{R_0}(Y_0)
\]
\[
L(V_{\text{down},\sigma}^\gamma_i)(Y) > 0, \quad L(V_{\text{down},\sigma}^\gamma_i)(Y) \text{ bounded, } j \neq i
\]
and the rest proof of the theorem goes through as given in Section 3, Theorem 15.

Now let $\tilde{\phi} := \min(V_{\text{down}}, \tilde{\phi})$, and we see that $\tilde{\phi} \in S^+(D_T, w_0)$, but $\tilde{\phi} < V_{\text{down}}$ at least one point in $D_T$. This is a contradiction, and this establishes Claim 3. \(\square\)

**Claim 5.** $U_{\text{up}}$ is a viscosity solution of $\Psi^+(U_{\text{up}}) \geq 0$ in $D_T^0$.

**Proof.** This is a mirror image of the proof of Claim 3, mutis mutandis. \(\square\)

**Claim 6.** $U_{\text{up}}$ is the lower semicontinuous regularization of $V_{\text{down}}$, and $V_{\text{down}}$ is the upper semicontinuous regularization of $U_{\text{up}}$.

**Proof.** We prove the second statement only, as the proof of the first statement is entirely analogous. Let $(U_{\text{up}})^*$ be the upper semicontinuous regularization of $U_{\text{up}}$. Note that $u_\# \leq U_{\text{up}} \leq (U_{\text{up}})^* \leq V_{\text{down}} \leq v_\#$. (Here, to see that $(U_{\text{up}})^* \leq V_{\text{down}}$ we have used the definition of $(U_{\text{up}})^*$, the fact that $V_{\text{down}}$ is upper semicontinuous, and that $V_{\text{down}} \geq U_{\text{up}}$.

Now, that by the definition of $(U_{\text{up}})^*$, we have $(U_{\text{up}})^*(0,x) = w_0(x)$ on $\text{Base}(D_T)$. Now, it follows by the difference criterion for viscosity supersolutions that $(U_{\text{up}})^*$ is a viscosity supersolution on $D_T^0$. But this contradicts the definition of $V_{\text{down}}$ unless $(U_{\text{up}}) = V_{\text{down}}$. \(\square\)

**Claim 7.** In $D_T^0$, $\Psi^-(V_{\text{down}}) = 0$, and $\Psi(V_{\text{down}}) = 0$.

**Proof.** Note that $\Psi^+(V_{\text{down}}) \leq \Psi^-(V_{\text{down}}) \leq 0$ by Claim 4. We prove the first equality of Claim 7.

If this equality is not valid, then there exists a point $P \in D_T^0$ and an $i \in \{1, 2, \ldots, N\}$ with
\[
[\Psi^-(V_{\text{down}})]^i(P) < 0, \quad [\Psi^+(V_{\text{down}})]^j(P) \leq 0, \quad j \in \{1, 2, \ldots, N\}, \quad j \neq i\quad (5)
\]

By the proof of Theorem 20, there exists a $C^{2+\alpha}(D_T, R^N) \cap H^{2,2}(D_T, R^N)$ function $\tilde{\phi}$, with $\tilde{\phi}(0,x) = w_0(x)$, with $\tilde{\phi}$ a subsolution of $L(\tilde{\phi}) \leq 0$ in $D_T^0$, and with $\tilde{\phi} > V_{\text{down}} > U_{\text{up}}$ at least on point in $D_T^0$. Now, let $\tilde{\phi} := \max(\tilde{\phi}, U_{\text{up}})$. We note, by the difference criterion for viscosity solutions, that $\tilde{\phi}$ is a viscosity subsolution of $\Psi^+(\tilde{\phi}) \leq 0$ in $D_T^0$. We have $\tilde{\phi}(0,x) = w_0(x)$. Note that $\tilde{\phi} \in S^-(D_T, w_0)$. But, $\tilde{\phi} > U_{\text{up}}$ at least one point in $D_T^0$, which is a contradiction. \(\square\)

**Claim 8.** In $D_T^0$, $\Psi^+(U_{\text{up}}) = \Psi(U_{\text{up}}) = 0$.
Proof. This is the mirror reflection of the proof of Claim 7, using the proof of Theorem 20. □


Claim 9. $U_{\uparrow} = V_{\downarrow} =: U$ and $U$ is a continuous viscosity solution of $\mathcal{L}(U) = 0$ in $D_T^0$.

Proof. Consider $V_{\downarrow}$ and note that $V_{\downarrow}$ is a viscosity solution in $D_T^0$ of $\mathcal{L}^\pm(V_{\downarrow}) = \mathcal{L}(V_{\downarrow}) = 0$, with $V_{\downarrow}(0, x) = w_0(x)$. We already know that $V_{\downarrow}$ is upper semicontinuous on $D_T$, but we do not yet know that $V_{\downarrow}$ is continuous on $D_T$. Similarly, consider $U_{\uparrow}$ and note that $U_{\uparrow}$ is a viscosity solution of $\mathcal{L}^\pm(U_{\uparrow}) = \mathcal{L}(U_{\uparrow}) = 0$. We know that $U_{\uparrow}$ is lower semicontinuous on $D_T$, but we do not yet know that $U_{\uparrow}$ is continuous on $D_T$. We will prove now that $U_{\uparrow} = V_{\downarrow} := U$, which will show that $U$ is continuous on $D_T$ and that $\mathcal{L}(U) = 0$ on $D_T^0$.

We already have that $U_{\uparrow} \leq V_{\downarrow}$ on $D_T$. We would like to apply the comparison theorem for semicontinuous subsolutions to $V_{\downarrow}$ and $U_{\uparrow}$ to show that $U_{\uparrow} \leq V_{\downarrow}$. We already know that, in $D_T^0$, $\mathcal{L}(U_{\uparrow}) = \mathcal{L}(V_{\downarrow})$, so that $U_{\uparrow}$ is a viscosity supersolution of $\mathcal{L}(U_{\uparrow}) = 0$, and $V_{\downarrow}$ is a viscosity subsolution of $\mathcal{L}(V_{\downarrow}) = 0$, except that $U_{\uparrow}$ is not a-priori upper semicontinuous and $V_{\downarrow}$ is not a-priori lower semicontinuous. We replace this comparison theorem by a slight variant which we prove within the body of this proof.

Let $\varepsilon > 0$. Let $P = (t_0, x_0)$ be an arbitrary point in $D_T^0$. Let $\bar{V}(t, x) := V_{\downarrow}(t, x) - \beta_1 t e^{-k_7(x-x_0)^2} e^{-k_8(t-t_0)^2}$, with $0 < \beta_1 < \beta_0$, $0 < k_7$, $0 < k_8$, chosen so that $-\varepsilon^2 \leq \mathcal{L}^\pm(V)(P) < 0$. We can do this because of the exponential factor, the (uniform) $C^1$-ness of the coefficients of $L$ in their arguments, and because $\mathcal{L}^\pm(V_{\downarrow}) = 0$ in $D_T^0$.

Similarly, let $\bar{U}(t, x) := U_{\uparrow}(t, x) + \beta_2 t e^{-k_7(x-x_0)^2} e^{-k_8(t-t_0)^2}$ with $0 < \beta_2 < \beta_1$, $0 < k_9$, $0 < k_{10}$, chosen so that $0 \leq \mathcal{L}^\pm(U)(P) \leq \varepsilon^2$. We can do this because of the exponential factor, the $C^1$-ness of the coefficients of $L$ in their arguments, and because $\mathcal{L}^\pm(U_{\uparrow}) = 0$ in $D_T$. Choose $\varepsilon_1 > 0$. Note that $\bar{U}(0, x) = \bar{V}(0, x) = w_0(x)$ and that we may choose $\gamma_0, \sigma_0 > 0$, so that for $0 < \sigma < \sigma_0$ and $0 < \gamma < \gamma_0$, we have:

$$
\bar{U}(P) - \varepsilon_1 \leq \bar{U}_{\gamma, \sigma}(P) \leq \bar{U}(P) + \varepsilon_1,
$$
$$
\bar{V}(P) - \varepsilon_1 \leq \bar{V}_{\gamma, \sigma}(P) \leq \bar{V}(P) + \varepsilon_1
$$

and that $\varepsilon^2 \geq \mathcal{L}^\pm(\bar{U})(P) \geq 0 \geq \mathcal{L}^\pm(\bar{V})(P) \geq -\varepsilon^2$.

We now use the same proof as that of the comparison theorem for semicontinuous viscosity sub(super)solution to show that for $\varepsilon_2 > 0$ and small enough, there exists a $C^{2+\varepsilon_0}(D_T, R^N) \cap H^{2,2}(D_T, R^N)$ supersolution $\phi_1$ in $D_T^0$, with $\phi_1(0, x) = w_0(x)$; and there exists a $C^{2+\varepsilon_0}(D_T, R^N) \cap H^{2,2}(D_T, R^N)$ subsolution $\phi_2$, with $\phi_2(0, x) = w_0(x)$ and these satisfy:

$$
\varepsilon_2 - \bar{V}(P) \leq \phi_1(P) \leq \bar{V}(P) + \varepsilon_2,
$$
$$
\bar{U}(P) - \varepsilon_2 \leq \phi_2(P) \leq \bar{U}(P) + \varepsilon_2,
$$
$$
\phi_1(P) \geq \phi_2(P).
$$

(7)
Let $\epsilon_2 \downarrow 0$ and let $\beta_0 \downarrow 0$ and we see that $U_{up}(P) \geq V_{down}(P)$. Thus, $V_{down}(P) = U_{up}(P)$, and since $P$ was arbitrary in $D_T$, we have that $U_{up} = V_{down} =: U$ in $D_T$. This implies that $U$ is continuous in $D_T$. □

This completes the proof of Claim 9 and the proof of Theorem 1.

3. Approximation of viscosity supersolutions by smooth supersolutions

**Theorem 15.** Let $L$ be a quasi-linear hyperbolic first order system, with admissible $A^i$ and admissible $f$, of the type considered in Section 1 (with $A^0 = I$), in a slab domain $D_T$. Let $\epsilon_0 \in (0, 1)$. Let $s > 0$. Let $X_0 = (t_0, x_0) \in D^0_T$. Let $v \in (D_T, \mathbb{R}^N)$ be an upper semicontinuous supersolution, bounded and with bounded jump, of $\mathcal{L}(v)(X_0) \geq 0$. Let $w \in C^{2+\epsilon_0}(\mathbb{R}^n, \mathbb{R}^N) \cap H^{3,2}(\mathbb{R}^n, \mathbb{R}^N)$ and let $v(0, x) = w_0(x)$. Then there exists $\phi_2 \in C^{2+\epsilon_0}(D_T, \mathbb{R}^N) \cap H^{3,2}(D_T, \mathbb{R}^N)$ satisfying:

1. $v(X_0) < \phi_2(X_0) < v(X_0) + \epsilon_1$,
2. $\phi_2(0, x) = w(x)$,
3. $\mathcal{L}(\phi_2)(X_0) \geq 0$,
4. $L(\phi_2) \geq 0$ in $D^0_T$.

(8)

**Proof.** Let $\gamma_0, \sigma_0 > 0$ be chosen so that $v^{\gamma, \sigma}(X_0)$ satisfies $v(X_0) < v^{\gamma, \sigma}(X_0) + 100\epsilon_1$, and also such that $\exists c_0 \in \mathbb{R}, \forall \gamma, \sigma$ such that $0 < \sigma < \sigma_0$ and $0 < \gamma < \gamma_0$, $L(v^{\gamma, \sigma})(X_0) > c_0^2$.

We will show the existence of $\phi_2$ by solving some auxiliary PDE system boundary value problems (BVP).

Let $\Delta_t$ denote the Laplacian in $\mathbb{R} \times \mathbb{R}^n$ with $t$ denoting the first variable. We denote by $x_\alpha$ or $x_\beta$, where $\alpha, \beta \in \{0, 1, 2, \ldots, N\}$, the co-ordinates of $\mathbb{R} \times \mathbb{R}^n$ i.e. $t = x_0$. □

**Step 1.** For each $i \in \{1, 2, \ldots, N\}$, let $(\eta^i_j) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ be orthogonal to the kernel of the $\Delta_t$ with vanishing initial data (note that this orthogonality condition is actually vacuous because if follows from the fact $w$ goes to zero at infinity, and the maximum principle for the Laplacian in the upper half space [4], that this kernel is trivial. We have included the condition only to clarify this point), where $\eta^i_j : D_T \to \mathbb{R}$ is a $C^\infty$, Schwartz class function, with $\eta^i_j \in H^{m,2}(D_T, \mathbb{R}^N) \cap C^\infty(D_T, \mathbb{R}^N)$ for all $m \geq 2$. For $i \in \{1, 2, \ldots, N\}$ we solve the elliptic system BVP (in $\mathbb{R}^+ \times \mathbb{R}^n$):

$$\Delta_t(\phi_0^i) = (\eta^i_j),$$
$$\phi_0^i(0, x) = w^i(x).$$

(9)

This is standard, using the Duhamel formula corresponding to the Poisson Integral formula, applied componentwise, which gives bounded maps from $C^{k_0,\epsilon_0}(\mathbb{R}^n, \mathbb{R}^N) \to C^{k,\epsilon_0}(\mathbb{R}^{n+1}, \mathbb{R}^N)$, and bounded maps from $H^{3,2}(\mathbb{R}^n, \mathbb{R}^N) \to H^{3,2}(\mathbb{R}^{n+1}, \mathbb{R}^N)$. Note that it follows from the Sobolev trace theorem (using a countable partition of unity in the proof of the Sobolev trace theorem, because our domain is unbounded) that for any fixed
\( t > 0, \varphi_0 \in L^2(\mathbb{R}^{n+1} \cap \{ t = \bar{t} \}, R^N), \frac{\partial \varphi_0}{\partial x_a} \in L^2(\mathbb{R}^{n+1} \cap \{ t = \bar{t} \}, R^N), \frac{\partial^2 \varphi_0}{\partial x_a \partial x_\beta} \in L^2(\mathbb{R}^{n+1} \cap \{ t = \bar{t} \}, R^N); \) also \( \varphi_0 \in C^2(\mathbb{R}^{n+1} \cap \{ t = \bar{t} \}, R^N), \frac{\partial \varphi_0}{\partial x_a} \in C^1(\mathbb{R}^{n+1} \cap \{ t = \bar{t} \}, R^N), \frac{\partial^2 \varphi_0}{\partial x_a \partial x_\beta} \in C^0(\mathbb{R}^{n+1} \cap \{ t = \bar{t} \}, R^N). \)

**Step 2.** We choose \( \alpha_0 > 0 \) so small that if \( 0 < \alpha < \alpha_0 \) we have in \( D_T \):

\[
A_1(\varphi_0)^i := - (\Delta_i(\varphi_0^i))^2 + \alpha [L(\varphi_0)]^i(\eta^i)^2 \in B_{(\eta^i)^2/2}(-(\eta^i)^2),
\]

\[i \in \{1, 2, \ldots, N\}, \tag{10}\]

Here, we have used the regularity of the coefficients of \( L \), and that the inhomogeneous term of \( L \) is bounded as \( |x| \to \infty \).

Note also that it follows from the local regularity for elliptic systems [1,7] that \( \varphi_0 \in C^{m,\alpha}_{loc}(\mathbb{R}^n, \mathbb{R}^n) \) for all \( m \in \mathbb{Z}^+ \).

**Case I.** We prove the theorem in the case that there exists a \( T_0 \) (to be determined below) with \( 0 < T < T_0 < 1 \). We now choose a special form for \( \eta \), and add a function \( G_0 \) to \( \varphi_0 \) that will preserve the boundary data at \( t = 0 \), and the regularity, and will make \( A_1 \) of the sum non-negative.

Let \( \eta = \gamma_0^2 e^{-k_1^2(t-t_0)^2} e^{-k_2^2(\|x-x_0\|^2)} \), where \( \| \cdot \| \) is the spatial Euclidean norm on \( \mathbb{R}^n \).

Here \( \gamma_0, k_1, k_2 \) are positive real constants.

Let

\[
G_1(t, x, \beta, k_3, k_4) := \beta^2 \left( \frac{t}{1} \right)^2 e^{-k_3^2(\frac{t-t_0}{1})^2} e^{-k_4^2(\|x-x_0\|^2)}
\]

where \( k_3, k_4, \beta \) are positive real constants.

Note that for \( \alpha, \gamma_0 \) small enough, \( k_1, k_2 \) large enough, we can choose \( k_3, k_4 \) large enough, such that we can choose any small enough \( \beta \) (depending on \( \gamma_0, k_1, k_2, T_0 \)) so that we have \( \alpha L(\varphi_0 + G_1(t, x, \beta, k_3, k_4)) \geq 0 \). We see this by looking at the magnitude of \( \Delta_i(\varphi_0 + G_1(t, x, \beta, k_3, k_4)) \). Here, we have used that the dominant term in \( \Delta_i(\varphi_0 + G_1(t, x, \beta, k_3, k_4)) \) due to the perturbation from \( \Delta_i(\varphi_0) \) (except for helpful non-negative terms coming from second spatial derivatives of the spatial Gaussian) arises from the highest time derivative, and the regularity of the coefficients of \( L \). We have used that \( T_0 \) is sufficiently small that the highest time derivative is the dominant term in the perturbation. We also see that for \( \alpha > 0 \) small enough, the \( [\alpha L(\varphi_0 + G_1(t, x, \beta, k_3, k_4)) - \alpha L(\varphi_0)]^i(\eta^i)^2 \) is negligible compared to \( (\eta^i)^2 \). After making these choices we let \( \tilde{\varphi}_0 := \varphi_0 + G_1(t, x, \beta, k_3, k_4) \). Note that \( \tilde{\varphi}_0 \) has the same regularity properties as \( \varphi_0 \) and the same initial data at \( t = 0 \).

**Step 3.** Let \( \Omega_1 := \{ DT \cap \{ \frac{\gamma_0}{2} < t \leq T \} \}. \) Let \( \Omega_2 := D_T - \Omega_1. \) Let \( R_0 > 0 \) be small enough that \( B_{R_0}(X_0) \subset \Omega_2. \) Let \( \Omega_3 := \Omega_2 - B_{R_0}(X_0). \)

**Definition 16.** Let \( \xi : \Omega_3 \to R^N \). We say that \( \xi \) has Step 2 boundary data iff for each \( i \in \{1, 2, \ldots, N\} \):
\(\xi^i = \varphi^i_0\), \(\frac{\partial (\xi^i)}{\partial N} = \frac{\partial (\varphi^i_0)}{\partial N}\), \(\frac{\partial^2 (\xi^i)}{\partial N^2} = \frac{\partial^2 (\varphi^i_0)}{\partial N^2}\) on \(\partial \Omega_2\):

\[
\xi^i = v^{i \gamma}_{,\alpha} + G_1(t, x, \beta, k_3, k_4) \quad \text{and} \quad \frac{\partial (\xi^i)}{\partial N} = \frac{\partial (v^{i \gamma}_{,\alpha} + G_1(t, x, \beta, k_3, k_4))}{\partial N},
\]

\[
\frac{\partial^2 (\xi^i)}{\partial N^2} = \frac{\partial^2 (v^{i \gamma}_{,\alpha} + G_1(t, x, \beta, k_3, k_4))}{\partial N^2} \quad \text{on} \quad \partial B_{R_0}(X_0). \tag{11}
\]

Here the normal derivatives are taken inward.

**Definition 17.** For each \(i \in \{1, 2, \ldots, N\}\), let \(\xi \) have Step 2 boundary data. Let \(\xi \in H^{3,2}(\Omega_3, R^N)\). Let \(g \in H^{3,2}(\Omega_3, R^N)\) be such that the \(H^{3,2}\)-trace of \(g\) is equal to the \(H^{3,2}\)-trace of \(\xi\) on \(\partial \Omega_3\) and let \(\xi - g \in H^{0,3,2}(\Omega_3, R^N)\). We say then that \(\xi\) has Agmon Step 2 data.

For each \(i \in \{1, 2, \ldots, N\}\), let \(\zeta : DT \to R^N\) with \(\zeta \in L^2(\Omega_3, R^N)\), where \(\zeta : \Omega_3 \to R^N\) is a \(C^\infty\), Schwartz class function in \(L^2(\Omega_3, R^N) \cap C_{\epsilon 0}(\Omega_3)\).

**Remark 18.** We now solve the elliptic system BVP for \(i \in \{1, 2, \ldots, N\}\):

\[
\Delta_3^3 (\varphi^i_0) = \xi^i \quad \text{in} \quad \Omega_3,
\]

\(\varphi_1\) has Agmon Step 2 boundary data. \(\tag{12}\)

Note that \(v^i_{,\alpha} \in C^\infty\) on \(\partial B_{R_0}(X_0)\). To solve this elliptic system BVP we use the Lax–Milgram method of [1, p. 98]. We solve the generalized Dirichlet problem [1, p. 98] (using its notation):

\[
u \in H^{3,2}(\Omega_3, R^N), \quad g \in H^{3,2}(\Omega_2, R^N),
\]

\(u\) has Agmon Step 2 boundary data,

\(u - g \in H^{0,3,2}(\Omega_3, R^N), \quad B(\varphi, u) = (\varphi, f) \quad \forall \varphi \in H^{0,3,2}(\Omega_3, R^N), \quad f \in L^2(\Omega_3, R^N). \tag{13}\)

**Remark 19.** Without loss of generality, we assume that \(g \in C^{6+\epsilon_0}(\Omega_3, R^N)\) and \(g\) has Agmon Step 2 boundary data. This does not affect Agmon’s argument and is useful to us later in proving our theorem.

To apply [1, Theorem 8.2, p. 99] we need that (using its notation and its positive constant \(c_0\))

\[
|B(\Psi, \Psi)| \geq c_2 |\Psi|_{H^{3,2}(\Omega_3, R^N)}^2 \quad \forall \Psi \in H^{0,3,2}(\Omega_3, R^N).
\]

Here \(|\Psi|_{H^{3,2}(\Omega_3, R^N)}\) denotes the highest homogeneity part of the \(H^{3,2}(\Omega_3, R^N)\) norm of \(\Psi\). But, note that, since \(\Psi \in H^{0,3,2}(\Omega_3, R^N)\), it follows by Poincare’s inequality that all the lower order homogeneity parts of the \(H^{2,3}(\Omega_3, R^N)\) norm are estimated from above by the highest homogeneity part. Thus Theorem 8.2 of [1] obtains.
Note that we have used the Poincare inequality for $\Psi$. This inequality holds for our domain by the (Friedrich’s proof) of integrating the fundamental theorem of calculus for the time derivative in our domain with respect to spatial variables and using Hölder’s inequality. To obtain an estimate on $\|u\|_{H^{3,2}(\Omega_3, R^N)}$ we slightly modify the proof of [1, line 9, p. 101]. Let $u_0 := u - g$. We have:

$$B(\varphi, u_0) = (\varphi, f) - B(\varphi, g), \quad f \in L^2(\Omega_3, R^N), \quad \forall \varphi \in H^{0,3,2}(\Omega_3, R^N), \quad u_0 = u - g \in H^{0,3,2}(\Omega_3, R^N).$$

(14)

Let $Tf =: u_0$, which defines $T$. Then

$$B(\varphi, Tf) = (\varphi, f) - B(\varphi, g) \quad \forall \varphi \in H^{0,3,2}(\Omega_3, R^N).$$

Note that $T$ maps $L^2(\Omega_3, R^N) \to H^{0,3,2}(\Omega_3, R^N)$ so that we take $\varphi = Tf$. We have (with $c_0, c_1 \in R^+$):

$$c_0\|Tf\|_{H^{3,2}(\Omega_3, R^N)}^2 \leq \left| B(Tf, Tf) \right| \leq \left| B(Tf, g) \right| + \left| Tf, f \right| \leq \|Tf\|_{L^2(\Omega_3, R^N)}\|f\|_{L^2(\Omega_3, R^N)} + c_1\|Tf\|_{H^{3,2}(\Omega_3, R^N)}\|g\|_{H^{3,2}(\Omega_3, R^N)}$$

(15)

and this implies:

$$\|Tf\|_{H^{3,2}(\Omega_3, R^N)} \leq \frac{1}{c_0}\left[\|f\|_{L^2(\Omega_3, R^N)} + c_1\|g\|_{H^{3,2}(\Omega_3, R^N)}\right].$$

(16)

$$\|u\|_{H^{3,2}(\Omega_3, R^N)} \leq \|g\|_{H^{3,2}(\Omega_3, R^N)} + \frac{1}{c_0}\left[\|f\|_{L^2(\Omega_3, R^N)} + c_1\|g\|_{H^{3,2}(\Omega_3, R^N)}\right].$$

(17)

(We have used: $\|u\|_{H^{3,2}(\Omega_3, R^N)} - \|g\|_{H^{3,2}(\Omega_3, R^N)} \leq \|u - g\|_{H^{3,2}(\Omega_3, R^N)}$. This is our global bound! Note that this also gives a global bound on $\|u\|_{L^2(\Omega_3, R^N)}$.

Next, we prove that we have global bounds on the $C^{2+\epsilon_0}$-norm of $u$.

If we are in a collar neighborhood of fixed width of $\partial \Omega_3$, we can apply a countable partition of unity and the global $C^{2+\epsilon_0}$ estimates on linear systems [1,7] to see this. On the other hand, if we are in $\Omega_3$, but outside this collar neighborhood, we apply the local Sobolev regularity estimate (as in [7, Theorem 11.1, p. 379]), which also holds for systems of our type, to see that $u$ is locally in any Sobolev space of any order, with bounds from above on the Sobolev norms given by (16). We apply the Sobolev embedding theorem (locally) and the bound of (16) to see that $u \in C^{2+\epsilon_0}(\Omega_3 - \text{collar neighborhood of } \partial \Omega_3, R^N)$. Combining these estimates (and relabeling $u$ as our $\varphi_1$), we see that $\varphi_1 \in C^{2+\epsilon_0}(\Omega_3, R^N) \cap H^{2,2}(\Omega_3, R^N)$ as required. We have used the usual trick to estimate the Hölder norm of second derivatives of dividing into two cases: (1) the points are close enough to be in a smaller half disc of a boundary half disc (or similarly in a smaller ball in an interior ball); and (2) the distance between the two points is bounded from below.

**Step 4.** Now choose $\alpha_1 > 0$ so small that if we choose $0 < \alpha < \min(\alpha_0, \alpha_1)$, we have in $\Omega_3$,

$$[\Lambda_2(\varphi_1)]^i := -\left(\Delta^3(\varphi^i_1)\right)^2 + \alpha \left[ L(\varphi) \right]^i(\xi^i)^2 \in B_{-\frac{1}{3}(\xi^i)^2}(-\xi^i)^2.$$

(18)
At this point in the proof, we now assume that $\zeta = \eta$.

Now, we add a function $G_2$ to $\varphi_1$ to make $\Lambda_2$ of the sum non-negative. We define

$$G_2(t, x, \beta_2, k_5, k_6) := \beta_1^2 \left( \frac{t - (t_0/2)}{1} \right)^6 e^{-k_5^2 \left( \frac{t - (t_0/2)}{1} \right)^2} e^{-k_6^2 \left( \frac{\|x - x_0\|}{1} \right)^2}$$

in $\Omega_3$ and

$$G_1(t, x, \beta, k_3, k_4) = 0 \quad \text{in } \Omega_1.$$

Here $\beta_1, k_5, k_6$, are positive real constants. Let $\tilde{\varphi}_1 := \varphi_1 + G_2(t, x, \beta_2, k_5, k_6)$. Note that for $\alpha, \gamma_0$ small enough, $k_1, k_2$ large enough, we can choose $k_5, k_6$ large enough, such that we can choose any small enough $\beta$ (depending on $\gamma_0, k_1, k_2, T_0$) so that we have $\alpha L(\varphi_1 + G_2(t, x, \beta, k_5, k_6)) \geq 0$. Again, we see this by estimating the magnitude of $\Delta_1^3 (\varphi_1 + G_2(t, x, \beta, k_5, k_6))$. Here, we have used that the dominant term (except for helpful non-negative terms arising from even spatial derivatives of the spatial Gaussian) in $\Delta_1^3 (\varphi_1 + G_2(t, x, \beta, k_5, k_6))$—due to the perturbation from $\Delta_1^3 (\varphi_1)$—arises from the highest time derivative, and the regularity of the coefficients of $L$. We have used that $T_0$ is sufficiently small that the highest time derivative is the dominant term in the perturbation. We have used that for small enough $\alpha > 0$, the $[\alpha L(\varphi_1 + G_2(t, x, \beta, k_5, k_6)) - \alpha L(\varphi_1)]^j \eta_1^j$ term is negligible compared to $((\eta_1^j)^2$. After making these choices we let $\tilde{\varphi}_1 := \varphi_0 + G_2(t, x, \beta, k_5, k_6)$. Note that $\tilde{\varphi}_1$ has the same regularity properties as $\varphi_1$ and the same initial data at $t = 0$.

$$L(\tilde{\varphi}_1) \geq 0 \quad \text{in } \Omega_3$$

and similarly we had

$$L(\tilde{\varphi}_0) \geq 0 \quad \text{in } \Omega_1.$$ (19)

Define

$$\tilde{\varphi}_1 \quad \text{in } \Omega_3,$$

$$\varphi_2 := \varphi_0' + G_1(t, x, \beta, k_3, k_4) \quad \text{in } B_{R_0}(X_0),$$

$$\tilde{\varphi}_0 \quad \text{in } \text{closure } (D_T - \Omega_1).$$ (20)

Note that $\alpha L(\varphi_2) \geq 0$ in $D_T^0$ for small enough positive $\gamma, \sigma, \beta$. By choosing $\eta > 0$ small we can choose $\beta$ as small as we wish. By choosing $\eta$ small we see that the $G_1$ adds a small positive constant to $\nu_{\sigma}'$ at $X_0$. By choosing $\gamma$ and $\sigma$ small, we see that $\varphi_2(X_0) > \nu(X_0)$ but can be made as close as we wish. The function $\varphi_5$ satisfies the conclusions of our theorem, except that it is in $C^2(D_T, R^n)$ but perhaps not in $C^{2+\epsilon_0}$. Consider the smoothing process of [3, Theorem 3, p. 127], which smooths up to the boundary. We do this for $D_T \cap \{t_0/3 < t\}$ using cubes at the boundary and notice that the process still works (using a countable partition of unity) for the unbounded domain $D_T$ under our hypothesis. We denote this smoothing process by $\text{EVS}(-, \tilde{\sigma})$, where $\tilde{\sigma}$ is the smoothing parameter. Let $\varphi_2 := \text{EVS}(\varphi, \tilde{\sigma})$. For sufficiently small positive $\tilde{\sigma}$, that all the conclusion of the theorem hold for this $C^{2+\epsilon_0}(D_T, R^N)$. Case I is proved.
Case II. Now we let $T > 0$ be arbitrary. We will use an argument similar to that of Case I but we will divide the slab region $D_T$ into a finite number of time overlapping slab regions each of time width smaller than $T_0$ and will solve similar PDE systems to Case I in each slab.

We start by doing Step 1 and Step 2 on $D_{\tilde{T}}$, where $\tilde{T} := \min(T_0/10, t_0/10)$. Choosing $\epsilon_1 > 0$, small enough, we do Steps 3 and 4 on successive domains $D_{T_i}, T_{i+1} := D_T \cap \{T_i - \epsilon_1 \leq T_{i+1}\}$ with $i = 1, 2, \ldots, i_{\text{final}} - 1$, $T_1 = \tilde{T} - \epsilon_1$, $T_{i+2} = T_{i+1} - \epsilon_1$, $T_{i+1} - T_i \leq T_0/20$, with the last $T_{i_{\text{final}}} = t_0 - \epsilon_1$. (In these domains we have no deleted ball, and no boundary conditions on such a ball, but otherwise the argument is the same as in Case I. At each stage we add a $G_i$, as in Steps 1, 2 and Steps 3, 4 of Case I to produce a supersolution.

We define:

$$G_1(t, x, \beta, k_3, k_4) := \beta^2 \left( \frac{t}{T} \right)^2 e^{-k_2^2 \left( \frac{u_{\min}}{t} \right)^2} e^{-k_2^2 \left( \frac{|x-x_0|}{t} \right)^2} D_{\tilde{T}},$$

$$G_i(t, x, \beta_i, k_{3+i}, k_{4+i}) := \beta_i^2 \left( \frac{t - (t_{i-1})}{1} \right)^6 e^{-k_{3+i}^2 \left( \frac{u_{\min}}{t-1} \right)^2} e^{-k_{3+i}^2 \left( \frac{|x-x_0|}{t-1} \right)^2} \quad \text{in} \ D_{T_{i-1}, T_i}, \ i = 2, 3, \ldots, i_{\text{final}},$$

$$G_i(t, x, \beta_i, k_{3+i}, k_{4+i}) := 0 \quad \text{for} \ t < t_{i-1}.$$  \hspace{1cm} (22)

We denote the supersolution obtained at stage $i$ by $\tilde{\phi}_i^0$.

Let $d = (T - (t_0 + R_0)$ At no loss of generality we assume $R_0 \leq \min(T_0/10, \epsilon_1/10)$. Now we consider the regions. Let $\Omega_{\epsilon_1} := \{D_T \cap \{T_{i_{\text{final}}} - \epsilon_1 < t \leq t_0 + R_0 + \epsilon_1\}$. Let $R_0 > 0$ be small enough that $B_{R_0}(X_0) \subset \Omega_{\epsilon_1}$. Let $\Omega_{\epsilon_1} := \Omega_{\epsilon_1} - B_{R_0}(X_0)$. Again, we repeat Steps 3 and 4 for this domain.

We call the supersolution obtained at this stage $\tilde{\phi}_1^0$. We now divide the region $D_{d, T} := D_T \cap \{d \leq t \leq T\}$ into a finite number of regions of the form $D_{t_j, t_{j+1}} := D_T \cap \{t_j - \epsilon_1 \leq t \leq t_{j+1}, j = 1, 2, \ldots, j_{\text{final}} - 1 \mid j^1 = d, j_{\text{final}} = T\}$, where $0 < t_{j+1} - t_j < T_0/10^3$. On each such region we carry out Steps 3 and 4 and in this process we add (in each region):

$$G^j(t, x, \beta_i, k_{3+i}, k_{4+i}) := \beta_j^2 \left( \frac{t - (t_{j-1})}{1} \right)^6 e^{-k_{3+j}^2 \left( \frac{u_{\min}}{t-1} \right)^2} e^{-k_{3+j}^2 \left( \frac{|x-x_0|}{t-1} \right)^2} \quad \text{in} \ D_{t_{j-1}, t_j}, \ j = 1, 2, \ldots, j_{\text{final}},$$

$$G^{j^1}(t, x, \beta_i, k_{3+i}, k_{4+i}) := 0 \quad \text{for} \ t < t_{j-1}.$$  \hspace{1cm} (23)

We call the supersolution that we obtain $\tilde{\phi}_1^j$. We now define:

$$\tilde{\phi}_2 := \begin{cases} 
\tilde{\phi}_1^j & \text{in} \ D_{t_{j-1}, t_j}, \ j = 1, 2, \ldots, j_{\text{final}}, \\
G^j(t, x, \beta, k_3, k_4) & \text{in} \ B_{R_0}(X_0), \\
\tilde{\phi}_2^0 & \text{in} \ \Omega_{\epsilon_1} \\
\tilde{\phi}_0^0 & \text{in} \ D_{T_{i-1}, T_i}, \ i = 2, 3, \ldots, i_{\text{final}}.
\end{cases}$$  \hspace{1cm} (24)

Note that $\alpha L(\phi_2) \geq 0$ in $D_{\epsilon_1}^0$ for positive $\beta, \gamma, \sigma$ small enough. By choosing $\eta$ small, we can choose $\beta$ as small as we wish. By choosing $\eta$ small we see that $G_1$ adds a small positive
constant to $\nu^{\gamma}_\sigma$ at $X_0$. By choosing $\gamma$ and $\sigma$ small, we see that $\varphi_2(X_0) > \nu(X_0)$ but can be made as close as we wish. Function $\varphi_2$ satisfies all the conclusions of our theorem except it may be in $C^2(D_T, R^N)$ but not in $C^{2+\epsilon_0}(D_T, R^N)$. As at the end of Case I, we remedy this with an Evans smoothing in $D_T \cap \{t_0/3 < t\}$ with small enough positive parameter.

We call that function $\varphi_2$. Case II is proved.

4. Approximation of viscosity subsolutions by smooth subsolutions

**Theorem 20.** Let $L$ be a quasi-linear hyperbolic first order system of the type considered in Section 1 (with $A^0 = I$), with $A^i$ and $f$ admissible, in a slab domain $D_T$. Let $\epsilon_0 \in (0, 1)$. Let $\epsilon_1 > 0$. Let $X_0 = (t_0, x_0) \in D_T^0$. Let $u \in (D_T^0, R^N)$ be an lower semi-continuous viscosity subsolution, bounded and with bounded jump, of $\mathcal{E}^-(u)(X_0) \leq 0$. Let $w \in C^{2+\epsilon_0}(R^n, R^N) \cap H^{2,2}(R^n, R^N)$ and let $u(0, x) = w_0(x)$. Then there exists $\varphi_3 \in C^{2+\epsilon_0}(D_T, R^N) \cap H^{2,2}(D_T, R^N)$ satisfying

1. $u(X_0) - \epsilon_1 < \varphi_3(X_0) < u(X_0)$,
2. $\varphi_3(0, x) = w(x)$,
3. $\mathcal{E}^-(\varphi_3)(X_0) \leq 0$,
4. $L(\varphi_3) \leq 0$ in $D_T^0$.

**(25)**

**Proof.** The mirror image of the proof of Theorem 15, mutis mutandis. □

5. Eternal solutions

We note that in the case that $D_T = D_\infty$, Theorem 1 is still true. This is because Theorem 15 still holds because we may iterate the method of its proof—that is to say we use a countable number of strips in its proof and note that the strips have a time width uniformly bounded from below. Thus Theorems 15 and 20 both still hold. Also note that the Banach spaces we used have the property that when $t = \infty$, they are contained in the Banach spaces of the same type when $t$ is a finite $T$. Thus the comparison theorems of Section 5 of Part I [6] and the Difference Criterion of Section 6 of Part I [6] still hold. Then, we see that all our results still hold.

References