Pergamon

Computers Math. Applic. Vol. 35, No. 10, pp. 9–14, 1998

0898-1221/98 \$19.00 + 0.00

PII: S0898-1221(98)00067-4

A New Completely General Class of Variational Inclusions with Noncompact Valued Mappings

NAN-JING HUANG
Department of Mathematics, Sichuan University
Chengdu, Sichuan 610064, P.R. China

(Received August 1997; accepted January 1998)

Abstract—In this paper, we introduce and study a new completely general class of variational inclusions with noncompact valued mappings and construct some new iterative algorithms. We prove the existence of solutions for the completely general class of variational inclusions and the convergence of iterative sequences generated by the algorithms. © 1998 Elsevier Science Ltd. All rights reserved.

Keywords—Variational inclusion, Set-valued mapping, Algorithm, Existence, Convergence.

1. INTRODUCTION

Variational inequalities, introduced by Hartman and Stampacchia [1] in the early sixties, are a very powerful tool of the current mathematical technology. These have been extended and generalized to study a wide class of problems arising in mechanics, physics, optimization and control, nonlinear programming, economics, and transportation equilibrium and engineering sciences, etc. Quasivariational inequalities are generalized form of variational inequalities in which the constraint set depend on the solution. These were introduced and studied by Bensoussan, Goursat and Lions [2]. For further details, we refer to [3–8].

In 1991, Chang and Huang [9,10] introduced and studied some new class of complementarity problems and variational inequalities for set-valued mappings with compact values in Hilbert spaces. In the recent paper [11], Adly has studied a new general class of variational inclusions, which included many variational inequalities, quasi-variational inequalities, and explicit and implicit complementarity problems considered by Noor [12–14], Isac [15], Siddiqi and Ansari [16,17], and Hassouni and Moudafi [18] as special cases.

In this paper, we first introduce a new completely general class of variational inclusions for set-valued mapping. Motivated and inspired by the methods of Adly [11] and Huang [19], we construct some new iterative algorithms for the new completely general class of variational inclusions with noncompact valued mappings. We also prove the existence of solutions for the completely general class of variational inclusions and the convergence of iterative sequences generated by the algorithms.

2. PRELIMINARIES

Let H be a real Hilbert space endowed with a norm $\|\cdot\|$, and inner product $\langle\cdot,\cdot\rangle$. Given set-valued mappings $T, A, G : H \to 2^H$ (where 2^H denotes the family of all nonempty subsets of H),

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TeX a set-valued maximal monotone mapping $M : H \to 2^H$ and single-valued mappings $f, p : H \to H$ with $\text{Range}(G) \bigcap \text{dom}(M) \neq \emptyset$, we consider the following problem:

Find $u \in H,\ w \in Tu,\ y \in Au,\ z \in Gu$ such that

$$0 \in f(w) - p(y) + M(z). \tag{2.1}$$

This problem is called a completely general class of variational inclusion for set-valued mappings.

An equivalent formulation of the original problem (2.1) is to find $u \in H,\ w \in Tu,\ y \in Au,$ $z \in Gu$ such that

$$\langle v^* + f(w) - p(y), v - z\rangle \geq 0, \qquad \forall\ (v, v^*) \in \text{Graph}\,(M). \tag{2.2}$$

Since M is maximal monotone, $u \in H,\ w \in Tu,\ y \in Au,\ z \in Gu$ are the solutions of the problem (2.2) if and only if $u \in H,\ w \in Tu,\ y \in Au,\ z \in Gu$ such that $p(y) - f(w) \in M(z)$.

If $G : H \to H$ is a single-valued mapping, then problem (2.1) is equivalent to finding $u \in H$, $w \in Tu,\ y \in Au$ such that

$$0 \in f(w) - p(y) + M(G(u)). \tag{2.3}$$

This problem is called a general class of variational inclusion for set-valued mapping.

If $T, A : H \to H$ are two identity mappings and $G : H \to H$ is a single-valued mapping, then problem (2.1) is equivalent to finding $u \in H$ such that

$$0 \in f(u) - p(u) + M(G(u)). \tag{2.4}$$

Variational inclusion like (2.4) have been studied in [11].

If $G : H \to H$ is a single-valued mapping and $M := \partial\varphi$, where $\partial\varphi$ denotes the subdifferential of a proper, convex, and lower semicontinuous function $\varphi : H \to R \cup \{+\infty\}$, then problem (2.1) is equivalent to finding $u \in H,\ w \in Tu,\ y \in Au$ such that

$$\begin{aligned} &G(u) \bigcap \text{dom}\,(\partial\varphi) \neq \emptyset, \\ &\langle f(w) - p(y), v - G(u)\rangle \geq \varphi(G(u)) - \varphi(v), \qquad \forall\, v \in H, \end{aligned} \tag{2.5}$$

which is called a set-valued nonlinear generalized variational inclusion studied in [19].

It is clear that the completely general class of variational inclusion (2.1) includes many kinds of variational inequalities, quasi-variational inequalities, and explicit and implicit complementarity problem of [1,6–11,15,18–23] as special cases.

3. ITERATIVE ALGORITHM

LEMMA 3.1. *u, w, y, and z are solutions of problem (2.1) if and only if there exists $w \in Tu$, $y \in Au$, and $z \in Gu$ such that*

$$u = u - z + J_\alpha^M(z - \alpha(f(w) - p(y))),$$

where $\alpha > 0$ is a constant and $J_\alpha^M = (I + \alpha M)^{-1}$ is the so-called proximal mapping on H.

PROOF. From the definition of the proximal mapping J_α^M one has

$$z - \alpha(f(w) - p(y)) \in z + \alpha M(z),$$

hence,

$$p(y) - f(w) \in M(z).$$

Thus, u, w, y, and z are solutions of problem (2.1). This completes the proof.

REMARK 3.1. We note that when $G : H \to H$ is a single-valued mapping and $M := \partial\varphi$, Lemma 3.1 is similar to Lemma 2.1 in [19].

Based on Lemma 3.1, we proceed our algorithm.

Let $T, A, G : H \to CB(H)$ (where $CB(H)$ denotes the family of all nonempty closed bounded subsets of H). For given $u_0 \in H$, let $w_0 \in Tu_0$, $y_0 \in Au_0$, $z_0 \in Gu_0$, and

$$u_1 = u_0 - z_0 + J_\alpha^M(z_0 - \alpha(f(w_0) - p(y_0))).$$

By [26] there exist $w_1 \in Tu_1$, $y_1 \in Au_1$, and $z_1 \in Gu_1$ such that

$$\begin{aligned} \|w_1 - w_0\| &\le (1+1)\widehat{\mathbf{H}}(Tu_1, Tu_0), \\ \|y_1 - y_0\| &\le (1+1)\widehat{\mathbf{H}}(Au_1, Au_0), \\ \|z_1 - z_0\| &\le (1+1)\widehat{\mathbf{H}}(Gu_1, Gu_0), \end{aligned}$$

where $\widehat{\mathbf{H}}$ is the Hausdorff metric on $CB(H)$. By induction, we can obtain our algorithm as following.

ALGORITHM 3.1. Let $T, A, G : H \to CB(H)$, and $f, p : H \to H$. For given $u_0 \in H$, we can get an algorithm for (2.1) as following:

$$\begin{aligned} &u_{n+1} = u_n - z_n + J_\alpha^M(z_n - \alpha(f(w_n) - p(y_n))), \\ w_n \in Tu_n, \quad &\|w_{n+1} - w_n\| \le \left(1 + (1+n)^{-1}\right) \widehat{\mathbf{H}}(Tu_{n+1}, Tu_n), \\ y_n \in Au_n, \quad &\|y_{n+1} - y_n\| \le \left(1 + (1+n)^{-1}\right) \widehat{\mathbf{H}}(Au_{n+1}, Au_n), \\ z_n \in Gu_n, \quad &\|z_{n+1} - z_n\| \le \left(1 + (1+n)^{-1}\right) \widehat{\mathbf{H}}(Gu_{n+1}, Gu_n), \\ &n = 0, 1, 2, \dots. \end{aligned} \tag{3.1}$$

From Algorithm 3.1, we can get the following algorithm.

ALGORITHM 3.2. Let $T, A : H \to CB(H)$, and $f, p, G : H \to H$. For given $u_0 \in H$, we can get an algorithm for (2.3) as following:

$$\begin{aligned} &u_{n+1} = u_n - G(u_n) + J_\alpha^M(G(u_n) - \alpha(f(w_n) - p(y_n))), \\ w_n \in Tu_n, \quad &\|w_{n+1} - w_n\| \le \left(1 + (1+n)^{-1}\right) \widehat{\mathbf{H}}(Tu_{n+1}, Tu_n), \\ y_n \in Au_n, \quad &\|y_{n+1} - y_n\| \le \left(1 + (1+n)^{-1}\right) \widehat{\mathbf{H}}(Au_{n+1}, Au_n), \\ &n = 0, 1, 2, \dots. \end{aligned} \tag{3.2}$$

REMARK 3.2. The Algorithms 3.1 and 3.2 include several known algorithms of [6,9,10,12–14,16, 17,19,20,22–24] as special cases.

4. EXISTENCE AND CONVERGENCE

DEFINITION 4.1. *A mapping $g : H \to H$ is said to be*

(i) *strongly monotone if there exists some $\delta > 0$ such that*

$$\langle g(u_1) - g(u_2), u_1 - u_2 \rangle \ge \delta \|u_1 - u_2\|^2, \qquad \forall\, u_i \in H, \quad i = 1, 2;$$

(ii) *Lipschitz continuous if there exists some $\sigma > 0$ such that*

$$\|g(u_1) - g(u_2)\| \le \sigma \|u_1 - u_2\|, \qquad \forall\, u_i \in H, \quad i = 1, 2.$$

DEFINITION 4.2. *A set-valued mapping $T : H \to 2^H$ is said to be*

(i) *strongly monotone if there exists some $\xi > 0$ such that*

$$\langle w_1 - w_2, u_1 - u_2 \rangle \geq \xi \|u_1 - u_2\|^2, \qquad \forall\, u_i \in H, \quad w_i \in Tu_i, \quad i = 1, 2;$$

(ii) *strongly monotone with respect to a mapping $f : H \to H$ if there exists some $\beta > 0$ such that*

$$\langle f(w_1) - f(w_2), u_1 - u_2 \rangle \geq \beta \|u_1 - u_2\|^2, \qquad \forall\, u_i \in H, \quad w_i \in Tu_i, \quad i = 1, 2;$$

(iii) *$\widehat{\mathbf{H}}$-Lipschitz continuous if there exists some $\gamma > 0$ such that*

$$\widehat{\mathbf{H}}(Tu_1, Tu_2) \leq \gamma \|u_1 - u_2\|, \qquad \forall\, u_i \in H, \quad i = 1, 2.$$

THEOREM 4.1. *Let $G : H \to CB(H)$ be strongly monotone and $\widehat{\mathbf{H}}$-Lipschitz continuous, $f, p : H \to H$ be Lipschitz continuous, $T, A : H \to CB(H)$ be $\widehat{\mathbf{H}}$-Lipschitz continuous and T be strongly monotone with respect to f. If the following conditions hold:*

$$\left| \alpha - \frac{\beta + \epsilon\mu(k-1)}{\eta^2\gamma^2 - \epsilon^2\mu^2} \right| < \frac{\sqrt{(\beta + (k-1)\epsilon\mu)^2 - (\eta^2\gamma^2 - \epsilon^2\mu^2)\, k(2-k)}}{\eta^2\gamma^2 - \epsilon^2\mu^2}, \tag{4.1}$$

$$\beta > (1-k)\epsilon\mu + \sqrt{(\eta^2\gamma^2 - \epsilon^2\mu^2)\, k(2-k)}, \qquad \eta\gamma > \epsilon\mu, \tag{4.2}$$

$$\alpha\mu\epsilon < 1 - k, \qquad k = 2\sqrt{1 - 2\delta + \sigma^2}, \qquad k < 1, \tag{4.3}$$

where β and δ are strongly monotone constants of T and G respectively, γ, μ, and σ are $\widehat{\mathbf{H}}$-Lipschitz constants of T, A, and G respectively, η and ϵ are the Lipschitz constants of f and p, respectively, then there exist $u \in H$, $w \in Tu$, $y \in Au$, and $z \in Gu$, which are solutions of problem (2.1). Moreover,

$$u_n \to u, \quad w_n \to w, \quad y_n \to y, \quad z_n \to z, \qquad n \to \infty,$$

where $\{u_n\}$, $\{w_n\}$, $\{y_n\}$, and $\{z_n\}$ are defined in Algorithm 3.1.

PROOF. From (3.1), we have

$$\|u_{n+1} - u_n\| = \left\| u_n - u_{n-1} - (z_n - z_{n-1}) + J_\alpha^M(h(u_n)) - J_\alpha^M(h(u_{n-1})) \right\|,$$

where $h(u_n) = z_n - \alpha(f(w_n) - p(y_n))$. Also, we have

$$\begin{aligned} \left\| J_\alpha^M(h(u_n)) - J_\alpha^M(h(u_{n-1})) \right\| &\leq \|h(u_n) - h(u_{n-1})\| \\ &\leq \|u_n - u_{n-1} - \alpha(f(w_n) - f(w_{n-1}))\| \\ &\quad + \|u_n - u_{n-1} - (z_n - z_{n-1})\| + \alpha\|p(y_n) - p(y_{n-1})\|. \end{aligned}$$

That is

$$\begin{aligned} \|u_{n+1} - u_n\| \leq{} & 2\|u_n - u_{n-1} - (z_n - z_{n-1})\| \\ & + \|u_n - u_{n-1} - \alpha(f(w_n) - f(w_{n-1}))\| + \alpha\|p(y_n) - p(y_{n-1})\|. \end{aligned} \tag{4.4}$$

By $\widehat{\mathbf{H}}$-Lipschitz continuity and strongly monotonicity of G, we obtain

$$\|u_n - u_{n-1} - (z_n - z_{n-1})\|^2 \leq \left(1 - 2\delta + \left(1 + n^{-1}\right)^2 \sigma^2\right) \|u_n - u_{n-1}\|^2. \tag{4.5}$$

Also from $\widehat{\mathbf{H}}$-Lipschitz continuity and strongly monotonicity of T, and Lipschitz continuity of f, we have

$$\|u_n - u_{n-1} - \alpha(f(w_n) - f(w_{n-1}))\|^2 \le \left(1 - 2\beta\alpha + \alpha^2\eta^2\left(1+n^{-1}\right)^2\gamma^2\right)\|u_n - u_{n-1}\|^2. \quad (4.6)$$

By $\widehat{\mathbf{H}}$-Lipschitz continuity of A, Lipschitz continuity of p and (3.1), we know

$$\alpha\|p(y_n) - p(y_{n-1})\| \le \alpha\epsilon\left(1+n^{-1}\right)\mu\|u_n - u_{n-1}\|. \quad (4.7)$$

So by combining (4.4)–(4.7), we get

$$\|u_{n+1} - u_n\| \le \theta_n\|u_n - u_{n-1}\|,$$

where

$$\theta_n := 2\sqrt{1 - 2\delta + \left(1+n^{-1}\right)^2\sigma^2} + \sqrt{1 - 2\beta\alpha + \alpha^2\eta^2\left(1+n^{-1}\right)^2\gamma^2} + \alpha\epsilon\left(1+n^{-1}\right)\mu.$$

Letting

$$\theta := 2\sqrt{1 - 2\delta + \sigma^2} + \sqrt{1 - 2\beta\alpha + \alpha^2\eta^2\gamma^2} + \alpha\epsilon\mu,$$

we know that $\theta_n \searrow \theta$. It follows from (4.1)–(4.3) that $\theta < 1$. Hence $\theta_n < 1$, for n sufficiently large. Therefore, $\{u_n\}$ is a Cauchy sequence and we can suppose that $u_n \to u \in H$.

Now we prove that

$$w_n \to w \in Tu, \qquad y_n \to y \in Au, \qquad z_n \to z \in Gu.$$

In fact, it follows from the Algorithm 3.1 that

$$\begin{aligned} \|w_n - w_{n-1}\| &\le \left(1+n^{-1}\right)\gamma\|u_n - u_{n-1}\|, \\ \|y_n - y_{n-1}\| &\le \left(1+n^{-1}\right)\mu\|u_n - u_{n-1}\|, \\ \|z_n - z_{n-1}\| &\le \left(1+n^{-1}\right)\sigma\|u_n - u_{n-1}\|, \end{aligned}$$

i.e., $\{w_n\}$, $\{y_n\}$, and $\{z_n\}$ are Cauchy sequences. Let $w_n \to w$, $y_n \to y$, $z_n \to z$. Further, we have

$$\begin{aligned} d(w, Tu) &= \inf\{\|w - z\| : z \in Tu\} \\ &\le \|w - w_n\| + d(w_n, Tu) \\ &\le \|w - w_n\| + \widehat{\mathbf{H}}(Tu_n, Tu) \\ &\le \|w - w_n\| + \gamma\|u_n - u\| \to 0. \end{aligned}$$

Hence, $w \in Tu$. Similarly, $y \in Au$ and $z \in Gu$. This completes the proof.

From Theorem 4.1, we can obtain the following theorem.

THEOREM 4.2. *Let $G : H \to H$ be strongly monotone and Lipschitz continuous, $f, p : H \to H$ be Lipschitz continuous, $T, A : H \to CB(H)$ be $\widehat{\mathbf{H}}$-Lipschitz continuous and T and be strongly monotone with respect to f. Let β and δ be strongly monotone constants of T and G respectively, γ and μ be $\widehat{\mathbf{H}}$-Lipschitz constants of T and A respectively, σ, η, and ϵ are the Lipschitz constants of G, f, and p, respectively. If conditions (4.1)–(4.3) hold, then there exist $u \in H$, $w \in Tu$, and $y \in Au$, which are solutions of problem (2.3). Moreover,*

$$u_n \to u, \quad w_n \to w, \quad y_n \to y, \qquad n \to \infty,$$

where $\{u_n\}$, $\{w_n\}$, and $\{y_n\}$ are defined in Algorithm 3.2.

REMARK 4.1. For a suitable choice of the mappings G, T, A, f, p, and M, we can obtain several known results [6,9,10,12–17,19,20,22–24] as special cases of the main results of this paper.

REFERENCES

1. P. Hartman and G. Stampacchia, On some nonlinear elliptic differential functional equations, *Acta. Math.* **115**, 271–310, (1966).
2. A. Bensoussan, M. Goursat and J.L. Lions, Control impulsinnel et inequations quasivariationalles stationaries, *C. R. Acad. Sci.* **276**, 1279–1284, (1973).
3. C. Baiocchi and A. Capelo, *Variational and Quasivariational Inequalities, Application to Free Boundary Problems*, Wiley, New York, (1984).
4. A. Bensoussan, *Stochastic Control by Functional Analysis Method*, North-Holland, Amsterdam, (1982).
5. A. Bensoussan and J.L. Lions, *Impulse Control and Quasivariational Inequalities*, Gauthiers-Villers, Bordas, Paris, (1984).
6. S. Chang, *Variational Inequality and Complementarity Problem Theory with Applications*, Shanghai Scientific and Tech. Literature Publishing House, Shanghai, (1991).
7. G. Isac, Complementarity problems, In *Lecture Notes in Mathematics*, Springer-Verlag, Heidelberg, (1992).
8. U. Mosco, Implicit variational problems and quasi-variational inequalities, In *Lecture Notes in Mathematics*, Volume 543, pp. 83–156, Springer-Verlag, Berlin, (1976).
9. S. Chang and N. Huang, Generalized strongly nonlinear quasi-complementarity problems in Hilbert spaces, *J. Math. Anal. Appl.* **158**, 194–202, (1991).
10. S. Chang and N. Huang, Generalized multivalued implicit complementarity problems in Hilbert spaces, *Math. Japonica* **36**, 1093–1100, (1991).
11. S. Adly, Perturbed algorithm and sensitivity analysis for a general class of variational inclusions, *J. Math. Anal. Appl.* **201**, 609–630, (1996).
12. M.A. Noor, Strongly nonlinear variational inequalities, *C. R. Math. Rep. Acad. Sci. Canada* **4**, 213–218, (1982).
13. M.A. Noor, On the nonlinear complementarity problem, *J. Math. Anal. Appl.* **123**, 455–460, (1987).
14. M.A. Noor, Quasivariational inequalities, *Appl. Math. Lett.* **1** (4), 367–370, (1988).
15. G. Isac, A special variational inequality and the implicit complementarity problem, *J. Fac. Sci. Univ. Tokyo* **37**, 109–127, (1990).
16. A.H. Siddiqi and Q.H. Ansari, Strongly nonlinear quasivariational inequalities, *J. Math. Anal. Appl.* **149**, 444–450, (1990).
17. A.H. Siddiqi and Q.H. Ansari, General strongly nonlinear variational inequalities, *J. Math. Anal. Appl.* **166**, 386–392, (1992).
18. A. Hassouni and A. Moudafi, A perturbed algorithm for variational inclusions, *J. Math. Anal. Appl.* **185**, 706–712, (1994).
19. N. Huang, Generalized nonlinear variational inclusions with noncompact valued mapping, *Appl. Math. Lett.* **9** (3), 25–29, (1996).
20. S. Chang and N. Huang, Generalized quasi-complementarity problems for a pair of fuzzy mappings, *J. Fuzzy Math.* **4**, 343–354, (1996).
21. X.P. Ding, Generalized strongly nonlinear quasivariational inequalities, *J. Math. Anal. Appl.* **173**, 577–587, (1993).
22. N. Huang, Completely Generalized strongly nonlinear quasi-complementarity problems for fuzzy mappings, *Indian J. Pure Appl. Math.* **28**, 23–32, (1997).
23. N. Huang, Completely generalized nonlinear variational inclusions for fuzzy mappings, *Czech. Math. J.* (to appear).
24. L. Zheng, Completely generalized strongly nonlinear quasicomplementarity problems in Hilbert spaces, *J. Math. Anal. Appl.* **193**, 706–714, (1995).
25. L. Zheng, Iterative algorithm for finding approximate solutions to completely generalized strongly nonlinear quasivariational inequalities, *J. Math. Anal. Appl.* **201**, 180–194, (1996).
26. S.B. Nadler, Jr., Multi-valued contraction mappings, *Pacific J. Math.* **30**, 475–488, (1969).