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Robust stability for fractional-order systems with structured and unstructured uncertainties[☆]

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ABSTRACT

The issues of robust stability for uncertain fractional-order systems of two types of order $\alpha \in (0, 1)$ are dealt with in this paper. For the polytope-type uncertainty case, a less conservative sufficient condition for robust stability is given; for the norm-bounded uncertainty case, a sufficient and necessary condition for robust stability is presented. Both of these conditions can be checked by solving sets of linear matrix inequalities. Two numerical examples are presented to confirm the proposed conditions.

1. Introduction

As an application background of fractional calculus [1], fractional-order control systems [2–14] have attracted increasing attention in the past few decades. The problems of robust stability for interval uncertain fractional-order systems were investigated systematically in [15–20]. For instance, an experimentally verified Kharitonov-like procedure for checking robust stability for interval fractional-order linear time invariant (FO-LTI) systems described by a transfer function was proposed in [15]. In [16], the robust stability problem for interval FO-LTI systems with order $0 < \alpha < 1$ described by a state-space form was discussed. In [17], a complex Lyapunov inequality was utilized to find the maximum eigenvalue of a Hermitian matrix, and a robust stability checking method for interval FO-LTI systems with order $1 < \alpha < 2$ was proposed. As an extension of [17], a necessary and sufficient condition for robust stability for uncertain FO-LTI systems with order $1 \leq \alpha < 2$ was proposed in [18]. Further, in [19], necessary and sufficient conditions for stability and stabilization of fractional-order interval systems with order $1 < \alpha < 2$ were presented. A sufficient and necessary condition for the robust asymptotical stability of fractional-order interval systems with order $0 < \alpha < 1$ was presented in [20], and a sufficient condition for the robust asymptotical stabilization was also derived. All the results in [19,20] were obtained in terms of linear matrix inequalities. For more knowledge about stability conditions for interval fractional-order systems, please refer to [21–24].

For more general uncertainties in control systems theory, polytope-type uncertainty and norm-bounded uncertainty are two representative forms of structured and unstructured uncertainties. Norm-bounded [25] uncertainty is mainly used along with the small gain theory in the robust stability analysis, and the polytope-type [26] uncertainty is primarily used for quadratic stability analysis. To the best of our knowledge, there exist no results about robust stability for fractional-order systems with these two kinds of uncertainties. With motivation from the fact mentioned above, robust stability for fractional-order systems of order $\alpha \in (0, 1)$ with structured and unstructured uncertainties is discussed in the current paper.

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This paper is organized as follows. Some preliminary results are recalled in Section 2. The main results on robust stability for fractional-order systems with structured and unstructured uncertainties are addressed in Section 3. Two numerical examples are given to verify the proposed conditions in Section 4, and a conclusion is given in Section 5.

2. Preliminaries

In this paper, we adopt the well-known Caputo definition for the fractional derivative [4],

$$D^{\alpha}f(t) := {}_0D_t^{\alpha}f(t) = \frac{1}{\Gamma(\alpha-n)}\int_0^t \frac{f^{(n)}(\tau)d\tau}{(t-\tau)^{\alpha+1-n}}$$

with $\alpha \in R^{+}$ satisfying $n-1<\alpha<n, f(t) \in C^n(0,\infty)$.

In general, FO-LTI systems can be described by the transfer function [4]

$$G(s) = \frac{b_m s^{\beta_m} + b_{m-1}s^{\beta_{m-1}} + \cdots + b_1 s^{\beta_1} + b_0}{a_n s^{\alpha_n} + a_{n-1}s^{\alpha_{n-1}} + \cdots + a_1 s^{\alpha_1} + a_0} \tag{1}$$

where $a_i, \alpha_i, b_j, \beta_j \in R$ $(i=0,1,\ldots,n, j=0,1,\ldots,m)$, $a_n \neq 0, \alpha_n > \alpha_{n-1} > \cdots > \alpha_1 > 0, \beta_m > \beta_{m-1} > \cdots > \beta_1 > 0$ and $\alpha_n \geqslant \beta_m$.

Assume that the FO-LTI system (1) can be converted into a commensurate form under some algebraic operations:

$$G(s) = \frac{b_m s^{m/v} + b_{m-1}s^{(m-1)/v} + \cdots + b_1 s^{1/v} + b_0}{a_n s^{n/v} + a_{n-1}s^{(n-1)/v} + \cdots + a_1 s^{1/v} + a_0}, \quad (v>1)$$

We focus our attention on the commensurate FO-LTI systems in the current paper.

The stability issues of FO-LTI systems were given in [3] as the following.

Lemma 1. *An FO-LTI system described by the transfer function*

$$G(s) = \frac{\sum_{k=0}^{m} b_k (s^{\alpha})^k}{\sum_{k=0}^{n} a_k (s^{\alpha})^k} = \frac{Q(s^{\alpha})}{P(s^{\alpha})}, \quad (0<\alpha<2)$$

is asymptotically stable if and only if $|\arg(\sigma_i)| > \alpha\frac{\pi}{2}$, *with* σ_i *being the ith root of the pseudo-polynomial* $P(\sigma)$, $\sigma = s^{\alpha}$.

Lemma 2. *An FO-LTI system described by the state-space form*

$$D^{\alpha}x(t) = Ax(t), \quad (0<\alpha<2)$$

is asymptotically stable if and only if $|\arg(eig(A))| > \alpha\frac{\pi}{2}$, *where* $eig(A)$ *are eigenvalues of the matrix A.*

The stable regions denoted as the D^{α}-stable regions for $0<\alpha<1, \alpha=1$ and $1<\alpha<2$ are shown in Fig. 1.

To proceed with our discussion about the robust stability for the uncertain fractional-order systems, the following lemmas need to be recalled.

Lemma 3 ([27])**.** *An FO-LTI system described by*

$$D^{\alpha}x(t) = Ax(t), \quad (0<\alpha<1)$$

is asymptotically stable if and only if there exist two positive definite Hermitian matrices Q_1 *and* Q_2 *such that*

$$e^{i\theta}Q_1A^* + e^{-i\theta}AQ_1 + e^{-i\theta}Q_2A^* + e^{i\theta}AQ_2 < 0,$$

where $\theta = (1-\alpha)\pi/2$.

Lemma 4 ([20])**.** *An FO-LTI system described by*

$$D^{\alpha}x(t) = Ax(t), \quad (0<\alpha<1)$$

is asymptotically stable if and only if there exist two real symmetric positive definite matrices Q_{11} *and* Q_{21}, *and two skew-symmetric matrices* Q_{12} *and* Q_{22}, *such that*

$$\sum_{i=1}^{2}\sum_{j=1}^{2} sym\{\Theta_{ij} \otimes (AQ_{ij})\} < 0,$$

$$\begin{bmatrix} Q_{11} & Q_{12} \\ -Q_{12} & Q_{11} \end{bmatrix} > 0, \qquad \begin{bmatrix} Q_{21} & Q_{22} \\ -Q_{22} & Q_{21} \end{bmatrix} > 0$$

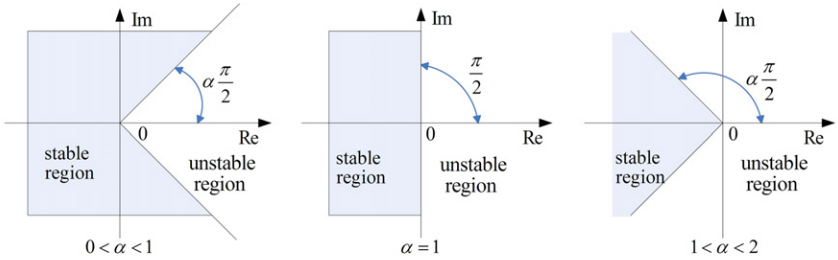

Fig. 1. Stable regions for $0 < \alpha < 1$, $\alpha = 1$ and $1 < \alpha < 2$.

where $\theta = (1-\alpha)\pi/2$, *and the* Θ_{ij} $(i, j = 1, 2)$ *are defined as follows:*

$$\Theta_{11} = \begin{bmatrix} \sin\left(\frac{\pi\alpha}{2}\right) & -\cos\left(\frac{\pi\alpha}{2}\right) \\ \cos\left(\frac{\pi\alpha}{2}\right) & \sin\left(\frac{\pi\alpha}{2}\right) \end{bmatrix}, \qquad \Theta_{12} = \begin{bmatrix} \cos\left(\frac{\pi\alpha}{2}\right) & \sin\left(\frac{\pi\alpha}{2}\right) \\ -\sin\left(\frac{\pi\alpha}{2}\right) & \cos\left(\frac{\pi\alpha}{2}\right) \end{bmatrix}, \tag{2}$$

$$\Theta_{21} = \begin{bmatrix} \sin\left(\frac{\pi\alpha}{2}\right) & \cos\left(\frac{\pi\alpha}{2}\right) \\ -\cos\left(\frac{\pi\alpha}{2}\right) & \sin\left(\frac{\pi\alpha}{2}\right) \end{bmatrix}, \qquad \Theta_{22} = \begin{bmatrix} -\cos\left(\frac{\pi\alpha}{2}\right) & \sin\left(\frac{\pi\alpha}{2}\right) \\ -\sin\left(\frac{\pi\alpha}{2}\right) & -\cos\left(\frac{\pi\alpha}{2}\right) \end{bmatrix}. \tag{3}$$

3. The main results

In this section, robust stability for fractional-order systems of order $\alpha \in (0, 1)$ with structured and unstructured uncertainties are analyzed.

3.1. The polytope-type uncertainty case

Consider an uncertain FO-LTI system described by

$$D^{\alpha}x(t) = Ax(t), \quad (0 < \alpha < 1) \tag{4}$$

where $x(t) \in R^n$, and $A \in R^{n\times n}$ belongs to a polytope-type domain $\mathbb{A}$. Any matrix inside the domain $\mathbb{A}$ can be written as a convex combination of the known vertices A_i of the uncertainty polytope [26], i.e.,

$$\mathbb{A} := \left\{ A(\xi) : A(\xi) = \sum_{i=1}^{N} \xi_i A_i; \ \sum_{i=1}^{N} \xi_i = 1; \ \xi_i \geq 0 \right\}. \tag{5}$$

Remark 1. Polytopes of matrices have been established as one of standard representations of uncertainties involved in control systems described by the state-space models [28]. It is natural to discuss the robust stability problem when the system matrices of uncertain systems are formulated in terms of a polytope of matrices.

On the basis of Lemma 3, the following definition is given.

Definition 1. The uncertain FO-LTI system (4) is said to be robustly stable if there exist positive definite Hermitian matrices $Q_1(\xi)$ and $Q_2(\xi)$ such that

$$e^{i\theta}Q_1(\xi)A^*(\xi) + e^{-i\theta}A(\xi)Q_1(\xi) + e^{-i\theta}Q_2(\xi)A^*(\xi) + e^{i\theta}A(\xi)Q_2(\xi) < 0,$$

for any $A(\xi) \in \mathbb{A}$, where $\theta = (1-\alpha)\pi/2$.

Due to convexity, robust stability of the uncertain FO-LTI system (4) can be easily verified through the existence of positive definite Hermitian matrices Q_1 and Q_2 such that

$$e^{i\theta}Q_1A_i^* + e^{-i\theta}A_iQ_1 + e^{-i\theta}Q_2A_i^* + e^{i\theta}A_iQ_2 < 0, \quad (i = 1, 2, \ldots, N), \tag{6}$$

where $\theta = (1-\alpha)\pi/2$.

Remark 2. Note that (6) yields a conservative result, as the Hermitian matrices $Q_1(\xi)$ and $Q_2(\xi)$ are restricted to constant positive definite Hermitian matrices Q_1 and Q_2. A less conservative alternative for testing the robust stability of the uncertain FO-LTI system (4) is given as the following theorem.

Theorem 1. *If there exist positive definite Hermitian matrices* Q_{1i} *and* Q_{2i} $(i = 1, 2, \ldots, N)$ *such that*

$$e^{i\theta}Q_{1i}A_i^* + e^{-i\theta}A_iQ_{1i} + e^{-i\theta}Q_{2i}A_i^* + e^{i\theta}A_iQ_{2i} < -I, \quad (i = 1, 2, \ldots, N), \tag{7}$$

$$e^{i\theta}Q_{1j}A_i + e^{-i\theta}A_i^*Q_{1j} + e^{i\theta}Q_{1i}A_j + e^{-i\theta}A_j^*Q_{1i} + e^{-i\theta}Q_{2j}A_i + e^{i\theta}A_i^*Q_{2j} + e^{-i\theta}Q_{2i}A_j + e^{i\theta}A_j^*Q_{2i} < \frac{2}{N-1}I$$
$$(i = 1, 2, \ldots, N-1,\ j = i+1, \ldots, N), \tag{8}$$

then the uncertain FO-LTI system (4) is robustly stable.

Proof. Let $Q_1(\xi) = \sum_{i=1}^N \xi_i Q_{1i}$, $Q_2(\xi) = \sum_{i=1}^N \xi_i Q_{2i}$, where $\sum_{i=1}^N \xi_i = 1$ and $\xi_i \geq 0$. It is clear that $Q_1(\xi)$ and $Q_2(\xi)$ are positive definite parameter dependent Hermitian matrices. From (5), the following equation can be obtained:

$$\begin{aligned}
&e^{i\theta}Q_1(\xi)A^*(\xi) + e^{-i\theta}A(\xi)Q_1(\xi) + e^{-i\theta}Q_2(\xi)A^*(\xi) + e^{i\theta}A(\xi)Q_2(\xi)\\
&\quad = e^{i\theta}\left(\sum_{i=1}^N \xi_i Q_{1i}\right)\left(\sum_{i=1}^N \xi_i A_i^*\right) + e^{-i\theta}\left(\sum_{i=1}^N \xi_i A_i\right)\left(\sum_{i=1}^N \xi_i Q_{1i}\right)\\
&\qquad + e^{-i\theta}\left(\sum_{i=1}^N \xi_i Q_{2i}\right)\left(\sum_{i=1}^N \xi_i A_i^*\right) + e^{i\theta}\left(\sum_{i=1}^N \xi_i A_i\right)\left(\sum_{i=1}^N \xi_i Q_{2i}\right)\\
&\quad = \sum_{i=1}^N \xi_i^2(e^{i\theta}Q_{1i}A_i^* + e^{-i\theta}A_iQ_{1i} + e^{-i\theta}Q_{2i}A_i^* + e^{i\theta}A_iQ_{2i}) + \sum_{i=1}^{N-1}\sum_{j=i+1}^N \xi_i\xi_j(e^{i\theta}Q_{1j}A_i^* + e^{-i\theta}A_iQ_{1j} + e^{i\theta}Q_{1i}A_j^*\\
&\qquad + e^{-i\theta}A_jQ_{1i} + e^{-i\theta}Q_{2j}A_i^* + e^{i\theta}A_iQ_{2j} + e^{-i\theta}Q_{2i}A_j^* + e^{i\theta}A_jQ_{2i}).
\end{aligned}$$

Since

$$(N-1)\sum_{i=1}^N \xi_i^2 - 2\sum_{i=1}^{N-1}\sum_{j=i+1}^N \xi_i\xi_j = \sum_{i=1}^{N-1}\sum_{j=i+1}^N (\xi_i - \xi_j)^2 \geq 0,$$

from conditions (7) and (8), one gets

$$e^{i\theta}Q_1(\xi)A^*(\xi) + e^{-i\theta}A(\xi)Q_1(\xi) + e^{-i\theta}Q_2(\xi)A^*(\xi) + e^{i\theta}A(\xi)Q_2(\xi) < -\left(\sum_{i=1}^N \xi_i^2 - \sum_{i=1}^{N-1}\sum_{j=i+1}^N \xi_i\xi_j\frac{2}{N-1}\right)I \leq 0.$$

This completes the proof of the theorem. □

Remark 3. It is important to note that if the polytope obtained in (5) is known to be robustly stable, then any positive combination of its vertices can also produce stable matrices for the uncertain FO-LTI system (4), since a strictly positive scalar number clearly does not affect the arguments of the eigenvalues of the system matrix.

3.2. The norm-bounded uncertainty case

Consider an uncertain FO-LTI system:

$$D^\alpha x(t) = A(\Delta)x(t), \quad (0 < \alpha < 1) \tag{9}$$

where $A(\Delta) \in \mathrm{A}_\Delta := \{A + D\Delta E \mid \|\Delta\| \leq 1\}$, A, D and E are known real matrices with appropriate dimensions.

On the basis of Lemma 4, we have the following definition.

Definition 2. The uncertain FO-LTI system (9) is said to be robustly stable if for any Δ satisfying $\|\Delta\| \leq 1$, there exist two real symmetric positive definite matrices Q_{11} and Q_{21}, and two skew-symmetric matrices Q_{12} and Q_{22}, such that

$$\sum_{i=1}^2\sum_{j=1}^2 sym\{\Theta_{ij} \otimes (A(\Delta)Q_{ij})\} < 0,$$

$$\begin{bmatrix} Q_{11} & Q_{12} \\ -Q_{12} & Q_{11} \end{bmatrix} > 0, \qquad \begin{bmatrix} Q_{21} & Q_{22} \\ -Q_{22} & Q_{21} \end{bmatrix} > 0,$$

where the Θ_{ij} $(i, j = 1, 2)$ satisfy (2)–(3).

On the basis of Definition 2, the following theorem is given.

Theorem 2. *The uncertain FO-LTI system (9) is robustly stable if and only if there exist two real symmetric positive definite matrices Q_{11} and Q_{21}, two skew-symmetric matrices Q_{12} and Q_{22}, and scalar constants $\lambda_{ij} > 0$ $(i, j = 1, 2)$ such that*

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} < 0 \tag{10}$$

$$\begin{bmatrix} Q_{11} & Q_{12} \\ -Q_{12} & Q_{11} \end{bmatrix} > 0, \qquad \begin{bmatrix} Q_{21} & Q_{22} \\ -Q_{22} & Q_{21} \end{bmatrix} > 0 \tag{11}$$

where the Θ_{ij} $(i, j = 1, 2)$ are defined in (2)–(3), $sym\{X\}$ is defined as $sym\{X\} := X + X^T$, and $P_{11} = \sum_{i=1}^2 \sum_{j=1}^2 \{sym\{\Theta_{ij} \otimes (AQ_{ij})\} + \lambda_{ij}(I_2 \otimes DD^T)\}$,

$$P_{12} = \begin{bmatrix} I_2 \otimes (EQ_{11}) \\ I_2 \otimes (EQ_{12}) \\ I_2 \otimes (EQ_{21}) \\ I_2 \otimes (EQ_{22}) \end{bmatrix}^T, \qquad P_{22} = -diag(\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}) \otimes I_{2n}.$$

Proof (*Sufficiency*). As

$$\sum_{i=1}^{2}\sum_{j=1}^{2} sym\{\Theta_{ij} \otimes (A(\Delta)Q_{ij})\} = \sum_{i=1}^{2}\sum_{j=1}^{2} sym\{\Theta_{ij} \otimes (AQ_{ij})\} + \sum_{i=1}^{2}\sum_{j=1}^{2} sym\{\Theta_{ij} \otimes (D\Delta EQ_{ij})\},$$

then for any given $\|\Delta\| \le 1$, i.e., $\Delta\Delta^T \le I$, one has

$$(I_2 \otimes \Delta)\,(I_2 \otimes \Delta)^T = (I_2 \otimes \Delta)\,(I_2 \otimes \Delta^T) = (I_2 \otimes \Delta\Delta^T) \le I. \tag{12}$$

Note that $\Theta_{ij}\Theta_{ij}^T = I_2$ $(i, j = 1, 2)$. It is known from (12) and Lemma A.3 that for any real scalars $\lambda_{ij} > 0$ $(i, j = 1, 2)$,

$$\begin{aligned} sym\{\Theta_{ij} \otimes (D\Delta EQ_{ij})\} &= sym\{(\Theta_{ij} \otimes D)\,(I_2 \otimes \Delta)\,(I_2 \otimes EQ_{ij})\} \\ &\le \lambda_{ij}(\Theta_{ij} \otimes D)(I_2 \otimes \Delta)(I_2 \otimes \Delta)^T(\Theta_{ij} \otimes D)^T + \lambda_{ij}^{-1}(I_2 \otimes EQ_{ij})^T(I_2 \otimes EQ_{ij}) \\ &\le \lambda_{ij}(I_2 \otimes DD^T) + \lambda_{ij}^{-1}(I_2 \otimes EQ_{ij})^T(I_2 \otimes EQ_{ij}). \end{aligned}$$

Then

$$\begin{aligned} \sum_{i=1}^{2}\sum_{j=1}^{2} sym\{\Theta_{ij} \otimes (A(\Delta)Q_{ij})\} &\le \sum_{i=1}^{2}\sum_{j=1}^{2}(sym\{\Theta_{ij} \otimes (AQ_{ij})\} + \lambda_{ij}(I_2 \otimes DD^T)) \\ &\quad + \sum_{i=1}^{2}\sum_{j=1}^{2} \lambda_{ij}^{-1}(I_2 \otimes EQ_{ij})^T(I_2 \otimes EQ_{ij}). \end{aligned} \tag{13}$$

As (10) and (11) hold, then by using the Schur complement [29] of (13), one has

$$\sum_{i=1}^{2}\sum_{j=1}^{2} sym\{\Theta_{ij} \otimes (A(\Delta)Q_{ij})\} < 0.$$

Then it is known from Definition 2 that the uncertain FO-LTI system (9) is robustly stable.

(Necessity) As the uncertain FO-LTI system (9) is robustly stable, then it is known from Definition 2 that there exist two real symmetric positive definite matrices Q_{11} and Q_{21}, and two skew-symmetric matrices Q_{12} and Q_{22}, such that

$$\sum_{i=1}^{2}\sum_{j=1}^{2} sym\{\Theta_{ij} \otimes (A(\Delta)Q_{ij})\} < 0, \tag{14}$$

where the Θ_{ij} $(i, j = 1, 2)$ satisfy (2)–(3). From (14), one has

$$T_{00} := \sum_{i=1}^{2}\sum_{j=1}^{2} sym\{\Theta_{ij} \otimes (AQ_{ij})\} + \sum_{i=1}^{2}\sum_{j=1}^{2} sym\{(\Theta_{ij} \otimes D)\,(I_2 \otimes \Delta)\,(I_2 \otimes EQ_{ij})\} < 0. \tag{15}$$

Let $\Phi_{11} = T_{00} - sym\{(\Theta_{11} \otimes D)\,(I_2 \otimes \Delta)\,(I_2 \otimes EQ_{11})\}$; then from (15) one knows that for any $\xi_1, \xi_2 \in R^n$, $\xi = [\xi_1^T, \xi_2^T]^T \ne 0$,

$$\xi^T\Phi_{11}\xi < -2\xi^T(\Theta_{11} \otimes D)\,(I_2 \otimes \Delta)\,(I_2 \otimes EQ_{11})\xi.$$

Define $\Gamma := \{\xi^T(\Theta_{11} \otimes D)\,(I_2 \otimes \Delta)\,(I_2 \otimes EQ_{11})\xi,\ \|\Delta\| \le 1\}$; then $-x \in \Gamma$ for any $x \in \Gamma$. So one has

$$\max_{\|\Delta\|\le 1}\{\xi^T(\Theta_{11} \otimes D)\,(I_2 \otimes \Delta)\,(I_2 \otimes EQ_{11})\xi\} \ge 0.$$

Therefore,

$$\xi^T\Phi_{11}\xi < -2\max_{\|\Delta\|\le 1}\{\xi^T(\Theta_{11} \otimes D)\,(I_2 \otimes \Delta)\,(I_2 \otimes EQ_{11})\xi\} \le 0.$$

And for any given $\xi \in R^{2n}$, $\xi \neq 0$, one obtains

$$(\xi^T\Phi_{11}\xi)^2 > 4\left(\max_{\|\Delta\|\le 1}\{\xi^T(\Theta_{11} \otimes D)\,(I_2 \otimes \Delta)\,(I_2 \otimes EQ_{11})\xi\}\right)^2,$$

and then

$$\left(\max_{\|\Delta\|\le 1}\{\xi^T(\Theta_{11} \otimes D)\,(I_2 \otimes \Delta)\,(I_2 \otimes EQ_{11})\xi\}\right)^2 = \max_{\|\Delta\|\le 1}(\xi^T(\Theta_{11} \otimes D)\,(I_2 \otimes \Delta)\,(I_2 \otimes EQ_{11})\xi)^2. \tag{16}$$

On the basis of Lemma A.1, it is known from (12) and (16) that

$$(\xi^T\Phi_{11}\xi)^2 > 4(\xi^T(\Theta_{11} \otimes D)(\Theta_{11} \otimes D)^T\xi)(\xi^T(I_2 \otimes EQ_{11})^T(I_2 \otimes EQ_{11})\xi). \tag{17}$$

Then it is known from Lemma A.2 that there exists a real scalar $\lambda_{11} > 0$ such that

$$\lambda_{11}^2(\Theta_{11} \otimes D)\,(\Theta_{11} \otimes D)^T + \lambda_{11}\Phi_{11} + (I_2 \otimes EQ_{11})^T(I_2 \otimes EQ_{11}) < 0.$$

Thus,

$$\begin{aligned} T_{11} &:= \Phi_{11} + \lambda_{11}(\Theta_{11} \otimes D)\,(\Theta_{11} \otimes D)^T + \lambda_{11}^{-1}(I_2 \otimes EQ_{11})^T(I_2 \otimes EQ_{11}) \\ &= T_{00} - sym\{(\Theta_{11} \otimes D)\,(I_2 \otimes \Delta)\,(I_2 \otimes EQ_{11})\} + \lambda_{11}(\Theta_{11} \otimes D)\,(\Theta_{11} \otimes D)^T + \lambda_{11}^{-1}(I_2 \otimes EQ_{11})^T(I_2 \otimes EQ_{11}) < 0. \end{aligned}$$

Let $\Phi_{12} = T_{11} - sym\{(\Theta_{12} \otimes D)\,(I_2 \otimes \Delta)\,(I_2 \otimes EQ_{12})\}$; from the above inequality one knows that for all $\xi_1, \xi_2 \in R^n$, $\xi = [\xi_1^T, \xi_2^T]^T \neq 0$,

$$\xi^T\Phi_{12}\xi < -2\xi^T(\Theta_{12} \otimes D)\,(I_2 \otimes \Delta)\,(I_2 \otimes EQ_{11})\xi,$$

and hence,

$$\xi^T\Phi_{12}\xi < -2\max_{\|\Delta\|\le 1}\{\xi^T(\Theta_{12} \otimes D)\,(I_2 \otimes \Delta)\,(I_2 \otimes EQ_{12})\xi\} \le 0.$$

Therefore, for any given $\xi \in R^{2n}$, $\xi \neq 0$, one has

$$\begin{aligned} (\xi^T\Phi_{12}\xi)^2 &> 4\left(\max_{\|\Delta\|\le 1}\{\xi^T(\Theta_{12} \otimes D)(I_2 \otimes \Delta)\,(I_2 \otimes EQ_{12})\xi\}\right)^2 \\ &= 4\max_{\|\Delta\|\le 1}(\xi^T(\Theta_{11} \otimes D)\,(I_2 \otimes \Delta)\,(I_2 \otimes EQ_{11})\xi)^2. \end{aligned}$$

On the basis of Lemma A.1, one has that

$$(\xi^T\Phi_{12}\xi)^2 > 4(\xi^T(\Theta_{12} \otimes D)(\Theta_{12} \otimes D)^T\xi)(\xi^T(I_2 \otimes EQ_{12})^T(I_2 \otimes EQ_{12})\xi).$$

Then it is known from Lemma A.2 that there exists a real scalar $\lambda_{12} > 0$ such that

$$\lambda_{12}^2(\Theta_{12} \otimes D)\,(\Theta_{12} \otimes D)^T + \lambda_{12}\Phi_{12} + (I_2 \otimes EQ_{12})^T(I_2 \otimes EQ_{12}) < 0.$$

Thus,

$$\begin{aligned} T_{12} &:= \Phi_{12} + \lambda_{12}(\Theta_{12} \otimes D)\,(\Theta_{12} \otimes D)^T + \lambda_{12}^{-1}(I_2 \otimes EQ_{12})^T(I_2 \otimes EQ_{12}) \\ &= T_{11} - sym\{(\Theta_{12} \otimes D)\,(I_2 \otimes \Delta)\,(I_2 \otimes EQ_{12})\} + \lambda_{12}(\Theta_{12} \otimes D)\,(\Theta_{12} \otimes D)^T + \lambda_{12}^{-1}(I_2 \otimes EQ_{12})^T(I_2 \otimes EQ_{12}) < 0. \end{aligned}$$

Similarly, let $\Phi_{21} = T_{12} - sym\{(\Theta_{21} \otimes D)\,(I_2 \otimes \Delta)\,(I_2 \otimes EQ_{21})\}$; it is known that there exists a real scalar $\lambda_{21} > 0$ such that

$$\lambda_{21}^2(\Theta_{21} \otimes D)\,(\Theta_{21} \otimes D)^T + \lambda_{21}\Phi_{21} + (I_2 \otimes EQ_{21})^T(I_2 \otimes EQ_{21}) < 0.$$

Thus,

$$\begin{aligned} T_{21} &:= \Phi_{21} + \lambda_{21}(\Theta_{21} \otimes D)\,(\Theta_{21} \otimes D)^T + \lambda_{21}^{-1}(I_2 \otimes EQ_{21})^T(I_2 \otimes EQ_{21}) \\ &= T_{12} - sym\{(\Theta_{21} \otimes D)\,(I_2 \otimes \Delta)\,(I_2 \otimes EQ_{21})\} + \lambda_{21}(\Theta_{21} \otimes D)\,(\Theta_{21} \otimes D)^T + \lambda_{21}^{-1}(I_2 \otimes EQ_{21})^T(I_2 \otimes EQ_{21}) < 0. \end{aligned}$$

Let $\Phi_{22} = T_{21} - sym\{(\Theta_{22} \otimes D)\,(I_2 \otimes \Delta)\,(I_2 \otimes EQ_{22})\}$. It is known that there exists a real scalar $\lambda_{22} > 0$ such that

$$\lambda_{22}^2(\Theta_{22} \otimes D)\,(\Theta_{22} \otimes D)^T + \lambda_{22}\Phi_{22} + (I_2 \otimes EQ_{22})^T(I_2 \otimes EQ_{22}) < 0.$$

Thus,

$$\begin{aligned} T_{22} &:= \Phi_{22} + \lambda_{22}(\Theta_{22} \otimes D)\,(\Theta_{22} \otimes D)^T + \lambda_{22}^{-1}(I_2 \otimes EQ_{22})^T(I_2 \otimes EQ_{22}) \\ &= T_{21} - sym\{(\Theta_{22} \otimes D)\,(I_2 \otimes \Delta)\,(I_2 \otimes EQ_{22})\} + \lambda_{22}(\Theta_{22} \otimes D)\,(\Theta_{22} \otimes D)^T + \lambda_{22}^{-1}(I_2 \otimes EQ_{22})^T(I_2 \otimes EQ_{22}), \end{aligned}$$

i.e.,

$$T_{22} = \sum_{i=1}^{2}\sum_{j=1}^{2} sym\{\Theta_{ij} \otimes (AQ_{ij})\} + \sum_{i=1}^{2}\sum_{j=1}^{2} \lambda_{ij}(\Theta_{ij} \otimes D)(\Theta_{ij} \otimes D)^T + \sum_{i=1}^{2}\sum_{j=1}^{2} \lambda_{ij}^{-1}(I_2 \otimes EQ_{ij})^T(I_2 \otimes EQ_{ij}) < 0. \tag{18}$$

As $\Theta_{ij}\Theta_{ij}^T = I_2$ $(i, j = 1, 2)$, it can be found from (18) that

$$T_{22} := \sum_{i=1}^{2}\sum_{j=1}^{2} sym\{\Theta_{ij} \otimes (AQ_{ij}) + \lambda_{ij}(I_2 \otimes DD^T)\} + \sum_{i=1}^{2}\sum_{j=1}^{2} \lambda_{ij}^{-1}(I_2 \otimes EQ_{ij})^T(I_2 \otimes EQ_{ij}) < 0.$$

By using the Schur complement of the above inequality, one obtains (10). This completes the proof of the theorem. □

Remark 4. Control systems with norm-bounded uncertainty are discussed frequently when no structured information about the uncertainties is assumed. In other words, the basic robust control theory addressing the norm-bounded uncertainty problem can accommodate uncertainties with arbitrary structure. On the basis of Lemma 4, LMI methods have been used for testing the robust stability for the uncertain FO-LTI system (9), and Theorem 2 provides a robust stability condition for (9) no matter what the structure of Δ is.

Remark 5. In [30], it is pointed out that the control problems of controlled systems with unstructured uncertainties are the main issues in nonlinear robust control, which is an important approach in engineering application, e.g., it is extensively applied in motion control of flying robots [31]. Therefore, the robust control of uncertain fractional-order systems with unstructured uncertainties should be a future topic of ours due to its significance both in theory and in application.

4. Numerical examples

In this section, two numerical examples are presented to demonstrate the effectiveness of the proposed theorems.

Example 1. Consider an uncertain FO-LTI system (4) with $\alpha = 0.5$, $N = 2$, $A_1 = \begin{bmatrix} -3.4885 & 2.8677 \\ -1.1614 & -10.5115 \end{bmatrix}$ and $A_2 = \begin{bmatrix} -3.1198 & 4.7278 \\ -5.1389 & -8.8802 \end{bmatrix}$.

By solving complex LMIs (7) and (8), one obtains positive definite Hermitian matrices

$$Q_{11} = \begin{bmatrix} 10.1770 & 1.4741 + j2.7719 \\ 1.4741 - j2.7719 & 3.5683 \end{bmatrix}, \qquad Q_{12} = \begin{bmatrix} 10.2520 & 2.5827 + j3.3649 \\ 2.5827 - j3.3649 & 5.8371 \end{bmatrix},$$

$$Q_{21} = \begin{bmatrix} 10.1770 & 1.4741 - j2.7719 \\ 1.4741 + j2.7719 & 3.5683 \end{bmatrix}, \qquad Q_{22} = \begin{bmatrix} 10.2520 & 2.5827 - j3.3649 \\ 2.5827 + j3.3649 & 5.8371 \end{bmatrix}.$$

By Theorem 1, one can conclude that the uncertain FO-LTI system (4) is robustly stable. For numerical confirmation, we chose 100 random numbers for ξ_1 in [0, 1] with the uniform distribution; then 100 matrix pairs $(A(\xi), Q_1(\xi), Q_2(\xi))$ were obtained for the following inequality to be satisfied for each matrix pair $(A(\xi), Q_1(\xi), Q_2(\xi))$:

$$e^{i\theta}Q_1(\xi)A(\xi) + e^{-i\theta}A^*(\xi)Q_1(\xi) + e^{-i\theta}Q_2(\xi)A(\xi) + e^{i\theta}A^*(\xi)Q_2(\xi) < 0.$$

Thus, Theorem 1 is confirmed numerically.

Example 2. Consider an uncertain FO-LTI system described by (9) with $\alpha = 0.5$, $A = \begin{bmatrix} -4.3228 & 0.4066 \\ 0.5306 & -3.6772 \end{bmatrix}$, $D = \begin{bmatrix} 0.6555 & 0.7060 \\ 0.1712 & 0.0318 \end{bmatrix}$, and $E = \begin{bmatrix} 0.6324 & 0.2785 \\ 0.0975 & 0.5469 \end{bmatrix}$. By solving (10) and (11), one obtains

$$Q_{11} = \begin{bmatrix} 4.4590 & 0.8436 \\ 0.8436 & 1.6910 \end{bmatrix}, \qquad Q_{12} = \begin{bmatrix} 0 & 0.0852 \\ -0.0852 & 0 \end{bmatrix},$$

$$Q_{21} = \begin{bmatrix} 4.4590 & 0.8436 \\ 0.8436 & 1.6910 \end{bmatrix}, \qquad Q_{22} = \begin{bmatrix} 0 & -0.0852 \\ 0.0852 & 0 \end{bmatrix},$$

and $\lambda_{11} = \lambda_{12} = \lambda_{21} = \lambda_{22} = 10.9045$.

Therefore, by Theorem 2, the uncertain FO-LTI system (9) is robustly stable.

To confirm this conclusion numerically, we choose 100 random uncertain matrices Δ satisfying $\|\Delta\| \leq 1$. It can be shown that $|\arg(eig(A(\Delta)))| > \alpha\frac{\pi}{2}$ holds for any chosen Δ. Thus, Theorem 2 is verified numerically.

5. Conclusions

Robust stability for uncertain FO-LTI systems of order $\alpha \in (0, 1)$ with structured and unstructured uncertainties was discussed in this paper. For fractional-order systems with polytope-type uncertainty, a less conservative sufficient condition was given: that the robust stability can be guaranteed by the existence of two positive definite parameter dependent Hermitian matrices. For fractional-order systems with norm-bounded uncertainty, a sufficient and necessary condition for robust stability was shown, in terms of two real symmetric positive definite matrices and two skew-symmetric matrices, which can be obtained by solving a set of linear matrix inequalities. Finally, two numerical examples were presented to illustrate the effectiveness of our results.

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Appendix

Lemma A.1 ([32]). *Given any $x \in R^n$, the following equation holds:*

$$\max_{\|\Delta\|\leq 1}\{(x^T D\Delta Ex)^2 : \Delta^T\Delta \leq I, \Delta \in R^{n\times n}\} = x^T DD^T xx^T E^T Ex.$$

Lemma A.2 ([32]). *Let X, Y and Z be given $n\times n$ real matrices such that $X \geq 0$, $Y < 0$ and $Z \geq 0$. Assume $(x^T Yx)^2 - 4x^T Xxx^T Zx > 0$ holds for all $x \in R^n$, with $x \neq 0$. Then there exists a constant $\lambda > 0$ such that*

$$\lambda^2 X + \lambda Y + Z < 0.$$

Lemma A.3 ([25]). *For any matrices X and Y with appropriate dimensions, we have*

$$X^T Y + Y^T X \leq \lambda X^T X + (1/\lambda)Y^T Y$$

valid for any $\lambda > 0$.

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